A characterization of isotropic immersions by extrinsic shapes of smooth curves

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Abstract

In this paper we show that an isometric immersion is isotropic in the sense of O’Neill if and only if it preserves logarithmic derivatives of first geodesic curvatures of some curves.

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1. Introduction

For an isometric immersion \( f : M \to \tilde{M} \) between Riemannian manifolds, we consider a function on the unit tangent bundle \( UM \) of \( M \) defined by \( \| \sigma_f(u, u) \| \) for \( u \in UM \) with the second fundamental form \( \sigma_f \) of \( f \). When this function can be regard as a function of \( M \), namely \( \| \sigma_f(u, u) \| \) does not depend on the choice of unit tangent vector \( u \in UxM \) at each point \( x \in M \), we call \( f \) isotropic in the sense of O’Neill [7]. When \( f \) is an isotropic immersion, we shall call this function \( \lambda_f \) the isotropy of \( f \). We say \( f \) is constant isotropic if the function \( \lambda_f \) is a constant function.

Isotropic condition on immersions is related to other geometric conditions. For example, we consider a parallel submanifold \( M \) of a standard sphere \( S^N \) through an immersion \( f \). Needless to say \( M \) is a compact Riemannian symmetric space. Moreover, by the classification theorem of parallel submanifolds in \( S^N \) we can see that \( M \) is of rank one if and only if the isometric immersion \( f : M \to S^N \) is isotropic (see [2,11]). That is, in this case the isotropic condition distinguishes symmetric spaces of rank one from those of higher rank.

It is clear that totally umbilic immersions are isotropic. But the converse does not hold. There exist many isotropic immersions which are not totally umbilic. It is known that equivariant immersions of symmetric spaces of rank one into...
homogeneous Riemannian manifolds are constant isotropic. We also have many examples of non-constant isotropic immersions. For example, we take holomorphic curves in a Kähler manifold and superminimal immersions of Riemann surfaces into a standard 4-sphere. They are generally non-constant isotropic (for detail, see the discussion before Proposition 2 in Section 4 and [11]).

It is an interesting problem to characterize some kind of isometric immersions by extrinsic shapes of some curves. For a smooth curve $\gamma$ on a Riemannian submanifold $M$ of $\widetilde{M}$ through $f$, we call the curve $f \circ \gamma$ its extrinsic shape. There are many results in this direction, [3,6,8,9] for example. In these papers extrinsic shapes of geodesics and circles are treated. In this paper we study curve-theoretic properties preserved by immersions and give a characterization of isotropic immersions in terms of logarithmic derivatives of first geodesic curvatures. As a consequence of this we characterize Veronese embeddings of complex projective spaces with complex dimension greater than one.

2. Isotropic immersions preserve some curvature logarithmic derivatives of curves

Let $\gamma$ be a smooth curve parameterized by its arclength. We call the function $\kappa_0 = \|\nabla \dot{\gamma} \gamma\|$ its first geodesic curvature. When this function $\kappa_0$ vanishes at $t_0$, the point $\gamma(t_0)$ is said to be an inflection point of $\gamma$. If $\gamma(t_0)$ is not an inflection point, we denote by $\ell_0(t_0)$ the logarithmic derivative $\kappa_0'(t_0)/\kappa_0(t_0)$ of first geodesic curvature, and call it curvature logarithmic derivative of $\gamma$ at $\gamma(t_0)$. In this paper we study curvature logarithmic derivatives of extrinsic shapes of curves through isometric immersions.

Let $f : M \rightarrow \tilde{M}$ be an isometric immersion. By the formula of Gauss, the first geodesic curvatures of a smooth curve $\gamma$ on $M$ and its extrinsic shape $\tilde{\gamma} = f \circ \gamma$ satisfy

$$\kappa_0^2 = \kappa^2_0 + \|\sigma_f(\dot{\gamma},\dot{\gamma})\|^2,$$

(2.1)

where $\sigma_f$ denotes the second fundamental form of $f$. Thus we see if $\gamma$ does not have inflection points then its extrinsic shape also does not have such points. Moreover, for a given direction $u \in U_x M$ at a given point $x \in M$, this shows that the correspondence between the set of squares of first geodesic curvatures of curves and those of their extrinsic shapes is just a translation, and it only tells the norm $\|\sigma_f(u,u)\|$. In order to study more on isometries by such curve-theoretic properties, we need to treat some other curve-theoretic properties. In [4,6,10], the property of “order 2” for curves was treated. Isometries preserving the property of order 2 are some kind of totally umbilic immersions. We here treat curvature logarithmic derivatives. We shall show that when an isometric immersion $f$ is isotropic it preserves some curvature logarithmic derivatives.

