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# On Iterated Semidirect Products of Finite Semilattices

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Let **Sl** denote the class of all finite semilattices and let $\mathbf{Sl}^n$ be the pseudovariety of semigroups generated by all semidirect products of $n$ finite semilattices. The main result of this paper shows that, for every $n \geqslant 3$, $\mathbf{Sl}^n$ is not finitely based. Nevertheless, a simple basis of identities is described for each $\mathbf{Sl}^n$. We also show that $\mathbf{Sl}^n$ is generated by a semigroup with $2n$ generators but, except for $n \leqslant 2$, we do not know whether the bound $2n$ is optimal. © 1991 Academic Press, Inc.

## 1. INTRODUCTION

Among the various operations which allow us to construct semigroups from simpler ones, the semidirect product has thus far received the most attention in the theory of finite semigroups. It plays a special role in the connections with language theory (Eilenberg [5], Pin [9]) and in the Krohn–Rodes decomposition theorem [5].

Questions dealing with semidirect products of finite semigroups are often properly dealt with in the context of pseudovarieties. For pseudovarieties **V** and **W**, their semidirect product $\mathbf{V} * \mathbf{W}$ is defined to be the pseudovariety generated by all semidirect products $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$. It is well known that this operation on pseudovarieties is associative and that $\mathbf{V} * \mathbf{W}$ consists of all homomorphic images of subsemigroups of $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$ [5].

To have a good knowledge of any pseudovariety, one should be able to provide the following information about it or to show such is not possible:

- a (finite) basis of pseudoidentities;
- a (low complexity) algorithm to decide when a finite semigroup lies in it;
- a (finite) set of generators;
- a (low complexity) algorithm to decide when a pseudoidentity



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holds in it (for identities, this means a solution of the word problem in the free objects).

Of course there are various relationships between the possible answers to these problems, some known, some conjectured, and possibly others not yet guessed (see [1, 3]). For instance, if a pseudovariety admits a finite basis of identities or if it is finitely generated, then it has a decidable membership problem.

This paper deals with semidirect powers $\mathbf{Sl}^n$ of the pseudovariety $\mathbf{Sl}$ of all finite semilattices. It is known that $\bigcup_{n \geq 1} \mathbf{Sl}^n$ is the class $\mathbf{R}$ of all finite $\mathscr{R}$-trivial semigroups (Stiffler [12]), that each $\mathbf{Sl}^n$ has a decidable membership problem (Pin [7]; see also Straubling [13]), and that the analog of $\mathbf{Sl}^n$ in the category of monoids has a simple finite basis of identities (Pin [8]). A case study for the above problems is proposed here through an approach to the semidirect product operation which was introduced in [2]. The essential ingredient in that approach is a semidirect product representation of the free objects in $\mathbf{V} * \mathbf{W}$ in case both $\mathbf{V}$ and $\mathbf{W}$ have finite free objects.

The first step consists in constructing an algorithm to decide when an identity holds in $\mathbf{Sl}^n$. This is then used to obtain a basis of identities for $\mathbf{Sl}^n$. This basis easily reduces to Pin's basis in [8] in case $n=2$, but it can also be used to show that $\mathbf{Sl}^n$ does not admit a finite basis of pseudoidentities for $n \geq 3$. It also yields easily that $\bigcup_{n \geq 1} \mathbf{Sl}^n = \mathbf{R}$. Further exploration of the basic algorithm for the identities of $\mathbf{Sl}^n$ leads to a proof that every $\mathbf{Sl}^n$ is finitely generated, which implies that it has a decidable membership problem. Specifically, $\mathbf{Sl}^n$ is generated by a semigroup on $2n$ generators. However, we only know that the upper bound $2n$ is minimal for $n \leq 2$.

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## 2. Preliminaries

A *pseudovariety* is a class of finite algebras of the same similarity type which is closed under the formation of homomorphic images, subalgebras, and finitary direct products. We will be dealing mainly with pseudovarieties of semigroups (for which the type involves only one binary operation) but it will be convenient to include some comments on pseudovarieties of monoids (and then a nullary operation also intervenes). For a general study of these classes see [5, 9].

In dealing with pseudovarietes, it is often convenient to refer back to the more classical concept of "variety" as introduced by Birkhoff. It will be assumed throughout this paper that the reader is familiar with the basics of the theory of varieties, say Chapter 2 of Burris and Sankappanavar [4].

In particular, it is well known that varieties admit free objects. The free object on the set $X$ in the variety generated by a pseudovariety **V** will be denoted by $F_X\mathbf{V}$. We will also write $F_n\mathbf{V}$ as an abbreviation for $F_{\{x_1, \ldots, x_n\}}\mathbf{V}$. Moreover, in case **V** is the pseudovariety of all finite semigroups, the notation $X^+$ is standard for $F_X\mathbf{V}$ and this semigroup is viewed as the set of all nonempty "words" on the "alphabet" $X$, multiplication being given by concatenation of words. If the empty word 1 is adjoined, then the resulting semigroup is the free monoid on $X$ and is usually denoted by $X^*$. For a word $w \in X^*$, let $c(w)$ denote the *content* of $w$, i.e., the set of all letters from $X$ that occur in $w$.

Fix a countable set $X$ of "variables" $x_1, x_2, \ldots, x_n, \ldots$. A formal equality $u = v$ between two words of $X^+$ is called a *semigroup identity*. For a set $\Sigma$ of semigroup identities, $[\Sigma]$ denotes the variety of all semigroups which satisfy all the identities in $\Sigma$. If $\mathscr{V} = [\Sigma]$, $\Sigma$ is said to be a *basis of identities* of $\mathscr{V}$. For $\Sigma$ and $u = v$ as above, write $\Sigma \vdash u = v$ if there is a *deduction* of $u = v$ from $\Sigma$, i.e., a finite sequence of words

$$(u=)\, w_0, w_1, \ldots, w_r\, (=v) \in X^+,$$

and, for $i = 0, \ldots, r-1$, there are words $a_i, b_i \in X^*$, $u_i, v_i \in X^+$, and a homomorphism $\varphi_i: X^+ \to X^+$ such that

$$w_i = a_i(\varphi_i u_i)\, b_i, \qquad w_{i+1} = a_i(\varphi_i v_i)\, b_i$$

and $(u_i = v_i) \in \Sigma$ or $(v_i = u_i) \in \Sigma$. Each identity $w_i = w_{i+1}$ is called an *elementary step* in the deduction. In case all $a_i = 1$, we say that the deduction is *left absorbing*. In case all $\varphi_i$ are the identity function, we say that the deduction *involves no substitutions*. By the completeness theorem of equational logic [4], $\Sigma \vdash u = v$ if and only if $[\Sigma] \models u = v$.

