Singular random matrix decompositions: distributions

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\textbf{Abstract}

Assuming that $Y$ has a singular matrix variate elliptically contoured distribution with respect to the Hausdorff measure, the distributions of several matrices associated to QR, modified QR, SV and polar decompositions of matrix $Y$ are determined, for central and non-central, non-singular and singular cases, as well as their relationship to the Wishart and pseudo-Wishart generalized singular and non-singular distributions. Some of these results are also applied to two particular subfamilies of elliptical distributions, the singular matrix variate normal distribution and the singular matrix variate symmetric Pearson type VII distribution.

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1. Introduction

It has been a common practice in the past, to eliminate variables or individuals to correct for dependencies among columns or rows when we sample from a multivariate distribution. This solution in part, was due to the fact of not having a distribution theory to handle all those cases. In a more formal way, let $Y \in \mathbb{R}^{N \times m}$, be a sample of $N$ individuals with $m$ variables under study, if there exist dependencies among rows (individuals) or columns (variables), $Y$ does not have a density with respect to the Lebesgue measure in $\mathbb{R}^{Nm}$. However, it is known that $Y$ has a density on a subspace $\mathcal{M} \subset \mathbb{R}^{Nm}$ on which it is possible to define a measure called the Hausdorff measure, which coincides with the Lebesgue measure when it is defined on $\mathcal{M}$ (for the cases studied in Section 3, $\mathcal{M}$ is an affine subspace). Details on this kind of problems can be found in [10,12]. They proposed expressions for the singular matrix variate normal distribution and singular matrix variate elliptically contoured distribution. In other words, we count now with a solution for the classical multivariate statistical analysis when based on the normal distributions and also for the more general case called the generalized multivariate statistical analysis based on the elliptically contoured distributions.

When $Y$ has a distribution with respect to the Lebesgue measure we could find different ways of deriving the Wishart distribution. Some are based on the QR decomposition, [29,32,33], others on the singular value decomposition (SVD), [23] and some others on the polar decomposition [2,22]. What all of these approaches through different factorizations are trying to do is to find an alternative coordinate system for the columns (or rows) for the matrix $Y$. For example, the coordinates obtained from the QR decomposition are called rectangular coordinates, [31, p. 597], for the polar decomposition, polar coordinates [2], etc. These matrices of coordinates, besides of being the key part for establishing the Wishart, pseudo-Wishart, $F$ and beta distributions, as well as distributions of $|Y^tY|$ and $\text{tr} \ Y^tY$ among others, play an important role in other areas of knowledge, in particular on the shape theory and pattern recognition. As an example, if $Y$ has a matrix variate normal distribution, it may be written as $Y = H_1^tT$, the QR decomposition. In the context of shape theory, the distribution of $T$ is called size-and-shape distribution, also known in the literature as the rectangular coordinates distribution, see [20,31, p. 597]. In the same setting of shape theory, when considering the SV ($Y = V_1 DW_1$) or polar ($Y = P_1 R$) decompositions, the matrices ($D, W_1$) and $R$ may both be thought of as an alternative coordinates system, in such a way that the corresponding distributions play the role of size-and-shape distributions, see [18,28]. Similarly, matrix $D$ is considered as yet another coordinate system, and its corresponding distribution is called size-and-shape cone distribution, see [11,12,20]. Some of these results were extended to the case in which $Y$ has a singular gaussian and a singular elliptically contoured distribution, see [10,11]. In the context of pattern recognition the role of some of these decomposition is also known, in particular the SV decomposition is known as the Karhunen–Lòeve expansion or decomposition [26].

In the present work some results on distributions of random matrices, for which their density function exist with respect to the Lebesgue measure, will be extended to the case in which $Y$ has a density with respect to the Hausdorff measure, and moreover, to the case in which $Y$ has a singular matrix variate elliptically contoured distribution. In Section 2, the densities of matrices associated to the QR, modified QR, SV and polar decompositions are found with respect to the Hausdorff measure, both for the non-central and central cases.
2. Notation and preliminary results