Theorem 1. Let $f : M \rightarrow \tilde{M}$ be an isometric immersion with the function of isotropy $\lambda_f$.

(1) For each unit tangent vector $u \in UM$ there is a smooth curve $\gamma : (-\epsilon, \epsilon) \rightarrow M$ without inflection points such that $\dot{\gamma}(0) = u$ and $\ell_0(0) = \ell_f(0)$.

(2) At a non-geodesic point $x \in M$, the curvature logarithmic derivative of a smooth curve with initial vector $u \in U_x M$ is preserved by $f$ if and only if it coincides with $u(\lambda_f)/\lambda_f(x)$.

(3) At a geodesic point, curvature logarithmic derivatives for all curves without inflection points are preserved by $f$.

Proof. The first assertion is a consequence of the second and the third assertions. Let $\gamma : (-\epsilon, \epsilon) \rightarrow M$ be a smooth curve with $\gamma(0) = x$ and $\dot{\gamma}(0) = u$. As we have $\kappa_0^2 = \kappa^2_0 + \lambda_f(\gamma)^2$ by (2.1), we see $\kappa_f^2 = \kappa_0^2 + \dot{\gamma}(\gamma)\lambda_f$. When $x$ is a geodesic point, as $\lambda_f(x) = 0$, we have $\kappa_f(0) = \kappa_0(0)$ and $\kappa_f(0)\kappa_f(0) = \kappa_0(0)\kappa_0(0)$, hence we get $\ell_f(0) = \ell_0(0)$. Thus we see all curvature logarithmic derivatives are preserved by $f$ at $x$.

When $x$ is not a geodesic point, we have

$$\ell_f(0) - \ell_0(0) = \frac{\kappa_f(0)\kappa_f(0)}{\kappa_0(0)^2} - \frac{\kappa_0(0)}{\kappa_0(0)} = \frac{\lambda_f(x)^2}{\kappa_0(0)^2 + \lambda_f(x)^2} \left( \frac{u(\lambda_f)}{\lambda_f(x)} - \frac{\kappa_f(0)}{\kappa_0(0)} \right).$$

Thus we find $\ell_f(0) = \ell_0(0)$ if and only if $\ell_f(0) = u(\lambda_f)/\lambda_f(x)$, and get the second assertion. □

For an isometric immersion $f : M \rightarrow \tilde{M}$ and an orthonormal pair of unit tangent vectors $u, v \in U_x M$, we denote by $F_f(u, v)$ a family of smooth curves $\gamma$’s on $M$ which are defined on some interval containing the origin, parameterized
by its arclength, without inflection points and satisfy \( \dot{y}(0) = u, \nabla_y \dot{y}(0)/\|\nabla_y \dot{y}(0)\| = v \) and \( \ell_2(0) = \ell_y(0) \). We put
\[
A_f(u,v) = \{ \ell_y(0) | y \in \mathcal{F}(u,v) \},
\]
\[
A_f(u) = \bigcup \{ A_f(u,v) | v \in U_x M, v \perp u \}.
\]
When \( f \) is isotropic, Theorem 1 tells that \( A_f(u,v) \neq \emptyset \) for all orthonormal pairs and that \( A_f(u,v) = \mathbb{R} \) at a geodesic point \( x \) and \( A_f(u) \) consists of a single value at non-geodesic point \( x \). In the latter case the value coincides with \( u(\lambda_f)/\lambda_f(x) \).

3. Isometric immersions preserving some curvature logarithmic derivatives of curves

In the previous section we showed that isotropic immersions preserve some curvature logarithmic derivatives of curves. In this section we study the converse. We shall show that isometric immersions which preserve some curvature logarithmic derivatives of curves are isotropic.

We say an isometric immersion \( f : M \to \tilde{M} \) is isotropic at \( x \in M \) when \( \|\sigma_f(u,u)\| \) dose not depend on the choice of \( u \in U_x M \).