For a set $\Sigma$ of semigroup identities, $[\![\Sigma]\!]$ denotes the pseudovariety of all finite semigroups which satisfy all the identities in $\Sigma$. We also use the same notation in case $\Sigma$ is a set of semigroup "pseudoidentities" and we recall that, by a theorem of Reiterman, every pseudovariety **V** of semigroups is of the form $[\![\Sigma]\!]$ for some such $\Sigma$ (Reiterman [10]; see also [3]) which is called a *basis* of **V**. A pseudovariety **V** is said to be *finitely based* if it admits a finite basis of pseudoidentities.

For $i \geqslant 0$, let $\Sigma_i = A_i \cup B_i$ where

$$A_i = \{u_i \cdots u_1 xy = u_i \cdots u_1 yx : u_1, \ldots, u_i \in X^+, x, y \in c(u_1) \subseteq \cdots \subseteq c(u_i)\}$$

$$B_i = \{u_i \cdots u_1 x^2 = u_i \cdots u_1 x : u_1, \ldots, u_i \in X^+, x \in c(u_1) \subseteq \cdots \subseteq c(u_i)\},$$

with $x = x_1$ and $y = x_2$. The following elementary properties of these sets of identities will be useful in the sequel.

LEMMA 2.1. (a) *If* $(u=v)\in\Sigma_i$ *and* $i>0$, *then* $(wu=wv)\in\Sigma_i$ *for any* $w\in X^*$.

(b) *If* $(u=v)\in\Sigma_i$ *and* $\varphi\colon X^+\to X^+$ *is a homomorphism, then there is a deduction of* $\varphi u=\varphi v$ *from* $\Sigma_i$ *involving no substitutions.*

*Proof.* The verification of (a) is straightforward. For instance, if $(u=v)\in B_i$ with $i>0$ and $w\in X^*$, then $(wu=wv)\in B_i$ since factorizations $u=u_i\cdots u_1x^2$ and $v=u_i\cdots u_1x$ with $x\in c(u_1)\subseteq\cdots\subseteq c(u_i)$ yield factorizations of the same type $wu=(wu_i)\,u_{i-1}\cdots u_1x^2$ and $wv=(wu_i)\,u_{i-1}\cdots u_1x$.
A complete proof of (b) is too tedious to be included here. Let us instead verify a particular case, say $u=u_i\cdots u_1xy$ and $v=u_i\cdots u_1\,yx$ with $x, y\in c(u_1)\subseteq\cdots\subseteq c(u_i)$ and the homomorphism $\varphi$ is such that $\varphi x=x$ and $\varphi y=yz$ $(z=x_3)$. We claim that then there is a deduction of the identity $\varphi u=\varphi v$ from $A_i$ involving no substitutions. Indeed, if we write $u'_k$ for $\varphi u_k$ $(k=1, ..., i)$, then we have factorizations $\varphi u=u'_i\cdots u'_1xyz$ and $\varphi v=u'_i\cdots u'_1\,yzx$ with $x$, $y$, $z\in c(u'_1)\subseteq\cdots\subseteq c(u'_i)$. Then the following sequence of identities yields the desired deduction:

$$u'_i\cdots u'_1xy\cdot z=u'_i\cdots u'_1\,yx\cdot z=u'_i\cdots u'_2(u'_1\,y)\,xz=u'_i\cdots u'_1\,yzx. \quad \blacksquare$$

Lemma 2.1 immediately yields the following.

LEMMA 2.2. *If* $\Sigma_i\vdash u=v$ *with* $i>0$, *then there is a left absorbing deduction of* $u=v$ *from* $\Sigma_i$ *involving no substitutions.*

*Proof.* It suffices to deal with each elementary step in a deduction of $u=v$. So, we may as well assume there are $a$, $b\in X^*$, $(u'=v')\in\Sigma_i$, and a homomorphism $\varphi\colon X^+\to X^+$ such that $u=a(\varphi u')b$ and $v=a(\varphi v')b$. By Lemma 2.1(b), there is a deduction of $\varphi u'=\varphi v'$ from $\sigma_i$ involving no substitutions. Then, applying Lemma 2.1(a), we may use this deduction to construct a left absorbing deduction of $u=v$ from $\Sigma_i$. ∎

The preceding syntactical observation allows us to deduce the following simple result which will be useful in induction proofs.

PROPOSITION 2.3. *For* $i>0$, *if* $\Sigma_i\vdash u=v$ *and* $c(u)\subseteq c(w)$, *then* $\Sigma_{i+1}\vdash wu=wv$.

*Proof.* Take a left absorbing deduction of $u=v$ from $\Sigma_i$ involving no substitutions according to Lemma 2.2. Then, appending $w$ to the left of each of its steps, we obtain a deduction of $wu=wv$ from $\Sigma_{i+1}$ since, by adding the prefix $w$, each identity from $\Sigma_i$ involved in elementary steps of the original deduction becomes an identity from $\Sigma_{i+1}$. ∎

For a semigroup $S$, $S^1$ denotes the semigroup $S \cup \{1\}$ obtained from $S$ by adjoining a neutral element if $S$ does not have one, and $S^1 = S$ otherwise.

Let us now turn our attention to semidirect products. Given semigroups $S$ and $T$ and a monoid homomorphism ("action" of $T$ on $S$) from $T^1$ into the monoid of (semigroup) endomorphisms of $S$ (composed on the left), the corresponding *semidirect product* $S * T$ is the cartesian product $S \times T$ endowed with the operation

$$(s_1, t_1)(s_2, t_2) = (s_1 + t_1 s_2, t_1 t_2),$$

where we adopt additive notation for $S$ (even if $S$ is not commutative) and multiplicative notation for $T$, and, for $t \in T^1$ and $s \in S$, we define $ts$ by $ts = \varphi(t)(s)$. This definition of semidirect product is not as general as the one considered by Eilenberg [5] as we only allow what he calls "left unitary actions." But it turns out that the study of more general actions in the context of pseudovarieties does not lead to more interesting phenomena but rather just imposes the consideration of numerous irrelevant and trivial cases. See Tilson [14] for a discussion of this question, although from a different perspective.

Next, consider two pseudovarieties of semigroups $\mathbf{V}$ and $\mathbf{W}$. Their *semidirect product* $\mathbf{V} * \mathbf{W}$ is defined to be the pseudovariety generated by all semidirect products $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$. The semidirect product $\mathscr{V} * \mathscr{W}$ of two varieties of semigroups $\mathscr{V}$ and $\mathscr{W}$ is defined analogously. It is well known that these operations on pseudovarieties and varieties are associative [5, 14].