Let \( \mathcal{L}_{m,N}(q) \) be the linear space of all \( N \times m \) real matrices of rank \( q \leq \min(N, m) \); \( \mathcal{L}^+_{m,N}(q) \) be the linear space of all \( N \times m \) real matrices of rank \( q \leq \min(N, m) \) with \( q \) distinct singular values. The set of matrices \( H_1 \in \mathcal{L}_{m,N}(m) \) such that \( H_1' H_1 = I_m \) is a manifold denoted by \( \mathcal{V}_{m,N} \), called Stiefel manifold. In particular, \( \mathcal{V}_{m,m} \) is the group of orthogonal matrices \( O(m) \). Denote by \( S^+_m(q) \), the \((mq - q(q - 1)/2)\)-dimensional manifold of rank \( q \) positive semidefinite \( m \times m \) symmetric matrices with \( q \) distinct positive eigenvalues; \( \mathcal{T}_m \) denote the group of \( m \times m \) upper triangular matrices and \( \mathcal{T}^+_m \) is the group of \( m \times m \) upper triangular matrices with positive diagonal elements, \( t_{ii} > 0, i = 1, \ldots, m \); \( \mathcal{T}^+_{m,N} \) the set of \( N \times m \) upper quasi-triangular matrices such that \( T = (T_1|T_2) \in \mathcal{T}^+_{m,N}, \) with \( T_1 \in \mathcal{T}^+_N \) and \( T_2 \in \mathcal{L}_{m-N,N}(q) \), with \( q = \min(N, m - N) \); \( \mathcal{T}^+_1 \) and \( \mathcal{T}^+_{m,N} \) denote the set of unit upper triangular or unit quasi-triangular matrices, respectively, such that \( t_{ii} = 1 \) for all \( i \), \( i = 1, \ldots, N \); \( D \in \mathcal{D}(m) \subset \mathcal{T}_m \) the diagonal matrices with diagonal elements \( d_{11}, \ldots, d_{mm} \).

**Definition 1.** (Matrix-variate singular elliptically contoured distribution). Let \( Y \in \mathcal{L}^+_{m,N}(q) \), such that \( Y \sim \mathcal{E}_{N \times m}^{k,r}(\mu, \Sigma, \Theta, h) \), with \( \Sigma : m \times m \) of rank \( r < m \) or \( \Theta : N \times N \) of rank \( k < N \). This distribution will be called a matrix-variate singular elliptically contoured distribution and will be denoted as

\[
Y \sim \mathcal{E}_{N \times m}^{k,r}(\mu, \Sigma, \Theta, h)
\]

omitting the supra-index when \( r = m \) and \( k = N \). In addition, its density function is given by

\[
\frac{1}{\left(\prod_{i=1}^r \tilde{\lambda}_i^{k/2}\left(\prod_{j=1}^k \tilde{\delta}_j^{r/2}\right)\right)} h \left(\text{tr} \Sigma^{-1}(Y - \mu)'\Theta^{-1}(Y - \mu)\right), \quad (1)
\]

\[
\begin{align*}
E_1'(Y - \mu)M_1' &= 0 \\
E_1'(Y - \mu)M_2' &= 0 \\
E_2'(Y - \mu)M_2' &= 0
\end{align*}
\]

\[
\text{a. s.} \quad (2)
\]

for some function \( h \). Where \( A^{-} \) is a symmetric generalized inverse, \( \tilde{\lambda}_i \) and \( \tilde{\delta}_j \) are the non-zero eigenvalues of \( \Sigma \) and \( \Theta \), respectively. Let \( E = (E_1|E_2) \in \mathcal{O}(N) \) and \( M = (M_1'|M_2') \in \mathcal{O}(m) \) be matrices associated with the spectral decomposition of matrices \( \Sigma \) and \( \Theta \), respectively, with \( E_1 \in \mathcal{V}_{k,N}, \ E_2 \in \mathcal{V}_{N-k,N}, \ M_1' \in \mathcal{V}_{r,m} \) and \( M_2' \in \mathcal{V}_{m-r,m} \), see [12].

Alternatively, this density can be written as (for the Normal distribution case, see [27])
Proof. The proof follows from [29, Lemma 9.5.3, p. 397]; [25, Eq. (22)].