**Theorem 2.** For an isometric immersion \( f : M \to \tilde{M} \) the following conditions are mutually equivalent at a point \( x \in M \).

1. \( f \) is isotropic at \( x \).
2. For each orthonormal pair \( (u,v) \in U_x M \times U_x M \) there are two smooth curves \( \gamma_1, \gamma_2 \in \mathcal{F}_f(u,v) \cup \mathcal{F}_f(u,-v) \) with \( \ell_{\gamma_1}(0) = \ell_{\gamma_2}(0) \) satisfying \( \nabla_{\dot{\gamma}_1} \gamma_1(0) \neq \nabla_{\dot{\gamma}_2} \gamma_2(0) \).

**Remark.** We here note on the second condition in Theorem 2.

1. When \( \gamma_1 \in \mathcal{F}_f(u,v), \gamma_2 \in \mathcal{F}_f(u,-v) \), the condition is equivalent to \( A_f(u,v) \cap A_f(u,-v) \neq \emptyset \).
2. When \( \gamma_1, \gamma_2 \in \mathcal{F}_f(u,v) \) or \( \gamma_1, \gamma_2 \in \mathcal{F}_f(u,-v) \), we need \( \kappa_{\gamma_1}(0) \neq \kappa_{\gamma_2}(0) \).

Let \( \nabla \) and \( \tilde{\nabla} \) denote the covariant differentiation on \( M \) and \( \tilde{M} \), respectively. For vector fields \( X, \xi \) on \( \tilde{M} \) which are tangent and normal to \( M \) respectively, we decompose \( \tilde{\nabla}_X \xi \) into tangential and normal components and denote as \( \tilde{\nabla}_X \xi = -A_X X + \nabla^\perp_X \xi \). We define the covariant differentiation \( \tilde{\nabla} \) of the second fundamental form \( \sigma \) with respect to the connection of \( TM \oplus TM^\perp \) by
\[
(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla^\perp_X (\sigma_f(Y, Z)) - \sigma_f(\nabla_X Y, Z) - \sigma_f(Y, \nabla_X Z).
\]
For a smooth curve \( \gamma \), by differentiating both sides of (2.1), we have
\[
k_{\gamma}^2 \kappa_{\gamma} = k_{\gamma}^2 \kappa_{\gamma} + (\langle \tilde{\nabla}_\dot{\gamma} \sigma(\dot{\gamma}, \dot{\gamma}), \sigma(\dot{\gamma}, \dot{\gamma}) \rangle + 2|\sigma(\tilde{\nabla}_\dot{\gamma} \gamma, \dot{\gamma})|, k_{\gamma}^2 \kappa_{\gamma} + 2|\sigma(\tilde{\nabla}_\dot{\gamma} \gamma, \dot{\gamma})|).
\]
By using (2.1) again we obtain the following.

**Lemma 1.** If \( \gamma \) does not have inflection points, curvature logarithmic derivatives of \( \gamma \) and of its extrinsic shape satisfy
\[
k_{\gamma}^2 (\ell_{\gamma} - \ell_y)^2 + \ell_{\gamma} \| \sigma(\dot{\gamma}, \dot{\gamma}) \|^2 = k_{\gamma}^2 (\ell_{\gamma} - \ell_y)^2 + \ell_y \| \sigma(\dot{\gamma}, \dot{\gamma}) \|^2
\]
\[
= (\langle \tilde{\nabla}_\dot{\gamma} \sigma(\dot{\gamma}, \dot{\gamma}), \sigma(\dot{\gamma}, \dot{\gamma}) \rangle + 2|\sigma(\tilde{\nabla}_\dot{\gamma} \gamma, \dot{\gamma})|, k_{\gamma}^2 \kappa_{\gamma} + 2|\sigma(\tilde{\nabla}_\dot{\gamma} \gamma, \dot{\gamma})|).
\]

By this lemma we find \( \mathcal{F}_f(u,v) \) is not empty. Hence we see the essential assumption in Theorem 2 is the condition that \( \ell_{\gamma_1}(0) = \ell_{\gamma_2}(0) \).