The following representation of free objects for $\mathbf{V} * \mathbf{W}$ obtained in [2] is crucial for the rest of the paper.

PROPOSITION 2.4. *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties such that $F_n\mathbf{V}$ and $F_n\mathbf{W}$ are finite for all $n$. Then so is $F_n(\mathbf{V} * \mathbf{W})$ and there is an embedding*

$$F_n(\mathbf{V} * \mathbf{W}) \hookrightarrow F_Y\mathbf{V} * F_n\mathbf{W}$$

$$x_i \mapsto ((1, x_i), x_i),$$

*where $Y = (F_n\mathbf{W})^1 \times \{x_1, ..., x_n\}$ and the action in the semidirect product of the free objects is given by $x_i(s, x_j) = (x_i s, x_j)$ $(s \in (F_n\mathbf{W})^1)$.*

As a first application of this result, let us consider the relationship between semidirect products of pseudovarieties of semigroups and the corresponding operation for pseudovarieties of monoids. Given monoids $M$ and $N$, to define a *monoidal semidirect product* $M * N$, proceed just as above in the case of semigroups with the only difference that the action of $N$ on $M$ is given by a homomorphism of $N$ into the monoid of **monoid** endomorphisms of $M$ (such an action will be called a *monoidal* action). Then, to define the semidirect product of two pseudovarieties of monoids, we proceed as in the case of pseudovarities of semigroups but only using monoidal semidirect products.

Call a pseudovariety $\mathbf{V}$ of semigroups *monoidal* if $S \in \mathbf{V}$ implies $S^1 \in \mathbf{V}$. Given a pseudovariety $\mathbf{V}$ of semigroups, define $\mathbf{V}_M$ to be the class of all monoids in $\mathbf{V}$. It is easy to see that $\mathbf{V}_M$ is always a pseudovariety of monoids.

PROPOSITION 2.5. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be monoidal pseudovarieties of semigroups. Then* $\mathbf{V} * \mathbf{W}$ *is also monoidal and* $(\mathbf{V} * \mathbf{W})_M = \mathbf{V}_M * \mathbf{W}_M$.

*Proof.* The inclusion $\supseteq$ is obvious. For the reverse inclusion and the remainder of the proof, we claim it suffices to consider the case when all finitely generated free objects for $\mathbf{V}$ and $\mathbf{W}$ are finite. Indeed, it is easily verified that a pseudovariety of semigroups is monoidal if and only if it is generated by its monoids. Hence, by standard arguments, a monoidal pseudovariety is the union of a chain of monoidal pseudovarieties each of which contains all corresponding finitely generated free objects. The claim follows by noting that the semidirect product and $\mathbf{U} \mapsto \mathbf{U}_M$ operations behave well with respect to unions of chains.

So, suppose $F_m\mathbf{V} \in \mathbf{V}$ and $F_n\mathbf{W} \in \mathbf{W}$ for all positive integers $m$ and $n$. Then $F_n(\mathbf{V} * \mathbf{W})$ embeds in $F_Y\mathbf{V} * F_n\mathbf{W}$ according to Proposition 2.4. Moreover, the action of $F_n\mathbf{W}$ on $F_Y\mathbf{V}$ is defined in such a way that it extends (uniquely) to a monoidal action of $(F_n\mathbf{W})^1$ on $(F_Y\mathbf{V})^1$. Thus, we have embeddings

$$F_n(\mathbf{V} * \mathbf{W}) \hookrightarrow F_Y\mathbf{V} * F_n\mathbf{W} \hookrightarrow (F_Y\mathbf{V})^1 * (F_n\mathbf{W})^1. \tag{1}$$

If $F_n(\mathbf{V} * \mathbf{W})$ is a monoid, then it must be a group since it is freely generated within a semigroup variety. Moreover, if $F_n(\mathbf{V} * \mathbf{W})$ is a nontrivial group for some $n$, then it must be a group for all $n$ and so $\mathbf{V} * \mathbf{W}$ contains only groups, whence the same is true of $\mathbf{V}$ and $\mathbf{W}$. If $F_n(\mathbf{V} * \mathbf{W})$ has only one element, then it is easy to see that the same is true of $(F_Y\mathbf{V})^1 * (F_n\mathbf{W})^1$. Hence, if $F_n(\mathbf{V} * \mathbf{W})$ is a monoid, the composite of the two embeddings in (1) is a monoid homomorphism. Thus, in any case, we have a monoid embedding

$$(F_n(\mathbf{V} * \mathbf{W}))^1 \hookrightarrow (F_Y\mathbf{V})^1 * (F_n\mathbf{W})^1, \tag{2}$$

where the last semidirect product is monoidal.

Consider now any $S \in \mathbf{V} * \mathbf{W}$. Then there is some $n$ for which there is an onto homomorphism $F_n(\mathbf{V} * \mathbf{W}) \to S$. Such an onto homomorphism extends to an onto monoid homomorphism $(F_n(\mathbf{V} * \mathbf{W}))^1 \to S^1$. By (2), it follows that $S^1 \in \mathbf{V}_M * \mathbf{W}_M$. This shows that $(\mathbf{V} * \mathbf{W})_M \subseteq \mathbf{V}_M * \mathbf{W}_M$ and also that $\mathbf{V} * \mathbf{W}$ is monoidal since, by the reverse inclusion, viewing $S^1$ as a semigroup, certainly $S^1 \in \mathbf{V} * \mathbf{W}$. ∎

As a last preliminary remark, consider the relationship between the finite basis property for a monoidal pseudovariety of semigroups $\mathbf{V}$ and the finite basis property for the corresponding $\mathbf{V}_M$. The following result is implicit in a side remark in [1].

PROPOSITION 2.6. *Let $\mathbf{V}$ be a monoidal pseudovariety of semigroups. Then $\mathbf{V}$ is finitely based if and only if $\mathbf{V}_M$ is finitely based.*

*Proof.* Let $\Sigma$ be a basis of (semigroup) pseudoidentities for $\mathbf{V}$. We claim $\Sigma$ is also a basis of pseudoidentities for $\mathbf{V}_M$. If $M \in \mathbf{V}_M$, then $M$, viewed as a semigroup, lies in $\mathbf{V}$ and so $M \models \Sigma$. Conversely, if $M \models \Sigma$, then $M$, viewed as a semigroup, lies in $\mathbf{V}$ and so $M \in \mathbf{V}_M$. Hence, if $\mathbf{V}$ is finitely based, then so is $\mathbf{V}_M$.

