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Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem

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Abstract

The authors consider the boundary value problem

$$\begin{cases} y'' - \rho^2 y + \lambda g(t) f(y) = 0, & 0 \leqslant t \leqslant 2\pi, \\ y(0) = y(2\pi), & y'(0) = y'(2\pi). \end{cases}$$

Under different combinations of superlinearity and sublinearity of the function f, various existence, multiplicity, and nonexistence results for positive solutions are derived in terms of different values of λ. The uniqueness of solutions and the dependence of solutions on the parameter λ are also studied. The results are illustrated with an example.

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Keywords: Dependence on a parameter; Existence of positive solutions; Krasnosel'skii's theorem; Multiplicity of positive solutions; Periodic boundary value problem

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1. Introduction

Krasnosel'skii's theorem in a cone has often been used to study the existence and multiplicity of positive solutions of periodic boundary value problems over the last several years. As recent

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examples, we mention the papers of Atici and Guseinov [3], Jiang et al. [5], Li [7], O'Regan and Wang [10], Torres [11], and Zhang and Wang [13]. Here, we consider the problem of existence, multiplicity, and nonexistence of positive solutions for the periodic boundary value problem

$$\begin{cases} y'' - \rho^2 y + \lambda g(t) f(y) = 0, & 0 \leqslant t \leqslant 2\pi, \\ y(0) = y(2\pi), & y'(0) = y'(2\pi), \end{cases} \tag{1.1}$$

where $\rho > 0$ is a constant and λ is a positive parameter. We will also examine the uniqueness of the solutions and their dependence on the parameter λ. Our basic assumptions here are:

(A1) $f : [0, \infty) \to [0, \infty)$ is continuous and $f(u) > 0$ for $u > 0$;
(A2) $g : [0, 2\pi] \to [0, \infty)$ is continuous and $\int_0^{2\pi} g(t)\, dt > 0$;
(A3) $f : [0, \infty) \to (0, \infty)$ is nondecreasing, and there exists $\theta \in (0, 1)$ such that

$$f(\kappa u) \geqslant \kappa^\theta f(u) \quad \text{for } \kappa \in (0, 1) \text{ and } u \in [0, \infty).$$

In the next section, we state our results for the problem (1.1). In Section 3 we present some preliminary lemmas and then prove the main results in Section 4. The final section of the paper contains an example to illustrate our results.

2. Main results

We begin by introducing the notations

$$f_0 = \lim_{u \to 0} \frac{f(u)}{u} \quad \text{and} \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

We will also need the function

$$f^*(u) = \max_{0 \leqslant t \leqslant u} \{ f(t) \}$$

and we let $f_0^* = \lim_{u \to 0} f^*(u)/u$ and $f_\infty^* = \lim_{u \to \infty} f^*(u)/u$. Our existence result is the following.

Theorem 2.1. *Assume that* (A1)–(A2) *hold.*

(a) *If* $f_0 = 0$ *or* $f_\infty = 0$*, then there exists* $\lambda_0 > 0$ *such that* (1.1) *has a positive solution for* $\lambda > \lambda_0$.
(b) *If* $f_0 = \infty$ *or* $f_\infty = \infty$*, then there exists* $\lambda_0 > 0$ *such that* (1.1) *has a positive solution for* $0 < \lambda < \lambda_0$.
(c) *If* $f_0 = f_\infty = 0$*, then there exists* $\lambda_0 > 0$ *such that* (1.1) *has at least two positive solutions for* $\lambda > \lambda_0$.
(d) *If* $f_0 = f_\infty = \infty$*, then there exists* $\lambda_0 > 0$ *such that* (1.1) *has at least two positive solutions for* $0 < \lambda < \lambda_0$.
(e) *If* $f_0 < \infty$ *and* $f_\infty < \infty$*, then there exists* $\lambda_0 > 0$ *such that* (1.1) *has no positive solutions for* $0 < \lambda < \lambda_0$.
(f) *If* $f_0 > 0$ *and* $f_\infty > 0$*, then there exists* $\lambda_0 > 0$ *such that* (1.1) *has no positive solutions for* $\lambda > \lambda_0$.