3.1. QR decomposition

examples on Section 4.

is the multivariate gamma function,

\[
\int_{H_1 \in V_{q,k}} (\text{tr}(XH_1))^{2t} (H_1' dH_1) = \frac{2^q \pi^{q/2}}{\Gamma_q[\frac{q}{2}]} \sum_{\kappa} \frac{4^t}{\left(\frac{1}{2}\right)_\kappa} C_k(XX'),
\]

where \(C_k(B)\) are the zonal polynomials of \(B\) corresponding to the partition \(\kappa = (t_1, \ldots, t_2)\) of \(t\), with \(\sum_{i=1}^2 t_i = q; (a)_\kappa = \prod_{j=1}^q (a - (j - 1)/2)_{t_j}, (a)_t = a(a+1) \cdots (a+t-1),\) being the generalized hypergeometric coefficients and \(\Gamma_\pi(a) = \pi^{(s-1)/4} \prod_{j=1}^s \Gamma(a - (j - 1)/2)\) is the multivariate gamma function, see [16], [25] or [29].

Lemma 2. Let \(X \in L_{k,m}(q)\) and \(H_1 \in V_{q,k}\), then

\[
dF_t(Y) = \frac{1}{\left(\prod_{i=1}^r \lambda_i^{k/i} \left(\prod_{j=1}^k \delta_j^{r/2}\right)\right)} h \left(\text{tr} \Sigma^-(Y - \mu)^\top \Sigma^-(Y - \mu)\right) (dY),
\]

where \((dY)\) is the Hausdorff measure, which coincides with that of Lebesgue measure when it is defined on the subspace \(\mathcal{M}\) given by hyperplanes (2), see [1, p. 247]; [3, p. 297]; [10]. Explicitly, if \(q = \min(r, k)\), \((dX)\) would be given by Eqs. (4), (7), (9) or (11).

Proof. The proof follows from [29, Lemma 9.5.3, p. 397]; [25, Eq. (22)].

Let \(Y \sim \mathcal{E}^{k,r}_{N \times m}(\mu, \Sigma, \Theta, h)\), and define the generalized Wishart \((N \geq m)\) or Pseudo-Wishart \((N < m)\) matrix as \(S = Y'\Theta^{-1}Y\). Let \(Q \in L_{N,k}(k)\), such that \(\Theta = Q'Q\), and define \(X = (Q')^t Y\). Then

\[
X \sim \mathcal{E}^{k,r}_{k \times m}(\mu_x, \Sigma, I_k, h)
\]

with \(\mu_x = (Q')^t \mu\) and \(S\) satisfies that

\[
S = Y'\Theta^{-1}Y = (Q')^t Y (Q')^{-1} Y = X'X.
\]

3. Density functions

In this section, we find the densities for matrices \(T, R, (C, \mathcal{G}), (D, W_1)\) and \(D\) associated with the QR, modified QR, SV, and polar decompositions of matrix \(X\), where \(X \sim \mathcal{E}^{k,r}_{k \times m}(\mu_x, \Sigma, I_k, h)\), under the assumption that \(h\) has a convergent power series expansion on some interval \(I\). This interval will depend on the specific distribution under study, see examples on Section 4.

3.1. QR decomposition

Let \(X \in L_{m,q}(q)\), then there exist \(H_1 \in V_{q,k}\) and \(T \in T_{m,q}\) with \(t_{ii} \geq 0, i = 1, 2, \ldots, \min(q, k - 1)\) such that \(X = H_1 T\). Let \(H_2 \in V_{m-k, k}\) (a function of \(H_1\)) such that \(H = (H_1 H_2) \in \mathcal{O}(k)\). Writing by columns, \(H_1 = (h_1 \cdots h_q), H_2 = (h_{q+1} \cdots h_k)\), see [17, Section 5.4]; [20,32, A.3.11, p. 149], then

\[
(dX) = \prod_{i=1}^q t_{ii}^{k-i} (H_1' dH_1)(dT), \quad (4)
\]
where

\[(dT) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i}^{m} dt_{ij}, \quad \text{with} \quad T = (t_{ij}) \tag{5}\]

and

\[(H'_{1}dH_{1}) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{k} h'_{j}dh_{j} \tag{6}\]

define an invariant measure on \(V_{q,k}\), see [16,23,29, Section 2.1.4]. The proof is given in [7], see also [8].