For the converse, suppose that $\Sigma$ is a basis of (monoid) pseudoidentities for $\mathbf{V}_M$. Let $\Sigma'$ be obtained from $\Sigma$ by: first, add all pseudoidentities which may be obtained from the ones in $\Sigma$ by substituting 1 for some of the variables; second, remove all occurrences of $1 = 1$; finally, replace each pseudoidentity of the form $u = 1$ by $uy = y = yu$ where $y$ is a variable which does not intervene in $u$. Then $\Sigma'$ is a set of semigroup pseudoidentities which is finite if and only if $\Sigma$ is finite. Moreover, for a finite semigroup $S$, it is easy to see that $S \models \Sigma'$ if and only if $S^1 \models \Sigma$. Since $\Sigma$ was assumed to be a basis for $\mathbf{V}_M$ and since $\mathbf{V}$ is monoidal, we conclude that $\Sigma'$ is a basis for $\mathbf{V}$ and the result follows. ∎

Of course there is nothing in the preceding proof that has anything to do directly with the finiteness of the algebras involved. Thus, there is an analogous result for varieties.

## 3. THE IDENTITIES OF $\mathbf{Sl}^n$

Recall that $\mathbf{Sl}$ denotes the pseudovariety of all finite semilattices, so $\mathbf{Sl} = [\![xy = yx,\ x^2 = x]\!]$. Represent by $\mathbf{Sl}^i$ the $i$th power of $\mathbf{Sl}$ for the semidirect product operation. For the case $i = 0$, we adopt the convention that $\mathbf{Sl}^0$ is the pseudovariety $\mathbf{I} = [\![x = y]\!]$ consisting of all one-point semigroups. This convention is convenient since it is easily checked that $\mathbf{I}$ is the neutral element for the semidirect product operation on pseudovarieties of semigroups.

For $n \geqslant 1$ and $i \geqslant 0$, let $p_{n,i}$: $\{x_1, ..., x_n\}^+ \to F_n\mathbf{Sl}^i$ be the canonical projection which maps the letter $x_j$ onto the generator $x_j$ of $F_n\mathbf{Sl}^i$. We proceed to examine more closely these mappings by means of repeated application of the representation of free objects on a semidirect product of pseudovarieties given by Proposition 2.4. Since it is well known that $F_n\mathbf{Sl}$ is finite for all $n \geqslant 1$, by Proposition 2.4, $F_n\mathbf{Sl}^i$ is also finite for all $n \geqslant 1$ and $i \geqslant 0$. The analysis of the mappings $p_{n,i}$ that follows may of course be carried out for other pseudovarieties which are semidirect products of pseudovarieties all of whose finitely generated free objects are finite. In [2] the reader will find other examples of this analysis based on Proposition 2.4.

DEFINITION 3.1. *For* $u \in \{x_1, ..., x_n\}^*$ *and* $i \geqslant 0$, *let*

$$c^i(u) = \{(p_i u', x) \in (F_n\mathbf{Sl}^i)^1 \times \{x_1, ..., x_n\} : \\ u = u'xu'' \text{ for some } u', u'' \in \{x_1, ..., x_n\}^*\}.$$

In the case of $i = 0$, $(F_n\mathbf{Sl}^0)^1 = \{1\}$ and so $c^0(u) = \{1\} \times c(u)$. Moreover, since $\mathbf{Sl}^i \subseteq \mathbf{Sl}^{i+1}$ for every $i \geqslant 0$, we have

$$c^{i+1}(u) = c^{i+1}(v) \Rightarrow c^i(u) = c^i(v).$$

The following proposition in fact shows that the binary relation on $\{x_1, ..., x_n\}^+$ defined by $c^i(u) = c^i(v)$ is precisely the kernel of the mapping $p_{n,i+1}$. A formulation in terms of identities satisfied by $\mathbf{Sl}^{i+1}$ is more suitable for the purposes of this paper.

PROPOSITION 3.2. *Let* $u, v \in \{x_1, ..., x_n\}^+$ *and let* $i \geqslant 0$. *Then* $\mathbf{Sl}^{i+1} \models u = v$ *if and only if* $c^i(u) = c^i(v)$.

*Proof.* Proceed by induction on $i$. For $i = 0$, the result is the well known characterization of the identities of $\mathbf{Sl}$. Suppose the result holds for $i-1$. Consider the embedding of Proposition 2.4:

$$F_n\mathbf{Sl}^{i+1} \hookrightarrow F_Y\mathbf{Sl} * F_n\mathbf{Sl}^i.$$

$$x_{i_1} \cdots x_{i_r} \mapsto ((1, x_{i_1}) + (x_{i_1}, x_{i_2}) + \cdots + (x_{i_1} \cdots x_{i_{r-1}}, x_{i_r}), x_{i_1} \cdots x_{i_r}).$$

Then, we have

$$\begin{aligned}
\mathbf{Sl}^{i+1} \models u = v &\Leftrightarrow p_{n,i+1}u = p_{n,i+1}v \\
&\Leftrightarrow c^i(u) = c^i(v) \quad \text{and} \quad \mathbf{Sl}^i \models u = v \\
&\qquad \text{composing with the above embedding} \\
&\Leftrightarrow c^i(u) = c^i(v) \quad \text{and} \quad c^{i-1}(u) = c^{i-1}(v) \\
&\qquad \text{by the induction hypothesis} \\
&\Leftrightarrow c^i(u) = c^i(v) \quad \text{by the above remarks.} \quad \blacksquare
\end{aligned}$$

To further study the identities of $\mathbf{Sl}^{i+1}$ we will need to look more closely at the definition of $c^i$ so as to relate it with $c^{i-1}$. The objective is to extract a simple basis of identities for the variety generated by $\mathbf{Sl}^{i+1}$. For this purpose it will be convenient to have the following technical lemma available.

LEMMA 3.3. *Let $u, v \in \{x_1, ..., x_n\}^+$ and $i > 0$ be such that $c^i(u) = c^i(v)$. Consider a letter $x \in c(u)$ and its first occurrence in $u$ (reading from left to right). Distinguish two cases according to whether or not $x$ is the last letter occurring for the first time in $u$.*

*Case* (a). *There is a factorization $u = u_1 x u_2$ with $u_1, u_2 \in \{x_1, ..., x_n\}^*$, $x \notin c(u_1)$, and $c(u_2) \subseteq c(u_1 x)$. Write also $v = v_1 x v_2$ with $v_1, v_2 \in \{x_1, ..., x_n\}^*$ and $x \notin c(v_1)$.*

*Case* (b). *There is a factorization $u = u_1 x u_2 y u_3$ with $u_1$, $u_2$, $u_3 \in \{x_1, ..., x_n\}^*$, $x \notin c(u_1)$, $c(u_2) \subseteq c(u_1 x)$, and $y \notin c(u_1 x u_2)$. Write also $v = v_1 x v_2 y v_3$ with $v_1$, $v_2$, $v_3 \in \{x_1, ..., x_n\}^*$, $c(v_2) \subseteq c(v_1 x)$, $x \notin c(v_1)$, and $y \notin c(v_1 x v_2)$.*

*Then, in both cases* (a) *and* (b), $c^{i-1}(u_2) = c^{i-1}(v_2)$.