Our next result concerns the uniqueness and dependence of solutions of (1.1) on the parameter λ. Let $\|u\| = \max_{t\in[0,2\pi]} |u(t)|$ for any continuous function $u(t)$ on $[0, 2\pi]$.

Theorem 2.2. *Assume that* (A1)–(A3) *hold. Then, for any* $\lambda \in (0, \infty)$, (1.1) *has a unique positive solution* $u_\lambda(t)$. *Furthermore, such a solution* $u_\lambda(t)$ *satisfies the following properties*:

(i) $u_\lambda(t)$ *is nondecreasing in* λ;
(ii) $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$ *and* $\lim_{\lambda\to\infty} \|u_\lambda\| = \infty$;
(iii) $u_\lambda(t)$ *is continuous in* λ, *that is, if* $\lambda \to \lambda_{\lambda_0}$, *then* $\|u_\lambda - u_{\lambda_0}\| \to 0$.

As a consequence of Theorem 2.2, we have the following result.

Corollary 2.1. *Assume that* (A1)–(A3) *hold. Then, for each* $M \in (0, \infty)$, *there exists* $\lambda^* \in (0, \infty)$ *such that* (1.1) *has a positive solution* $u^*(t)$ *with* $\|u^*\| = M$.

Remark 2.1. We note that:

(1) Results similar to Theorem 2.2 have been established by Li and Liu [8,9] for other types of boundary value problems. Some ideas of the proof of Theorem 2.2 are from [8,9].
(2) The problem of finding solutions of boundary value problems with given maximum has been studied by Agarwal, O'Regan, and Staněk. For more details on this study, we refer the reader to [1] for a higher order problem with Lidstone boundary conditions, and [2] for a second order problem with a nonlinear term in the equation and Dirichlet boundary conditions.

3. Preliminary lemmas

Our first lemma gives some relationships between the functions f and f^*.

Lemma 3.1. *(See [12].) Assume* (H1) *holds. Then* $f_0^* = f_0$ *and* $f_\infty^* = f_\infty$.

The following fixed-point theorem of cone expansion/compression type is crucial in the proofs of our results.

Lemma 3.2. *(See [4,6].) Let* X *be a Banach space and let* $K \subset X$ *be a cone in* X. *Assume* Ω_1, Ω_2 *are bounded open subsets of* X *with* $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ *and let*

$$F : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Fu\| \leqslant \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Fu\| \geqslant \|u\|$ *for any* $u \in K \cap \partial\Omega_2$,

or

(ii) $\|Fu\| \geqslant \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Fu\| \leqslant \|u\|$ *for any* $u \in K \cap \partial\Omega_2$.

Then F *has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We consider the function

$$G(t,s)=\begin{cases}\frac{e^{\rho(t-s)}+e^{\rho(2\pi-t+s)}}{2\rho(e^{2\rho\pi}-1)}, & 0\leqslant s\leqslant t\leqslant 2\pi,\\ \frac{e^{\rho(s-t)}+e^{\rho(2\pi-s+t)}}{2\rho(e^{2\rho\pi}-1)}, & 0\leqslant t\leqslant s\leqslant 2\pi.\end{cases}$$

Define

$$\hat{G}(x)=\frac{e^{\rho x}+e^{\rho(2\pi-x)}}{2\rho(e^{2\rho\pi}-1)}\quad \text{for } x\in[0,2\pi].$$

Then, it is easy to check that $\hat{G}$ is decreasing on $[0,\pi]$, increasing on $[\pi,2\pi]$, and $G(t,s)=\hat{G}(|t-s|)$. Thus,

$$\frac{e^{\rho\pi}}{\rho(e^{2\rho\pi}-1)}=\hat{G}(\pi)\leqslant G(t,s)\leqslant \hat{G}(0)=\frac{1+e^{\rho 2\pi}}{2\rho(e^{2\rho\pi}-1)}$$

for $s,t\in[0,2\pi]$.