**Theorem 3.** For \(k \geq m\) or \(k < m\), with \(q = \min(k, r)\), the density of \(T\) is given by

\[dF_{T}(T) = \frac{2^{q} \pi^{q/2} \prod_{i=1}^{q} t_{ii}^{k-i}}{\Gamma_{q} \left( \frac{1}{2} k \right) \left( \prod_{i=1}^{q} t_{ii}^{k/2} \right)} \sum_{t=0}^{\infty} \sum_{\kappa} \frac{h^{(2r)}(\text{tr}(\Sigma^{-T'T} + \Omega)) C_{\kappa}(\Omega^{-T'T})}{t!} \left( \frac{1}{2} k \right)_{\kappa} (dT),\]

where the measure \((dT)\) is defined by (5); \(\Omega = \Sigma^{-\mu'\Theta^{-}\mu}, C_{\kappa}(B)\) are the zonal polynomials of \(B\) corresponding to the partition \(\kappa = (t_{1}, \ldots, t_{x})\) of \(t\), with \(\sum_{1}^{\kappa} t_{i} = t\) and \(h^{(j)}(v)\) is the \(j\)th derivative of \(h\) with respect to \(v\).

**Proof.** The density of \(X\) is given by

\[dF_{X}(X) = \frac{1}{\left( \prod_{i=1}^{q} x_{i}^{k/2} \right)} h(\text{tr} \Sigma^{-}(X - \mu_{x})(X - \mu_{x}))(dX)\]

or

\[dF_{X}(X) = \frac{1}{\left( \prod_{i=1}^{q} x_{i}^{k/2} \right)} h(\text{tr} \Sigma^{-}(X'X + \mu_{x}'\mu_{x}) - 2 \text{tr} \Sigma^{-}X'\mu_{x}))(dX).\]

Factoring \(X = H_{1}T\) from (4), we have that the joint density of \(H_{1}\) and \(T\) is given by

\[dF_{H_{1},T}(H_{1},T) = \frac{\left( \prod_{i=1}^{q} t_{ii}^{k-i} \right)}{\left( \prod_{i=1}^{q} x_{i}^{k/2} \right)} h(\text{tr}(\Sigma^{-}T'T + \Omega)\text{tr} - 2 \text{tr} \Sigma^{-}T'H_{1}'\mu_{x}) (H'_{1}dH_{1})(dT),\]

where \(\Omega = \Sigma^{-\mu'\mu_{x}} = \Sigma^{-\mu'\Theta^{-}\mu}\). Assuming that \(h(\cdot)\) can be expanded in power series, (see [14]), i.e.,

\[h(a + v) = \sum_{t=0}^{\infty} \frac{h^{(t)}(a)}{t!} v^{t}.\]
Thus
\[
dF_{H_1,T}(H_1, T) = \frac{1}{(\prod_{i=1}^{q} k_i^{-i})} \sum_{r=0}^{\infty} \frac{1}{r!} h^{(r)}(\text{tr}(\Sigma^{-1}T' + \Omega)) \\
\times (\text{tr}(-2\mu \Sigma^{-1}T' H_1^{'})')^r(H_1'dH_1)(dT).
\]

Integrating on \( H_1 \in \mathcal{V}_{q,k} \) and noting that this integral equals zero when \( t \) is odd, (see [24, Lemma p. 876] or [25, Eqs. (44)–(46)]) the marginal density of \( T \) may be expressed as
\[
dF_T(T) = \frac{1}{(\prod_{i=1}^{q} k_i^{-i})} \sum_{r=0}^{\infty} \frac{1}{(2t)!} h^{(2t)}(\text{tr}(\Sigma^{-1}T' + \Omega)) \\
\times \left( \int_{H_1 \in \mathcal{V}_{q,k}} (\text{tr}(-2\mu \Sigma^{-1}T' H_1^{'})^2(H_1'dH_1)) (dT) \right).
\]

By Lemma 2, we have
\[
\int_{H_1 \in \mathcal{V}_{q,k}} (\text{tr}(-2\mu \Sigma^{-1}T' H_1^{'})^2(H_1'dH_1)) = \frac{2^q \pi^{q/2}}{\Gamma_q[\frac{1}{2}]} \sum_{\kappa} \left( \frac{1}{2} \right)_t \frac{4^t}{i^t} C_\kappa(\Omega \Sigma^{-1}T').
\]

Observing that \( \frac{1}{(2t)!} \cdot \frac{4^t}{t!} = 1 \), the result follows, see [29, p. 21].