*Proof.* Throughout this proof all words are taken from the set $\{x_1, ..., x_n\}^*$. We will also write $p_j$ instead of $p_{n,j}$ to simplify the notation. Note that the existence of the factorizations for $v$ of the indicated form given the existence of the corresponding factorizations for $u$ is an immediate consequence of the assumption that $c^i(u) = c^i(v)$.

Proceed by induction on $i \geqslant 1$.

Let $u_2 = u_2' z u_2''$ with $z \in \{x_1, ..., x_n\}$. Then $(p_i u_1 x u_2', z) \in c^i(u) = c^i(v)$ and so there is a factorization $v = v' z v''$ with $p_i u_1 x u_2' = p_i v'$. By Proposition 3.2, it follows that

$$c^{i-1}(u_1 x u_2') = c^{i-1}(v') \tag{3}$$

whence $x \in c(v')$ and, in case (b), $y \notin c(v')$. Therefore, the chosen occurrence of $z$ in $v = v' z v''$ must fall in $v_2$, i.e., there is a factorization $v_2 = v_2' z v_2''$ such that $v' = v_1 x v_2'$. This argument already yields $c(u_2) \subseteq c(v_2)$, whence the case $i = 1$ by symmetry.

In case $i > 1$, by the case $i - 1$ applied to the words $u_1 x u_2'$ and $v' = v_1 x v_2'$ (cf. (3)), we have $c^{i-2}(u_2') = c^{i-2}(v_2')$ since $c(u_2') \subseteq c(u_1 x)$. Hence $(p_{i-1} u_2', z) = (p_{i-1} v_2', z) \in c^{i-1}(v_2)$ by Proposition 3.2, and so $c^{i-1}(u_2) \subseteq c^{i-1}(v_2)$. By symmetry, $c^{i-1}(u_2) = c^{i-1}(v_2)$. ∎

Let $\mathscr{S}\ell$ denote the variety of all semilattices. It is well known that $\mathscr{S}\ell$ is locally finite. If, for a class $\mathscr{C}$ of algebras, we represent by $\mathscr{C}^F$ the subclass of all finite algebras in $\mathscr{C}$, then, by definition, $\mathbf{Sl} = \mathscr{S}\ell^F$. Hence, by [2, Corollary 2.3], for every $i \geqslant 1$, $(\mathscr{S}\ell^i)^F = \mathbf{Sl}^i$ and $\mathscr{S}\ell^i$ is locally finite. Thus, $\mathscr{S}\ell^i$ is the variety generated by $\mathbf{Sl}^i$ and so $F_n\mathbf{Sl}^i$ is the free object on $n$ generators in this variety.

The next result furnishes a basis of identities for $\mathscr{S}\ell^{i+1}$.

THEOREM 3.4. *For* $i \geqslant 0$, $\mathscr{S}\ell^{i+1} = [\Sigma_i]$.

*Proof.* Proceed by induction on $i$. The case $i=0$ is well known.

For the inclusion $\mathscr{S}\ell^{i+1} \subseteq [\Sigma_i]$, use Proposition 3.2 to show that, whenever $(u=v) \in \Sigma_i$, we have $c^i(u) = c^i(v)$. To show that

$$x, y \in c(u_1) \subseteq \cdots \subseteq c(u_i) \Rightarrow c^i(u_i \cdots u_1 xy) = c^i(u_i \cdots u_1 yx) \tag{4}$$

by symmetry it suffices to verify that, under the assumption that the hypothesis in the implication (4) holds, $c^{i-1}(u_i \cdots u_1) = c^{i-1}(u_i \cdots u_1 y)$, i.e., that $\mathscr{S}\ell^i \models u_i \cdots u_1 = u_i \cdots u_1 y$ by Proposition 3.2. By the induction hypothesis, this relation in turn reduces to $\Sigma_{i-1} \vdash u_i \cdots u_1 = u_i \cdots u_1 y$, which is easily checked. The case of identities in $B_i$ is similar.

For the reverse inclusion, applying Birkhoff's variety theorem [4], we wish to show that

$$c^i(u) = c^i(v) \Rightarrow \Sigma_i \vdash u = v.$$

For this purpose, assume that $c^i(u) = c^i(v)$. Let $x \in c(u)$ and isolate its first occurrence in $u$ and $v$. Denoting respectively by $u_2$ and $v_2$ the longest factors of $u$ and $v$ following the first occurrences of $x$ which do not involve any new letters, we have $c^{i-1}(u_2) = c^{i-1}(v_2)$ by Lemma 3.3. By the induction hypothesis we then conclude that $\Sigma_{i-1} \vdash u_2 = v_2$. As in Lemma 3.3, let $u_1$ and $v_1$ be the longest prefixes of $u$ and $v$, respectively, in which the letter $x$ does not occur. The idea of the proof is to assume inductively that $\Sigma_i \vdash u_1 = v_1$ and to use a deduction yielding $\Sigma_{i-1} \vdash u_2 = v_2$ to show that $\Sigma_i \vdash u_1 x u_2 = v_1 x v_2$.

To be precise, we prove that for any letter $x \in X$, if we let $u'$ and $v'$ be the longest prefixes of $u$ and $v$, respectively, which do not contain the letter $x$, then $\Sigma_i \vdash u' = v'$. If $x$ is the first letter in $u$ (and so also the first letter in $v$), then the identity $u' = v'$ becomes $1 = 1$. This is not really a semigroup identity, but it is trivial anyway, so we will argue as if it were a semigroup identity, writing $\Sigma_i \vdash 1 = 1$ as a convention and accepting the sequence 1, 1 as deduction of $1 = 1$. Now, in the notation of the preceding paragraph, we may assume inductively that $\Sigma_i \vdash u_1 = v_1$. By the induction hypothesis on $i$, we also know that $\Sigma_{i-1} \vdash u_2 = v_2$. Hence, by Proposition 2.3 there is a deduction of $u_1 x u_2 = u_1 x v_2$ from $\Sigma_i$. Hence $\Sigma_i \vdash u_1 x u_2 = u_1 x v_2 = v_1 x v_2$. Thus, the induction step allows us to proceed until the occurrence for the first time of another letter. Once every letter of $u$ has been found (i.e., in $|c(u)|$ steps), we obtain $\Sigma_i \vdash u = v$, as desired. ∎

Since $\mathbf{Sl}^i = (\mathscr{S}\ell^i)^F$, any basis of identities for $\mathscr{S}\ell^i$ is also a basis of identities for $\mathbf{Sl}^i$.