**Corollary 1.** Let \( f : M \to \tilde{M} \) be an isometric immersion. For an arbitrary orthonormal pair \( (u,v) \in U_x M \times U_x M \) and a positive \( k \), there is a smooth curve \( \gamma \in \mathcal{F}_f(u,v) \) with \( \kappa_{\gamma}(0) = k \). In particular, \( A_f(u,v) \neq \emptyset \).
Proof of Theorem 2. If we consider a function \( \varphi \) of \( U_x M \) defined by \( \varphi(u) = \|\sigma(u, u)\|^2 \), we see its differential is given as \( \nu(\varphi)_u = 4\langle \sigma(u, u), \sigma(u, u) \rangle \). Thus we find \( f \) is isotropic at \( x \) if and only if \( \langle \sigma(u, u), \sigma(u, v) \rangle = 0 \) for every orthonormal pair \( (u, v) \in U_x M \times U_x M \).

We shall show our theorem by use of this fact. When \( f \) is isotropic at \( x \), for every \( u \in U_x M \) we see by Lemma 1 that
\[
\kappa(\gamma)(0) \left( \ell(\gamma)(0) - \ell(\gamma)(0) \right) + \ell(\gamma)(0) \| \sigma(u, u) \|^2 = \left\langle (\nabla_u \sigma)(u, u), \sigma(u, u) \right\rangle
\]
for every smooth curve \( \gamma \) parameterized by its arc length with \( \dot{\gamma}(0) = u \). If \( \sigma(u, u) = 0 \) we find \( \ell(\gamma)(0) = \ell(\gamma)(0) \) for every \( \gamma \). If \( \sigma(u, u) \neq 0 \), for every smooth curve with
\[
\ell(\gamma)(0) = \left\langle (\nabla_u \sigma)(u, u), \sigma(u, u) \right\rangle \| \sigma(u, u) \|^2,
\]
we also have \( \ell(\gamma)(0) = \ell(\gamma)(0) \). Thus the second condition holds.

On the contrary we suppose the second condition holds. For every orthonormal pair \( (u, v) \), we take two smooth curves \( \gamma_1, \gamma_2 \) satisfying the assumption. By the equality in Lemma 1 we have
\[
\ell(\gamma_i)(0) \| \sigma(u, u) \|^2 = \left\langle (\nabla_u \sigma)(u, u), \sigma(u, u) \right\rangle \pm 2\kappa(\gamma_i)(0)\| \sigma(u, u) \|^2,
\]
for \( i = 1, 2 \), where the double sign takes positive or negative according to \( \nabla_{\gamma_i} \gamma_1(0)/\| \nabla_{\gamma_i} \gamma_1(0) \| = v - v \). We hence obtain \( \langle \sigma(u, u), \sigma(u, v) \rangle = 0 \), and \( f \) is isotropic at \( x \). □

Along our lines we can characterize geodesic points of isometric immersions in the following manner.

Proposition 1. For an isometric immersion \( f : M \rightarrow \tilde{M} \) the following conditions are mutually equivalent at a point \( x \in M \).

1. \( f \) is geodesic at \( x \).
2. For each orthonormal pair \( (u, v) \in U_x M \times U_x M \),
   a) there are two smooth curves \( \gamma_1, \gamma_2 \in F_f(u, v) \cup F_f(u, -v) \) with \( \ell(\gamma_1)(0) = \ell(\gamma_2)(0) \) satisfying \( \nabla_{\gamma_1} \gamma_1(0) \neq \nabla_{\gamma_2} \gamma_2(0) \),
   b) the set \( \mathcal{A}(u) \) does not consist of a single value.

Proof. If \( f \) is geodesic at \( x \), we see by Lemma 1 that \( \ell(\gamma)(0) = \ell(\gamma)(0) \) for every smooth curve \( \gamma \) with \( \gamma(0) = x, \nabla_{\gamma} \dot{\gamma}(0) \neq 0 \). Hence the second condition holds.

If the second condition holds, we see \( f \) is isotropic at \( x \) by Theorem 2. Therefore by the argument in the proof of that theorem we get \( \langle \sigma(u, u), \sigma(u, v) \rangle = 0 \) for every orthonormal pair \( (u, v) \in U_x M \times U_x M \). Since \( \mathcal{A}(u) \neq \emptyset \) by Corollary 1, we can take two smooth curves \( \gamma_3, \gamma_4 \in \bigcup \{ F_f(u, v) \mid v \in U_x M, v \perp u \} \) with \( \ell(\gamma_3)(0) \neq \ell(\gamma_4)(0) \), which means that \( \ell(\gamma_3)(0), \ell(\gamma_4)(0) \) are distinct values in \( \mathcal{A}(u) \). We then find by Lemma 1 that
\[
\ell(\gamma_3)(0) \| \sigma(u, u) \|^2 = \left\langle (\nabla_u \sigma)(u, u), \sigma(u, u) \right\rangle = \ell(\gamma_4)(0) \| \sigma(u, u) \|^2,
\]
which shows \( \sigma(u, u) = 0 \). We hence get the conclusion. □