Let X be the Banach space $C[0,2\pi]$ endowed with the norm

$$\|u\|=\max_{0\leqslant t\leqslant 2\pi}\bigl|u(t)\bigr|.$$

Define the cone K in X by

$$K=\Bigl\{u\in X\colon\ u(t)\geqslant 0 \text{ on } [0,2\pi] \text{ and } \min_{0\leqslant t\leqslant 2\pi}u(t)\geqslant\sigma\|u\|\Bigr\},$$

where $\sigma=2e^{\rho\pi}/(1+e^{2\rho\pi})$, and for $r>0$, let

$$\Omega_r=\bigl\{u\in K\colon\ \|u\|<r\bigr\}.$$

Define the map $T_\lambda : K\to X$ by

$$T_\lambda u(t)=\lambda\int_0^{2\pi}G(t,s)g(s)f\bigl(u(s)\bigr)\,ds,\quad 0\leqslant t\leqslant 2\pi.$$

Then the following lemma can be easily verified.

Lemma 3.3. *Assume* (A1)–(A2) *hold. Then $u\in K$ is a positive fixed point of T_λ if and only if u is a positive solution of* (1.1).

In the next lemma, we show that T_λ is completely continuous and maps K into itself.

Lemma 3.4. *Assume* (A1)–(A2) *hold. Then $T_\lambda(K)\subset K$ and $T_\lambda : K\to K$ is completely continuous.*

Proof. Let $u \in K$; then $T_\lambda u(t) \geqslant 0$ on $[0, 2\pi]$ and

$$\min_{0 \leqslant t \leqslant 2\pi} T_\lambda u(t) \geqslant \hat{G}(\pi)\lambda \int_0^{2\pi} g(s) f\big(u(s)\big)\, ds = \sigma \hat{G}(0)\lambda \int_0^{2\pi} g(s) f\big(u(s)\big)\, ds \geqslant \sigma \|T_\lambda u\|,$$

i.e., $T_\lambda(K) \subset K$. A standard argument can be used to show that $T_\lambda : K \to K$ is completely continuous. □

In the next two lemmas, we obtain lower and upper estimates on the operator T_λ. Define

$$\Gamma = \hat{G}(\pi)\sigma \int_0^{2\pi} g(s)\, ds.$$

Lemma 3.5. *Assume* (A1) *holds and let $\eta > 0$ be given. If $u \in K$ and $f(u(t)) \geqslant u(t)\eta$ for $t \in [0, 2\pi]$, then*

$$\|T_\lambda u\| \geqslant \lambda \Gamma \eta \|u\|.$$

Proof. From the definitions of $T_\lambda u$ and K, it follows that

$$\begin{aligned} \|T_\lambda u\| &\geqslant \lambda \hat{G}(\pi) \int_0^{2\pi} g(s) f\big(u(s)\big)\, ds \geqslant \lambda \hat{G}(\pi)\eta \int_0^{2\pi} g(s) u(s)\, ds \\ &\geqslant \lambda \hat{G}(\pi)\eta\sigma \|u\| \int_0^{2\pi} g(s)\, ds = \lambda \Gamma \eta \|u\|. \end{aligned}$$

This completes the proof. □

Lemma 3.6. *Assume* (A1) *holds and let $r > 0$ be given. If there exists $\varepsilon > 0$ such that $f^*(r) \leqslant \varepsilon r$, then*

$$\|T_\lambda u\| \leqslant \lambda \varepsilon \|u\| \hat{G}(0) \int_0^{2\pi} g(s)\, ds \quad \textit{for } u \in \partial\Omega_r.$$

Proof. From the definition of T_λ, we have that

$$\|T_\lambda u\| \leqslant \lambda \hat{G}(0) \int_0^{2\pi} g(s) f\big(u(s)\big)\, ds \leqslant \lambda \hat{G}(0) \int_0^{2\pi} g(s) f^*(r)\, ds \leqslant \lambda \varepsilon \|u\| \hat{G}(0) \int_0^{2\pi} g(s)\, ds$$

for $u \in \partial\Omega_r$. This completes the proof. □

The following two lemmas are weak forms of Lemmas 3.5 and 3.6.

Lemma 3.7. *Assume* (A1)–(A2) *hold. If $u \in \partial\Omega_r$, $r > 0$, then*

$$\|T_\lambda u\| \geqslant \lambda \hat{m}_r \hat{G}(\pi) \int_0^{2\pi} g(s)\,ds,$$

where $\hat{m}_r = \min_{r\sigma \leqslant t \leqslant r}\{f(t)\} > 0$.