The central case can easily be obtained from Theorem 3,

**Corollary 4.** The central density of \( T \) is
\[
dF_T(T) = \frac{2^q \pi^{q/2}}{\Gamma_q[\frac{1}{2}]} \sum_{\kappa} \left( \frac{1}{2} \right)_t \frac{4^t}{i^t} h(\text{tr} \Sigma^{-1}T')(dT).
\]

From the Wishart matrix (or generalized Wishart matrix, under an elliptically model), the \( S = T'T \) factorization is known in the literature as Bartlett decomposition (central or non-central). The density of \( T \) has been studied by different authors for the central non-singular case \( (q = m \leq k) \), as a function of both, the density of \( S \) and the density of \( X \) \((S = X'X)\), see [33, p. 74]; [29, p. 99]; [13, p. 314]; [15, p. 119], among others. In the normal, non-central, non-singular cases, Goodall and Mardia [19,20] study the density of \( T \) when \( q = \min(k, m) \), with \( k \geq m \) and \( k < m \), in the shape theory setting. Later Dahel and Giri [5], also under normal theory, found the density of \( T \) for the case when \( r(\mu) = 1 \).

### 3.2. Polar decomposition

Let \( X \in \mathcal{L}^+_{m,k}(q), k \geq m, \) and write \( X = P_1 R \), such that, \( P_1 \in \mathcal{V}_{m,k} \). Let \( P_2 \in \mathcal{V}_{k-m,k} \) (a function of \( P_1 \)) such that \( P = (P_1|P_2) \in \mathcal{O}(k), P_1 = (p_1 \cdots p_m), P_2 = (p_{m+1} \cdots p_k) \) are writing by columns. Let \( R \in \mathcal{S}^+_{m}(q), \) with \( R = J_1 L J_1' \) is the non-singular part of the spectral decomposition of \( R, J_1 \in \mathcal{V}_{q,m} \) and \( L \in \mathcal{D}(q), l_{11} > \cdots > l_{qq} > 0.\)
see [2,17,22], then

\[(dX) = \frac{|L|^{k-q}}{\text{Vol}(V_{m-q,k-q})} \prod_{i<j}^q (l_{ii} + l_{jj})(dR)(P_1'dP_1),\]  

(7)

where \(\text{Vol}(V_{m-q,k-q}) = \int_{K_1 \in V_{m-q,k-q}} (K_1' dK_1) = \frac{2^{(m-q)} \pi^{(m-q)(k-q)/2}}{\Gamma_{m-q}[\frac{1}{2}(k - q)]},\)

\[(dR) = 2^{-q} |L|^{m-q} \prod_{i<j}^q (l_{ii} - l_{jj})(dL)(J_1'dJ_1),\]  

(8)

(see [10,34]); \((P_1'dP_1)\) and \((J_1'dJ_1)\) are given by (6). The proof is given in [7], see also [8].

**Theorem 5.** Assuming that \(k \geq m\), with \(q = \min(k, r)\), the density of \(R\) is given by

\[dF_R(R) = \frac{2^q \pi^{q/2} |L|^{k-q} \prod_{i<j}^q (l_{ii} + l_{jj})}{\Gamma_q \left[\frac{1}{2}k\right] \left(\prod_{i=1}^q \lambda_i^{k/2}\right)} \sum_{t=0}^{\infty} \sum_{\kappa} h^{(2t)}(\text{tr}(\Sigma^{-1} R^2 + \Omega)) C_\kappa(\Omega^{-1} R^2) (dR),\]

where the measure \((dR)\) is defined by (8), \(\Omega = \Sigma^{-1} \mu' \Theta^{-1} \mu\), \(C_\kappa(B)\) are the zonal polynomials of \(B\) corresponding to the partition \(\kappa = (t_1, \ldots, t_\kappa)\) of \(t\), with \(\sum_{i=1}^\kappa t_i = t\) and \(h^{(j)}(v)\) is the \(j\)th derivative of \(h\) with respect to \(v\).