We here make mention of Maeda’s characterization on constant isotropic immersions in terms of extrinsic shapes of circles [3]. A smooth curve \( \gamma \) parameterized by its arc length is said to be a circle of (constant) geodesic curvature \( \kappa \geq 0 \) if it satisfies \( \nabla_{\gamma} \nabla_{\gamma} \dot{\gamma} = -\kappa^2 \dot{\gamma} \).

Suppose \( f \) is isotropic at \( x \). When \( \gamma \) is a smooth curve with null curvature logarithmic derivative at \( \gamma(0) \) satisfying \( \gamma(0) = u \in U_x M \) and \( \| \nabla_{\gamma} \gamma(0) \| = k(>0) \), in particular, when \( \gamma \) is a circle of geodesic curvature \( \kappa \) with \( \gamma(0) = u \), we have
\[
\ell(\gamma)(0) = \left\langle (\nabla_u \sigma)(u, u), \sigma(u, u) \right\rangle / \{ \kappa^2 + \| \sigma(u, u) \|^2 \}
\]
by Lemma 1. Thus one can easily see that our result is an extension of Maeda’s characterization of constant isotropic immersions.
Corollary 2. (Cf. [3.]) An isometric immersion \( f : M \to \tilde{M} \) of a connected Riemannian manifold \( M \) into a Riemannian manifold \( \tilde{M} \) is constant isotropic if and only if for every orthonormal pair \( (u, v) \in U_\epsilon M \times U_\epsilon M \) at an arbitrary point \( x \in M \), there is a circle \( \gamma \) of positive geodesic curvature with \( \dot{\gamma}(0) = u, \nabla_\gamma \dot{\gamma}(0)/\|\nabla_\gamma \dot{\gamma}(0)\| = v \) whose extrinsic shape has constant first geodesic curvature.

4. Veronese embeddings

As an application of our result we give a characterization of Veronese embeddings. We denote by \( M_n(c) \) a complex \( n \)-dimensional Kähler manifold of constant holomorphic sectional curvature \( c \), which is locally congruent (i.e. holomorphically isometric) to a complex projective space \( \mathbb{C} P^n(c) \) when \( c > 0 \), a complex Euclidean space \( \mathbb{C}^n \) when \( c = 0 \), and a complex hyperbolic space \( \mathbb{C} H^n(c) \) when \( c < 0 \). We denote by \( f_k : \mathbb{C} P^n(c) \to \mathbb{C} P^{m(k)}(\tilde{c}) \) the Kähler embedding given by

\[
[z_i]_{0 \leq i \leq n} \mapsto \left[\sqrt{k!/(k_0! \cdots k_n!)} \ z_0^{k_0} \cdots z_n^{k_n}\right]_{k_0 + \cdots + k_n = k}
\]

with homogeneous coordinates, where \( m(k) = (n + k)!/(n!k!) - 1 \). This embedding is called the \( k \)th Veronese embedding, and is constant isotropic with isotropy constant \( \lambda_f \equiv \tilde{c}(k - 1)/(2k) \). We here recall the following rigidity theorem for Kähler isometric immersions of \( M_n(c) \) into \( M_N(\tilde{c}) \).

Fact. (See [5.]) Let \( f : M_n(c) \to M_N(\tilde{c}) \) be a Kähler isometric immersion of a Kähler manifold of constant holomorphic sectional curvature \( c \) into a Kähler manifold of constant holomorphic sectional curvature \( \tilde{c} \). Then the following hold.

1. When \( \tilde{c} \leq 0 \), the immersion \( f \) is totally geodesic.
2. When \( \tilde{c} > 0 \), we see \( c > 0 \) and there exists such a positive integer \( k \) that \( \tilde{c} = kc \) and \( f \) is locally equivalent to the Kähler embedding

\[
\iota \circ f_k : \mathbb{C} P^n(c) \to \mathbb{C} P^{m(k)}(\tilde{c}) \to \mathbb{C} P^N(\tilde{c})
\]

given as the composition of the \( k \)th Veronese embedding \( f_k \) and a totally geodesic Kähler embedding \( \iota \).