Proof. Since $f(u(t)) \geqslant \hat{m}_r$ for $t \in [0, 2\pi]$, it follows that

$$\|T_\lambda u\| \geqslant \lambda \hat{G}(\pi) \int_0^{2\pi} g(s) f\bigl(u(s)\bigr)\,ds \geqslant \lambda \hat{m}_r \hat{G}(\pi) \int_0^{2\pi} g(s)\,ds.$$

This completes the proof. □

Lemma 3.8. *Assume* (A1)–(A2) *hold. If $u \in \partial\Omega_r$, $r > 0$, then*

$$\|T_\lambda u\| \leqslant \lambda \hat{M}_r \hat{G}(0) \int_0^{2\pi} g(s)\,ds,$$

where $\hat{M}_r = 1 + \max_{0 \leqslant t \leqslant r}\{f(t)\} > 0$.

Proof. Since $f(u(t)) \leqslant \hat{M}_r$ for $t \in [0, 2\pi]$, we have

$$\|T_\lambda u\| \leqslant \lambda \hat{G}(0) \int_0^{2\pi} g(s) f\bigl(u(s)\bigr)\,ds \leqslant \lambda \hat{M}_r \hat{G}(0) \int_0^{2\pi} g(s)\,ds$$

for $u \in \partial\Omega_r$. This completes the proof. □

Our final lemma in this section gives upper and lower estimates for the operator T_λ.

Lemma 3.9. *Assume* (A1)–(A3) *hold. Then, for any nonnegative $u \in X$, there exists $D_u \geqslant C > 0$ such that*

$$C L_\lambda \leqslant T_\lambda u(t) \leqslant D_u L_\lambda, \tag{3.1}$$

where

$$L_\lambda = \lambda \int_0^{2\pi} g(s)\,ds. \tag{3.2}$$

Proof. Recall that $f(0) > 0$ and f is nondecreasing. Then, for any nonnegative $u \in X$ and $t \in [0, 2\pi]$, we have

$$T_\lambda u(t) \geqslant \lambda f(0)\hat{G}(\pi)\int_0^{2\pi} g(s)\,ds = f(0)\hat{G}(\pi)L_\lambda := CL_\lambda.$$

Clearly, $C > 0$ and is independent of $u(t)$. Again, from the monotonicity of f, we have that

$$T_\lambda u(t) \leqslant \lambda\hat{G}(0)f\big(\|u\|\big)\int_0^{2\pi} g(s)\,ds = \hat{G}(0)f\big(\|u\|\big)L_\lambda := D_u L_\lambda.$$

It is obvious that $D_u \geqslant C$. This completes the proof. □

4. Proofs of the main results

Proof of Theorem 2.1. Part (a). Choose a number $r_1 > 0$. By Lemma 3.7, we have

$$\|T_\lambda u\| > \|u\| \quad \text{for } u \in \partial\Omega_{r_1} \text{ and } \lambda > \lambda_0,$$

where

$$\lambda_0 \geqslant \frac{r_1}{\hat{m}_{r_1}\hat{G}(\pi)\int_0^{2\pi} g(s)\,ds} > 0.$$

If $f_0 = 0$, then from Lemma 3.1, $f_0^* = 0$, and so we can choose $r_2 \in (0, r_1)$ so that $f^*(r_2) \leqslant \varepsilon r_2$, where $\varepsilon > 0$ satisfies

$$\lambda\varepsilon\hat{G}(0)\int_0^{2\pi} g(s)\,ds < 1. \tag{4.1}$$

Then, Lemma 3.6 implies that

$$\|T_\lambda u\| \leqslant \lambda\varepsilon\|u\|\hat{G}(0)\int_0^{2\pi} g(s)\,ds < \|u\| \quad \text{for } u \in \partial\Omega_{r_2}.$$