**Proof.** The proof is similar to that given for Theorem 3, by considering the factorization of the measure \((dX)\) given by (7).

If \(\mu_s = 0\), we obtained the central case.

**Corollary 6.** The central density of \(R\) is

\[dF_R(R) = \frac{2^q \pi^{q/2} |L|^{k-q} \prod_{i<j}^q (l_{ii} + l_{jj})}{\Gamma_q \left[\frac{1}{2}k\right] \left(\prod_{i=1}^q \lambda_i^{k/2}\right)} h(\text{tr} \Sigma^{-1} R^2)(dR).\]

Olkin and Rubin [30] study the density of \(R\) under a non-singular central normal distribution, expressing the eigenvalues of \(R\) as a function of the elements of \(S\), for the case when \(q = m = 2\).

### 3.3. Modified QR decomposition

Let \(X \in \mathcal{L}_{m,k}^+(q)\), then there exist \(H_1 \in \mathcal{V}_{q,k}\), \(C \in \mathcal{D}(q)\) and \(G \in \mathcal{T}_{m,q}^1\) with \(c_{ii} \geq 0\), \(i = 1, 2, \ldots, \min(q, k - 1)\) such that \(X = H_1 CG\). Let \(H_2 \in \mathcal{V}_{k-q,k}\) (a function of \(H_1\))
such that $H = (H_1 | H_2) \in O(k)$, where $H_1 = (h_1 \cdots h_q)$, $H_2 = (h_{q+1} \cdots h_k)$ are written by columns. For this decomposition we have

$$(dX) = 2^{-q} \prod_{i=1}^{q} c^{k+m-2i}_{ii} (H_1' dH_1) (dC) (dG),$$

(9)

where

$$(dG) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{m} dg_{ij}, \quad \text{with} \quad G = (g_{ij}); \quad (dC) \equiv \bigwedge_{i=1}^{q} dc_{ii}$$

and $(H_1' dH_1)$ is given by (6). The proof is given in [7], see also [8].

**Theorem 7.** For $k \geq m$ or $k < m$, with $q = \min(k, r)$, the joint density of $(C, G)$ is given by

$$dF_{C,G}(C, G) = \frac{2^q \pi^{k/2} q^{-k/2} \prod_{i=1}^{q} c^{k+m-2i}_{ii}}{\Gamma_q[\frac{1}{2} k]} \left( \prod_{i=1}^{q} \lambda_{ii}^{k/2} \right) \times \sum_{t=0}^{\infty} \sum_{\kappa} h(2t)(\text{tr}(\Sigma^{-\mu} G' C^2 G + \Omega)) C_k(\Omega \Sigma^{-\mu} G' C^2 G) \left( \frac{1}{2} k \right) \kappa \times (dC)(dG),$$

with the measures $(dG)$ and $(dC)$ defined by (10), $\Omega = \Sigma^{-\mu} \Theta^{-\mu}$, $C_k(B)$ are the zonal polynomials of $B$ corresponding to the partition $\kappa = (t_1, \ldots, t_k)$ of $t$, with $\sum_1^k t_i = t$ and $h^{(j)}(v)$ is the $j$th derivative of $h$ with respect to $v$.

**Proof.** The proof is similar to that given for Theorem 3, by considering the factorization of the measure $(dX)$ given by (9).

For the central case we have:

**Corollary 8.** The central joint density of $(C, G)$ is given by

$$dF_{C,G}(C, G) = \frac{2^q \pi^{k/2} q^{-k/2} \prod_{i=1}^{q} c^{k-i}_{ii}}{\Gamma_q[\frac{1}{2} k]} \left( \prod_{i=1}^{q} \lambda_{ii}^{k/2} \right) h(\text{tr} \Sigma^{-\mu} G' C^2 G) (dC)(dG).$$