By use of Theorem 2 and this rigidity theorem, we can characterize Veronese embeddings for the case \( n \geq 2 \) by the property that some curvature logarithmic derivatives are preserved.

Theorem 3. Let \( f : M \to M_N(\tilde{c}) \) be a non-totally geodesic Kähler isometric full immersion of a Kähler manifold \( M \) of complex dimension \( n \geq 2 \) into a Kähler manifold of constant holomorphic sectional curvature \( \tilde{c} \). Then the following conditions are equivalent:

1. There exists such a positive integer \( k \) that \( N = m(k) \), the ambient space is locally congruent to \( \mathbb{C} P^N(\tilde{c}) \) (i.e. \( \tilde{c} \) is positive), \( M \) is locally congruent to \( \mathbb{C} P^n(c)/k \) and \( f \) is locally equivalent to the \( k \)th Veronese embedding \( f_k \);
2. For every orthonormal pair \( (u, v) \) of tangent vectors of \( M \), there are two smooth curves \( \gamma_1, \gamma_2 \in \mathcal{F}_f(u, v) \cup \mathcal{F}_f(u, -v) \) with \( \ell_{\gamma_1}(0) = \ell_{\gamma_2}(0) \) satisfying \( \nabla_{\dot{\gamma}_1} \dot{\gamma}_1(0) \neq \nabla_{\dot{\gamma}_2} \dot{\gamma}_2(0) \).

Proof. Since every Veronese embedding \( f_k \) is positive constant isotropic, we see \( A_{f_k}(u, v) = \{0\} \) and the second condition holds if we choose two circles.

On the other hand, if \( f \) satisfies the second condition, we find \( f \) is isotropic by Theorem 2. By the Gauss equation we find holomorphic sectional curvatures \( H_M(u) \), \( H_{\tilde{M}}(u) \) of the complex lines spanned by \( u \) in \( T_x M \) and \( T_x \tilde{M} \) with \( \tilde{M} = M_N(\tilde{c}) \) are related as

\[
\tilde{c} = H_{\tilde{M}}(u) = H_M(u) + \langle \sigma(u, J_u) \rangle - \langle \sigma(u, u) \rangle \equiv \langle \sigma(u, J_u) \rangle = H_M(u) + 2\|\sigma(u, u)\|^2.
\]

Thus we see \( H_M(u) = \tilde{c} - 2\langle \sigma(u, u) \rangle \) does not depend on the choice of \( u \in T_x M \). By applying the complex version of Shur’s lemma, we find \( M \) is locally congruent to a complex space form. Hence the rigidity theorem leads us to the conclusion. \( \Box \)
We should note that we cannot extend Theorem 3 to the case \( n = 1 \) in this form. For a holomorphic curve \( f : M \to \tilde{M} \) in a Kähler manifold \( \tilde{M} \), we see
\[
\langle \sigma(u, u), \sigma(u, Ju) \rangle = \langle \sigma(u, u), J\sigma(u, u) \rangle = 0,
\]
hence it is always isotropic as \( u, Ju \in T_xM \) span \( T_xM \). In order to include the case \( n = 1 \), as was pointed out in [3], we need to investigate curves with null curvature logarithmic derivatives.

**Proposition 2.** (Cf. [3].) Let \( f : M \to M_N(\tilde{c}) (\tilde{c} > 0) \) be a full isometric immersed holomorphic curve. Then the following conditions are equivalent:

1. The immersion \( f \) is locally equivalent to the \( k \)th Veronese embedding \( f_k : \mathbb{C}P^{1}(\tilde{c}/k) \to \mathbb{C}P^{k}(\tilde{c}) \) for some positive integer \( k \);
2. For every orthonormal pair \((u, v)\) of tangent vectors of \( M \), there are two smooth curves \( \gamma_1, \gamma_2 \in F_f(u, v) \cup F_f(u, -v) \) with null curvature logarithmic derivatives which satisfy \( \nabla_{\dot{\gamma}_1} \dot{\gamma}_1(0) \neq \nabla_{\dot{\gamma}_2} \dot{\gamma}_2(0) \).

**References**