If $f_\infty = 0$, then from Lemma 3.1, $f_\infty^* = 0$. Hence, there exists $r_3 \in (2r_1, \infty)$ such that $f^*(r_3) \leqslant \varepsilon r_3$, where $\varepsilon > 0$ satisfies (4.1). Thus,

$$\|T_\lambda u\| \leqslant \lambda\varepsilon\|u\|\hat{G}(0)\int_0^{2\pi} g(s)\,ds < \|u\| \quad \text{for } u \in \partial\Omega_{r_3}.$$

Then, from Lemma 3.2, T_λ has a fixed point in $\overline{\Omega}_{r_1} \setminus \Omega_{r_2}$ or $\overline{\Omega}_{r_3} \setminus \Omega_{r_1}$ according to whether $f_0 = 0$ or $f_\infty = 0$, respectively. Consequently, (1.1) has a positive solution for $\lambda > \lambda_0$.

Part (b). Choose a number $r_1 > 0$. By Lemma 3.8, there exists $\lambda_0 > 0$ such that

$$\|T_\lambda u\| < \|u\| \quad \text{for } u \in \partial\Omega_{r_1} \text{ and } 0 < \lambda < \lambda_0.$$

If $f_0 = \infty$, then there exists $r_2 \in (0, r_1)$ such that $f(u) \geqslant \eta u$ for $0 \leqslant u \leqslant r_2$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1. \tag{4.2}$$

Clearly,

$$f\big(u(t)\big) \geqslant \eta u(t) \quad \text{for } u \in \partial\Omega_{r_2},\ t \in [0, 2\pi].$$

Then, from Lemma 3.5,

$$\|T_\lambda u\| \geqslant \lambda \Gamma \eta \|u\| > \|u\| \quad \text{for } u \in \partial\Omega_{r_2}.$$

If $f_\infty = \infty$, then there exists $\hat{H} > 0$ such that $f(u) \geqslant \eta u$ for $u \geqslant \hat{H}$, where $\eta > 0$ satisfies (4.2). Let $r_3 = \max\{2r_1, \hat{H}/\sigma\}$. If $u \in \partial\Omega_{r_3}$, then

$$\min_{0 \leqslant t \leqslant 2\pi} u(t) \geqslant \sigma \|u\| \geqslant \hat{H}.$$

As a result,

$$f\big(u(t)\big) \geqslant \eta u(t) \quad \text{for } t \in [0, 2\pi].$$

From Lemma 3.5, it follows that

$$\|T_\lambda u\| \geqslant \lambda \Gamma \eta \|u\| > \|u\| \quad \text{for } u \in \partial\Omega_{r_3}.$$

Then, Lemma 3.2 implies that T_λ has a fixed point in $\overline{\Omega}_{r_1} \setminus \Omega_{r_2}$ or $\overline{\Omega}_{r_3} \setminus \Omega_{r_1}$ according to whether $f_0 = \infty$ or $f_\infty = \infty$, respectively. Consequently, (1.1) has a positive solution for $0 < \lambda < \lambda_0$.

Part (c). Choose two numbers $0 < r_3 < r_4$. By Lemma 3.7, there exists $\lambda_0 > 0$ such that

$$\|T_\lambda u\| > \|u\| \quad \text{for } \lambda > \lambda_0,\ u \in \partial\Omega_{r_i},\ i = 3, 4.$$

Since $f_0 = 0$ and $f_\infty = 0$, from the proof of Theorem 2.1(a), it follows that we can choose $r_1 \in (0, r_3/2)$ and $r_2 \in (2r_4, \infty)$ such that

$$\|T_\lambda u\| < \|u\| \quad \text{for } u \in \partial\Omega_{r_i},\ i = 1, 2.$$

From Lemma 3.2, T_λ has two fixed points u_1 and u_2 such that $u_1 \in \overline{\Omega}_{r_3} \setminus \Omega_{r_1}$ and $u_2 \in \overline{\Omega}_{r_2} \setminus \Omega_{r_4}$. These are the desired distinct positive solutions of (1.1) for $\lambda > \lambda_0$ satisfying

$$r_1 \leqslant \|u_1\| \leqslant r_3 < r_4 \leqslant \|u_2\| \leqslant r_2. \tag{4.3}$$