COROLLARY 3.5. *For* $i \geqslant 0$, $\mathbf{Sl}^{i+1} = [\![\Sigma_i]\!]$.

Using Corollary 3.5, it is now easy to establish a classical result of Stiffer [12].

COROLLARY 3.6. *The smallest pseudovariety of semigroups closed under the formation of semidirect products containing* $\mathbf{Sl}$ *is* $\bigcup_{i \geqslant 0} \mathbf{Sl}^i = \mathbf{R}$.

*Proof.* Let $S \in \mathbf{Sl}^i$. Then $S \models x^{i+1} = x^i$, $(xy)^i x = (xy)^i$ since the first of these identities belongs to $\Sigma_{i-1}$ and the second is easily deduced from $\Sigma_{i-1}$. Hence $S \in \mathbf{R}$ [5].

Conversely, if $S \in \mathbf{R}$ with $|S| = k$, then for any $s_1, ..., s_k \in S$, there are indices $1 \leqslant i < j \leqslant k$ such that $s_i s_{i+1} \cdots s_j$ is an idempotent [5, Proposition III.9.2]. Whence, for $x \in c(u_1) \subseteq \cdots \subseteq c(u_k)$ and any homomorphism $\varphi: X^+ \to S$, there are indices $1 \leqslant i < j \leqslant k$ such that $e = \varphi(u_j \cdots u_{i+1} u_i)$ is an idempotent. But, since $S$ is $\mathscr{R}$-trivial, $e = e\varphi(t)$ for any $t \in c(u_j \cdots u_{i+1} u_i)$. In particular, $e = e\varphi(u_{i-1} \cdots u_1) = e\varphi(u_{i-1} \cdots u_1)\, \varphi(x)$ so that $S \models u_k \cdots u_1 x = u_k \cdots u_1$. Hence $S \models \Sigma_k$ and $S \in \mathbf{Sl}^{k+1}$ by Corollary 3.5. ∎

## 4. THE FINITE BASIS PROBLEM FOR $\mathbf{Sl}^n$

While $\Sigma_0 = \{xy = yx,\ x^2 = x\}$, it is not immediately obvious whether one may extract from any particular $\Sigma_i$ a finite basis of identities for the pseudovariety $\mathbf{Sl}^{i+1}$. More generally, one may ask whether $\mathbf{Sl}^{i+1}$ always admits a finite basis of pseudoidentities. However, as it is observed below, these questions have necessarily the same answer in the present case.

By [3], since $\mathbf{Sl}^i$ is generated by some $F_n\mathbf{Sl}^i$ (cf. Corollary 5.2), if $\mathbf{Sl}^i$ admits a finite basis of pseudoidentities, then it admits a finite basis of identities. Then, since $\mathscr{S}\ell^i$ is locally finite, $F_n\mathbf{Sl}^i$ also generates $\mathscr{S}\ell^i$ (as a variety). Hence $\mathscr{S}\ell^i$ must be finitely based by a result of Sapir [11]. Since $\mathscr{S}\ell^i = [\Sigma_{i-1}]$ by Theorem 3.5, we conclude that $\mathbf{Sl}^i$ admits a finite basis of identities if and only if it is possible to extract a finite basis of identities of $\mathbf{Sl}^i$ from $\Sigma_{i-1}$ (by the compactness theorem of equational logic [4]).

For $\Sigma_1$ it is not hard to extract a finite basis. Indeed, the reader may easily verify that the following does it.

PROPOSITION 4.1. *Thet set consisting of the identities*

$$xzytxy = xzytyx, \qquad xytxy = xytyx, \qquad xzyxy = xzyyx,$$

$$xyxy = xy^2x, \qquad xyx^2 = xyx, \qquad x^3 = x^2$$

*constitutes a basis of identities for* $\mathbf{Sl}^2$.

For monoids, this set reduces to the basis given by Pin [8] and Pin's result is a consequence of Propositions 4.1 and 2.5 (cf. the discussion at the end of the paper). We show below that all $\Sigma_i$ with $i \geqslant 2$ are not finitely based. The argument is based on ideas of Perkins [6] and may be applied in other situations. In what follows, fix $i \geqslant 2$.

For each $r \geqslant 1$, consider the identity

$$e_r : x\bar{z}x\vec{z}x^{i-1} = x\bar{z}x\vec{z}x^i,$$

where $\vec{z} = z_1 \cdots z_r$, $\bar{z} = z_r \cdots z_1$, $x = x_1$, and, say $z_j = x_{j+1}$. The factorizations

$$x\bar{z} \cdot x\vec{z} \cdot \underbrace{x \cdots x}_{i-2} \cdot x \qquad \text{and} \qquad x\bar{z} \cdot x\vec{z} \cdot \underbrace{x \cdots x}_{i-2} \cdot x^2$$

of the sides of $e_r$ show that $e_r$ lies in $B_i$ and so also in $\Sigma_i$.

LEMMA 4.2. *Let $\varphi: X^+ \to X^+$ be a homomorphism and let $u, v \in X^+$ be such that $\varphi u = x\vec{z}$, $\varphi v = x\bar{z}$, and $c(u) \subseteq c(v)$. Then $|c(u)| = r + 1$.*

*Proof.* Since $\varphi u$ and $\varphi v$ are products of distinct variables, $u$ and $v$ must also be products of distinct variables, say $u = t_1 \cdots t_s$. If $s < r + 1$, then $|\varphi t_i| > 1$ for some $i \in \{1, ..., s\}$. However, $t_i \in c(v)$ and the words $x\bar{z}$ and $x\vec{z}$ have no common factors of length greater than 1. Hence $s = r + 1$. ∎

PROPOSITION 4.3. *If $F \subseteq \Sigma_i$ and $F \vdash e_r$, then $F$ involves at least $r + 1$ variables.*

*Proof.* To apply some identity from $\Sigma_i$ to the left hand side of $e_r$, one needs to find a factor of $x\bar{z}x\vec{z}x^{i-1}$ of the form $u_i \cdots u_1 x'$ with $x' \in c(u_1) \subseteq \cdots \subseteq c(u_i)$. But, since $x$ is the only variable occurring $i + 1$ times in the word $x\bar{z}x\vec{z}x^{i-1}$, this factorization must be given by $u_i = x\bar{z}$, $u_{i-1} = x\vec{z}$, $u_{i-2} = \cdots = u_1 = x' = x$. Thus, to be able to apply some identity from $\Sigma_i$ to the left side of $e_r$, it must be of the form $uvx^i = uvx^{i-1}$ with $x \in c(v) \subseteq c(u)$ and there must be a homomorphism $\varphi: X^+ \to X^+$ such that $\varphi(u) = x\bar{z}$ and $\varphi(v) = x\vec{z}$. By Lemma 4.2, the identity from $\Sigma_i$ in question must involve at least $r + 1$ variables. ∎

COROLLARY 4.4. *The set of identities $\Sigma_i$ is not finitely based.*

*Proof.* Since $e_r \in \Sigma_i$ for all $r \geqslant 1$ and, by Proposition 4.3 there can be no finite $F \subseteq \Sigma_i$ such that $F \vdash e_r$ for all $r \geqslant 1$, the result follows. ∎

Combining Corollary 4.4 with Theorem 3.4 and in view of the discussion at the beginning of this section, we obtain the main result of this paper.