Part (d). Choose two numbers $0 < r_3 < r_4$. By Lemma 3.8, there exists $\lambda_0 > 0$ such that

$$\|T_\lambda u\| < \|u\| \quad \text{for } u \in \partial\Omega_{r_i},\ 0 < \lambda < \lambda_0,\ i = 3, 4.$$

Since $f_0 = \infty$ and $f_\infty = \infty$, from the proof of Theorem 2.1(b), we see that we can choose $r_1 \in (0, r_3/2)$ and $r_2 \in (2r_4, \infty)$ such that

$$\|T_\lambda u\| > \|u\| \quad \text{for } u \in \partial\Omega_{r_i},\ i = 1, 2.$$

From Lemma 3.2, T_λ has two fixed points u_1 and u_2 such that $u_1 \in \Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ and $u_2 \in \Omega_{r_2} \setminus \overline{\Omega}_{r_4}$, which are the desired distinct positive solutions of (1.1) for $0 < \lambda < \lambda_0$ satisfying (4.3).

Part (e). Since $f_0 < \infty$ and $f_\infty < \infty$, there exist positive numbers ε_1, ε_2, r_1, and r_2 such that $r_1 < r_2$, and

$$f(u) \leqslant \varepsilon_1 u \quad \text{for } u \in [0, r_1],$$

$$f(u) \leqslant \varepsilon_2 u \quad \text{for } u \in [r_2, \infty).$$

Let the positive number ε_3 be defined by

$$\varepsilon_3 = \max\left\{\varepsilon_1, \varepsilon_2, \max_{r_1 \leqslant u \leqslant r_2}\left\{\frac{f(u)}{u}\right\}\right\}.$$

Then,

$$f(u) \leqslant \varepsilon_3 u \quad \text{for } u \in [0, \infty).$$

Assume $v(t)$ is a positive solution of (1.1). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 = 1/(\varepsilon_3 \hat{G}(0) \int_0^{2\pi} g(s)\, ds)$. Since $T_\lambda v(t) = v(t)$ for $t \in [0, 1]$, by Lemma 3.6, we have that

$$\|v\| = \|T_\lambda v\| \leqslant \lambda \hat{G}(0)\varepsilon_3 \|v\| \int_0^{2\pi} g(s)\, ds < \|v\|,$$

which is a contradiction.

Part (f). Since $f_0 > 0$ and $f_\infty > 0$, there exist positive numbers η_1, η_2, r_1, and r_2 such that $r_1 < r_2$, and

$$f(u) \geqslant \eta_1 u \quad \text{for } u \in [0, r_1],$$

$$f(u) \geqslant \eta_2 u \quad \text{for } u \in [r_2, \infty).$$

Let the positive number ε_3 be defined by

$$\eta_3 = \min\left\{\eta_1, \eta_2, \min_{r_1 \leqslant u \leqslant r_2}\left\{\frac{f(u)}{u}\right\}\right\}.$$

Then,

$$f(u) \geqslant \eta_3 u \quad \text{for } u \in [0, \infty).$$

Assume $v(t)$ is a positive solution of (1.1). We will show that this leads to a contradiction for $\lambda > \lambda_0 = 1/(\Gamma \eta_3)$. Since $T_\lambda v(t) = v(t)$ for $t \in [0, 1]$, by Lemma 3.5, we have that

$$\|v\| = \|T_\lambda v\| \geqslant \lambda \Gamma \eta_3 \|v\| > \|v\|,$$

which is a contradiction. This completes the proof. □

Proof of Theorem 2.2. We first show that, for any fixed $\lambda \in (0, \infty)$, (1.1) has a solution. From (A3), we see that T_λ is nondecreasing and satisfies

$$\begin{aligned} T_\lambda\big(\kappa u(t)\big) &= \lambda \int_0^{2\pi} G(t,s) g(s) f\big(\kappa u(s)\big)\, ds \\ &\geqslant \kappa^\theta \lambda \int_0^{2\pi} G(t,s) g(s) f\big(u(s)\big)\, ds = \kappa^\theta T_\lambda u(t) \end{aligned} \tag{4.4}$$

for $u \in X$ with $u(t) \geqslant 0$ for $t \in [0, 2\pi]$. Let L_λ be defined by (3.2) and define $\bar{u}(t) = L_\lambda$ for $t \in [0, 2\pi]$. Then, $\bar{u}(t) \in X$ and $\bar{u}(t) > 0$ on $[0, 2\pi]$. Thus, by Lemma 3.9,

$$C L_\lambda \leqslant T_\lambda \bar{u}(t) \leqslant D_{L_\lambda} L_\lambda.$$

Let $\bar{C}$ and $\bar{D}$ be defined by

$$\bar{C} = \sup\big\{x\colon\ x L_\lambda \leqslant T_\lambda \bar{u}(t)\big\} \quad \text{and} \quad \bar{D} = \inf\big\{x\colon\ T_\lambda \bar{u}(t) \leqslant x L_\lambda\big\}.$$

Clearly, $\bar{C} \geqslant C$ and $\bar{D} \leqslant D_{L_\lambda}$. Choose $\hat{C}$ and $\hat{D}$ such that

$$0 < \hat{C} < \min\big\{1, (\bar{C})^{\frac{1}{1-\theta}}\big\} \quad \text{and} \quad \max\big\{1, (\bar{D})^{\frac{1}{1-\theta}}\big\} < \hat{D} < \infty.$$

Define two sequences $\{u_k(t)\}_{k=1}^\infty$ and $\{v_k(t)\}_{k=1}^\infty$ by

$$u_1(t) = \hat{C} L_\lambda, \qquad u_{k+1}(t) = T_\lambda u_k(t), \quad t \in [0, 2\pi],\ k = 1, 2, \ldots$$

and

$$v_1(t) = \hat{D} L_\lambda, \qquad v_{k+1}(t) = T_\lambda v_k(t), \quad t \in [0, 2\pi],\ k = 1, 2, \ldots.$$

Then, from the monotonicity of T_λ and (4.4), we obtain that

$$\hat{C} L_\lambda = u_1(t) \leqslant u_2(t) \leqslant \cdots \leqslant u_k(t) \leqslant \cdots \leqslant v_k(t) \leqslant \cdots \leqslant v_2(t) \leqslant v_1(t) = \hat{D} L_\lambda. \tag{4.5}$$

Let $d = \hat{C}/\hat{D}$. Then $d \in (0,1)$. We now claim that

$$u_k(t) \geqslant d^{\theta^k} v_k(t) \quad \text{for } t \in [0, 2\pi]. \tag{4.6}$$

In fact, it is obvious that $u_1(t) = dv_1(t)$ on $[0, 2\pi]$. Assume (4.6) holds for $k = n$, i.e., $u_n(t) \geqslant d^{\theta^n} v_n(t)$ for $t \in [0, 2\pi]$. Then, from the monotonicity of T_λ and (4.4), we see that

$$u_{n+1}(t) = T_\lambda u_n(t) \geqslant T_\lambda\big(d^{\theta^n} v_n(t)\big) \geqslant \big(d^{\theta^n}\big)^\theta T_\lambda v_n(t) = d^{\theta^{n+1}} v_{n+1}(t)$$

for $t \in [0, 2\pi]$. Hence, by induction, (4.6) holds. From (4.5) and (4.6), it follows that

$$0 \leqslant u_{k+l}(t) - u_k(t) \leqslant v_k(t) - u_k(t) \leqslant \big(1 - d^{\theta^k}\big) v_1(t) = \big(1 - d^{\theta^k}\big)\hat{D} L_\lambda$$

for $t \in [0, 2\pi]$, where l is a nonnegative integer. Thus,

$$\|u_{k+l} - u_k\| \leqslant \|v_k - u_k\| \leqslant \big(1 - d^{\theta^k}\big)\hat{D} L_\lambda.$$

Therefore, there exists a positive function $\tilde{u} \in X$ such that

$$\lim_{k\to\infty} u_k(t) = \lim_{k\to\infty} v_k(t) = \tilde{u}(t) \quad \text{for } t \in [0, 2\pi].$$

Clearly, $\tilde{u}(t)$ is a positive solution of (1.1).