THEOREM 4.5. *For $i \geqslant 3$, $\mathbf{Sl}^i$ does not admit a finite basis of pseudo-identities.*

## 5. GENERATORS FOR $\mathbf{Sl}^n$

The aim of this section is to obtain a semigroup $S \in \mathbf{Sl}^i$ that generates this pseudovariety. Of course, if there is such an $S$, then $F_n\mathbf{Sl}^i$ is a generator of $\mathbf{Sl}^i$ for any sufficiently large $n$. Moreover, since $\mathbf{Sl}^i = (\mathscr{S}\ell^i)^F$ and $\mathscr{S}\ell^i$ is locally finite, to find a generator for $\mathbf{Sl}^i$ is equivalent to determine a finite generator of $\mathscr{S}\ell^i$.

THEOREM 5.1. *Suppose that the variety $\mathscr{S}\ell^i$ is generated by $F_n\mathbf{Sl}^i$. Then the semigroup $F_{n+2}\mathbf{Sl}^{i+1}$ generates the pseudovariety $\mathscr{S}\ell^{i+1}$.*

*Proof.* For $i=0$, the result is certainly true since $F_2\mathbf{Sl}^1$ generates $\mathscr{S}\ell^1$. So, we assume $i>0$. We show that, for $u, v \in \{x_1, ..., x_m\}^+$,

$$F_{n+2}\mathbf{Sl}^{i+1} \models u=v \Rightarrow c^i(u)=c^i(v).$$

The result follows by Proposition 3.2. Thus, assume $F_{n+2}\mathbf{Sl}^{i+1} \models u=v$. We employ an argument similar to the one used in the proof of Theorem 3.4 based on Lemma 3.3 to deduce that $c^i(u)=c^i(v)$.

Given a letter $x \in c(u)$, isolate its first occurrence in $u$. As in Lemma 3.3, we distinguish two cases according to whether or not $x$ is the last letter to occur for the first time in $u$. In the following, all words are assumed to come from $\{x_1, ..., x_m\}^*$.

*Case* (a). We have $u=u_1xu_2$ with $c(u_2) \subseteq c(u_1x)$.

*Case* (b). We have $u=u_1xu_2yu_3$ with $c(u_2) \subseteq c(u_1x)$ and $y \notin c(u_1xu_2)$.

Since the two-point semilattice is a homomorphic image of $F_{n+2}\mathbf{Sl}^{i+1}$, certainly $c(u)=c(v)$ and so $x$ also occurs in $v$. Let $v_1$ be the longest factor of $v$ to the left of the first occurrence of $x$. If case (a) holds, then write $v=v_1xv_2$. If case (b) holds, then there must be no occurrence of $y$ in $v_1x$ since there is an occurrence of $x$ in $u$ without occurrences of $y$ to the left of it and $F_2\mathbf{Sl}^2$ is a homomorphic image of $F_{n+2}\mathbf{Sl}^{i+1}$. Hence, in case (b) we have a factorization $v=v_1xv_2yv_3$ such that $y \notin c(v_1xv_2)$ and, in both cases, $c(v_2) \subseteq c(v_1x)$.

If $x$ is the first letter of $u$ (and so also of $v$), then clearly $c^i(u_1)=c^i(v_1)$. So, we will apply the same induction scheme as in the final part of the proof of Theorem 3.4. The induction step consists in proving that, if $c^i(u_1)=c^i(v_1)$, then $c^i(u_1xu_2)=c^i(v_1xv_2)$.

We first claim that $F_n\mathbf{Sl}^i \models u_2=v_2$. Let $\varphi\colon \{x_1, ..., x_m\}^+ \to F_n\mathbf{Sl}^i$ be any homomorphism. Since $p_{n,i}\colon \{x_1, ..., x_n\} \to F_n\mathbf{Sl}^i$ is an onto homomorphism, we may construct a homomorphism $\psi\colon \{x_1, ..., x_m\}^+ \to \{x_1, ..., x_{n+2}\}^+$ by defining $\psi x = x_{n+1}$, in case (b) holds $\psi y = x_{n+2}$, and $\psi z$ so that $p_i \psi z = \varphi z$ for all other $z \in \{x_1, ..., x_m\}$. Then $x_{n+1} \notin c(\psi u_1) = c(\psi v_1)$ and, in case (b), $x_{n+2} \notin c(\psi u_1 x u_2) = c(\psi v_1 x v_2)$, $c(\psi u_2) \subseteq c(\psi u_1 x)$. Since $\psi u = \psi v$ is an identity obtained from $u = v$ by substitution of variables, we have $F_{n+2}\mathbf{Sl}^{i+1} \models \psi u = \psi v$, and therefore $\psi u = \psi v$ holds in $\mathbf{Sl}^{i+1}$ since it involves at most $n+2$ variables. Hence, by Lemma 3.3, $c^{i-1}(\psi u^2) = c^{i-1}(\psi v_2)$ and so $\mathbf{Sl}^i \models \psi u_2 = \psi v_2$. Let $\theta\colon F_{n+2}\mathbf{S} \to F_n\mathbf{Sl}^i$ be the homomorphism defined by $\theta x_{n+1} = \varphi x$, $\theta x_{n+2} = \varphi y$ in case (b), and $\theta z = p_i z$ for $z \in \{x_1, ..., x_n\}$. Then $\varphi = \theta \circ \psi$ so that $\varphi u_2$ and $\varphi v_2$ are obtained from the sides of the identity $\psi u_2 = \psi v_2$ by interpretation of the variables in $F_n\mathbf{Sl}^i$. Since the identity in question holds in $\mathbf{Sl}^i$, it follows that $\varphi u_2 = \varphi v_2$. Thus, we have shown that $F_n\mathbf{Sl}^i \models u_2 = v_2$, which proves the claim.