Next, we show the uniqueness of solutions of (1.1). Assume, to the contrary, that there exist two positive solutions $u_1(t)$ and $u_2(t)$ of (1.1); then $T_\lambda u_1(t) = u_1(t)$ and $T_\lambda u_2(t) = u_2(t)$ for $t \in [0, 2\pi]$. We note that there exists $\alpha > 0$ such that $u_1(t) \geqslant \alpha u_2(t)$ for $t \in [0, 2\pi]$. Let $\alpha_0 = \sup\{\alpha\colon\ u_1(t) \geqslant \alpha u_2(t)\}$. Then $0 < \alpha_0 < \infty$ and $u_1(t) \geqslant \alpha_0 u_2(t)$ for $t \in [0, 2\pi]$. We now show that $\alpha_0 \geqslant 1$. In fact, if $\alpha_0 < 1$, then, from (A3), $f(\alpha_0 u_2(t)) > \alpha_0 f(u_2(t))$ on $[0, 2\pi]$. This, together with the monotonicity of f, implies that

$$u_1(t) = T_\lambda u_1(t) \geqslant T_\lambda\big(\alpha_0 u_2(t)\big) > \alpha_0 T_\lambda u_2(t) = \alpha_0 u_2(t) \quad \text{for } t \in [0, 2\pi].$$

Thus, we can find $\tau > 0$ such that $u_1(t) \geqslant (\alpha_0 + \tau)u_2(t)$ on $[0, 2\pi]$, which contradicts the definition of α_0. Hence, $u_1(t) \geqslant u_2(t)$ for $t \in [0, 2\pi]$. Similarly, we can show that $u_2(t) \geqslant u_1(t)$ for $t \in [0, 2\pi]$. Therefore, (1.1) has a unique solution.

Using exactly the same argument as in the second part of the proof of [9, Theorem 6], we can show that (i), (ii), and (iii) hold. The details are omitted here. This completes the proof of the theorem. □

Proof of Corollary 2.1. The conclusion readily follows from Theorem 2.2. □

Remark 4.1. In Theorem 2.2, we have that f is nondecreasing and $f(0) > 0$, so $f_0 = \infty$. In addition, we see that condition (A3) implies

$$\frac{f(\kappa u)}{\kappa u} \geqslant \frac{\kappa^\theta f(u)}{\kappa u} = \kappa^{\theta-1}\frac{f(u)}{u},$$

and so

$$f_\infty \geqslant \kappa^{\theta-1} f_\infty.$$

Thus,

$$\left(1-\kappa^{\theta-1}\right) f_\infty \geqslant 0,$$

and hence $f_\infty = 0$ since $1-\kappa^{\theta-1} < 0$. It would then be easy to construct a proof using Lemma 3.2 to show that a positive solution to our problem exists for every $0 < \lambda < \infty$.

5. Example

As an example of our results in this paper, we have the following example.

Example 5.1. Consider the boundary value problem (1.1), where $\rho > 0$ is a constant, λ is a positive parameter, $g(t)$ is any nonnegative continuous function on $[0, 2\pi]$, $g(t) \not\equiv 0$ on $[0, 2\pi]$, and

$$f(u) = \sum_{i=1}^{n} u^{\alpha_i} + 1$$

with n an integer and $\alpha_i \in (0, 1)$, $i = 1, \ldots, n$. We claim that, for any $\lambda \in (0, \infty)$, the problem (1.1) has a unique solution $u_\lambda(t)$ satisfying the properties (i), (ii), and (iii) stated in Theorem 2.2, i.e., $u_\lambda(t)$ satisfies

(i) $u_\lambda(t)$ is nondecreasing in λ;
(ii) $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$, and $\lim_{\lambda\to\infty} \|u_\lambda\| = \infty$; and
(iii) $u_\lambda(t)$ is continuous in λ.

In fact, for the above functions g and f, (A1) and (A2) are trivially satisfied. Note that for $u \in [0, \infty)$, $f > 0$ and is nondecreasing. Moreover, for $\theta \in (\sup_i \alpha_i, 1)$, it is easy to see that

$$f(\kappa u) \geqslant \kappa^\theta f(u) \quad \text{for } \kappa \in (0, 1) \text{ and } u \in [0, \infty),$$

i.e., (A3) holds. The conclusion then follows from Theorem 2.2.

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