Since $F_n\mathbf{Sl}^i \models u = v$ and, by hypothesis $F_n\mathbf{Sl}^i$ generates $\mathbf{Sl}^i$, Proposition 3.2 yields $c^{i-1}(u_2) = c^{i-1}(v_2)$. By Theorem 3.4, we then have $\Sigma_{i-1} \vdash u_2 = v_2$ and so $\Sigma_i \vdash u_1 x u_2 = u_1 x v_2$ by Proposition 2.3. By the same token, $\Sigma_i \vdash u_1 = v_1$ in view of the induction hypothesis, and so $\Sigma_i \vdash u_1 x u_2 = u_1 x v_2 = v_1 x v_2$, whence $c^i(u_1 x u_2) = c^i(v_1 x v_2)$ by Theorem 3.4. In $|c(u)|$ induction steps, we obtain $c^i(u) = c^i(v)$, as desired. ∎

Corollary 5.2. (a) *The variety $\mathscr{S}\ell^n$ is generated by $F_{2n}\mathbf{Sl}^n$.*

(b) *The pseudovariety $\mathbf{Sl}^n$ is generated by $F_{2n}\mathbf{Sl}^n$.*

In particular, we obtain the following decidability result [7, Corollary 4.3].

Corollary 5.4. *For any $n \geqslant 0$, the pseudovariety $\mathbf{Sl}^n$ has a decidable membership problem.*

*Proof.* It is easy to show that a finitely generated pseudovariety has a decidable membership problem [1, 3]. ∎

Combining Corollaries 5.2 and 4.5, we obtain an infinite family of finite $\mathscr{R}$-trivial semigroups whose identities are not finitely based. The first example of such a semigroup appears in Perkins [6].

Corollary 5.2 naturally leads to the problem of determining the smallest $g = g(n)$ such that $F_g\mathbf{Sl}^n$ generates the variety $\mathscr{S}\ell^n$ (or the pseudovariety $\mathbf{Sl}^n$). We only have a very modest partial solution for this problem.

For $n = 1$, Corollary 5.2 already gives the best possible value, namely $g(1) = 2$. For $n = 2$, the inequality $g(2) \leqslant 4$ also follows from Proposition 4.1 since $F_4\mathbf{Sl}^2$ is finite.

Lemma 5.4. $F_3\mathbf{Sl}^2 \models xyxz^2yxt^2xyz = xyxz^2yt^2xyz$.

*Proof.* For convenience of notation, let $\{a, b, c\}$ represent the usual set of free generators for the relatively free semigroup $F_3\mathbf{Sl}^2$. We need to show that, for every $x, y, z, t \in F_3\mathbf{Sl}^2$, $xyxz^2yxt^2xyz = xyxz^2yt^2xyz$.

If $c(x) \subseteq c(yz)$, then we have the equality $xyxz^2yx = xyxz^2y$ since then the occurrences of variables (in $\{a, b, c\}$) in the rightmost occurrence of $x$ may be absorbed into the preceding factor $zy$. In particular, the case $c(yz) = \{a, b, c\}$ is eliminated. On the other hand, if $c(yz)$ has just one element, then the prefix $xyxz^2y$ allows the absorption of $x$. Thus, without loss of generality, we will assume that $c(yz) = \{a, b\}$.

We may remove from $t$ every variable which occurs in $x$, $y$, or $z$: the left factors $xyxz^2yx$ and $xyxz^2y$ contain all the variables needed to absorb the first occurrences of variables from $c(yz)$ in $t$; for the next occurrences, i.e., after some variable outside $c(yz)$ occurs (and, we already assumed that there is at most one such variable), using the fact that, to the left of them, everything already occurred, we may move them to the end of the factor $t^2$ and, there, absorb them in the factor $yz$. Moreoever, if $c(t) \subseteq c(yz)$, the desired equality is also clear since we just showed that, then, the factor $t^2$ may be removed. Hence, we may assume that $c(t) = \{c\}$.

Finally, by a similar argument, we may remove from the extra occurrence of $x$ on the left side everything which occurs in either $yz$ or $t$. Hence this occurrence is removable and we do have equality. ∎

PROPOSITION 5.5. $g(2) = 4$.

*Proof.* Since $g(2) \leqslant 4$ and $F_3\mathbf{Sl}^2$ satisfies the identity in Lemma 5.4, it suffices to show that this same identity is not valid in $\mathbf{Sl}^2$. For this purpose, just observe that $xyxz^2yxt^2xyz$ has a prefix $ux$ with $c(u) = \{x, y, z\}$ while this is not true of $xyxz^2yt^2xyz$, and apply Proposition 3.2. ∎

We conclude with a few remarks. The embedding of Proposition 2.4 is in general very uneconomical. For instance, a simple hand calculation using Proposition 3.2 shows that $F_2\mathbf{Sl}^2$ has 20 elements while Proposition 2.4 embeds it in a semigroup with $765 = (2^{4 \times 2} - 1) \times 3$ elements.

The semigroup $F_{g(n)}\mathbf{Sl}^n$ is probably not the smallest generator of $\mathbf{Sl}^n$. For example, $(F_1\mathbf{Sl})^1$ generates $\mathbf{Sl}$. Using Proposition 3.2, one may also show that $(F_3\mathbf{Sl}^2)^1$ generates $\mathbf{Sl}^2$. This points in the direction of one needing only $2n-1$ generators for a finite monoid generating $\mathbf{Sl}^n$.

Although it is natural to expect that $g(n)$ is an increasing function of $n$, we have found no definite reason why this should happen.

We have not completely solved all of the four problems for the pseudovarieties $\mathbf{Sl}^n$ proposed in the Introduction. Proposition 3.2 provides an algorithm to decide when an identity holds in $\mathbf{Sl}^n$. But this is a recursive algorithm with high complexity on the length of the input identity.

The algorithm to decide when a finite semigroup lies in $\mathbf{Sl}^n$ issuing from

Corollary 5.3 is also of very high complexity. In [3] we proposed a conjecture that implies that a pseudovariety not admitting a finite basis of pseudoidentities does not have a membership problem solvable in polynomial time. So, we suspect that, for $n \geqslant 3$, $\mathbf{Sl}^n$ is not decidable in polynomial time.

In this paper we worked essentially with semigroups. But, since **Sl** is a monoidal pseudovariety, Propositions 2.5 and 2.6 (and the proof of the latter) show us how to translate results on $\mathbf{Sl}^i$ to results on semidirect powers $\mathbf{Sl}^i_M$ of the pseudovariety of monoid semilattices. This translation was already mentioned for the basis of identities of $\mathbf{Sl}^2$ but the same thing could be done with Corollaries 3.5 and 3.6 and Theorem 4.5. In particular, we obtain a negative solution to a problem proposed by Pin [9, p. 113] asking whether $\mathbf{Sl}^3_M$ may be defined by a finite set of identities.

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