### 3.4. SV decomposition

Let $X \in \mathcal{L}^+_{m,k}(q)$, then there exist $V_1 \in \mathcal{V}_{q,k}$, $W_1 \in \mathcal{V}_{q,m}$ and $D \in \mathcal{D}(q)$, $D_{11} > \cdots > D_{qq} > 0$, such that $X = V_1 DW_1'$, this factorization is called the non-singular part of the SVD, [31, p. 42]; [13, p. 58]. Let $V_2 \in \mathcal{V}_{k-q,k}$ (a function of $V_1$) and $W_2 \in \mathcal{V}_{m-q,m}$ (a function of $W_1$) such that $V = (V_1 | V_2) \in O(k)$ and $W = (W_1 | W_2) \in O(m)$; $V_1 = (v_1 \cdots v_q)$, $V_2 = (v_{q+1} \cdots v_k)$, $W_1 = (w_1 \cdots w_q)$ and $W_2 = (w_{q+1} \cdots w_m)$ are writing by
columns. Then, we have that
\[
(dx) = 2^{-q} |D|^{k+m-2q} \prod_{i<j}^q (D_{ii}^2 - D_{jj}^2) (dD)(V'_1 dV_1)(W'_1 dW_1),
\] (11)

where \((dD)\) and \((V'_1 dV_1)\) and \((W'_1 dW_1)\) are defined as in (10) and (6) respectively. The proof is given in [10].

**Theorem 9.** (i) The joint density of \(D\) and \(W_1\) is
\[
dF_{D,W_1}(D,W_1) = \frac{2^{-q} \pi^{q(k-m)/2} \Gamma_q \left[ \frac{1}{2} m \right] |D|^{k+m-2q} \prod_{i<j}^q (D_{ii}^2 - D_{jj}^2)}{\Gamma_q \left[ \frac{1}{2} k \right] \left( \prod_{i=1}^r \lambda_i^{k/2} \right)} \times \sum_{t=0}^\infty \sum_{\kappa} \frac{h^{(2r)}(\text{tr}(\Sigma^- W_i D^2 W'_i + \Omega)) C_\kappa(\Omega \Sigma^- W_1 D^2 W'_i)}{t!} (1/2k)^\kappa (dW_1)(dD),
\]
where \((dW_1) = \frac{(W'_1 dW_1)}{\text{Vol}(V_{q,m})}\) with \(\text{Vol}(V_{q,m}) = \frac{2^q \pi^m}{\Gamma_q (1/2m)}\).

(ii) The density of \(D\) is given by
\[
dF_D(D) = \frac{2^{-q} \pi^{q(k+m)/2} \prod_{i=1}^q D_{ii}^{k+m-2q} \prod_{i<j}^q (D_{ii}^2 - D_{jj}^2)}{\Gamma_q \left[ \frac{1}{2} k \right] \Gamma_q \left[ \frac{1}{2} m \right] \left( \prod_{i=1}^r \lambda_i^{k/2} \right)} \times \sum_{t=0}^\infty \sum_{\theta, \kappa, \phi \in \theta, \kappa} \frac{h^{(2t)}(\text{tr}(\Sigma^-) C_\phi(\Sigma^-) C_\phi^*(\Sigma^-))}{t!} \left( \frac{1}{2k} \right)^\kappa (dD),
\]
where the measure \((dD)\) is defined as in (10); \(\Omega = \Sigma^- \mu \Theta^- \mu\), \(C_k(B)\) are the zonal polynomials of \(B\) corresponding to the partition \(\kappa = (t_1, \ldots, t_2)\) of \(t\), with \(\sum_{i=1}^2 t_i = t\); \(h^{(j)}(v)\) is the \(j\)th derivative of \(h\) with respect to \(v\) and the operators \(\Delta_\phi^{\theta, \kappa}\), \(C_\phi^{\theta, \kappa}\) and the notation for the sums are given in [6], see also [4].

**Proof.**
(i) The proof is similar to that given to Theorem 3 applying Lemma 2 and considering the decomposition of the measure \((dx)\) given in (11).

(ii) From Theorem 9(1), by rewriting it as function of \((W'_1 dW_1)\), we have that
\[
dF_D(D) = \left( \int_{W_1 \in V_{q,m}} f_{D,W_1}(D,W_1) (W'_1 dW_1) \right) (dD)
\]
\[
= \frac{\pi^{k/2} |D|^{k+m-2q} \prod_{i<j}^q (D_{ii}^2 - D_{jj}^2)}{\Gamma_q \left[ \frac{1}{2} k \right] \left( \prod_{i=1}^r \lambda_i^{k/2} \right)} \times \sum_{t=0}^\infty \sum_{\kappa} \frac{h^{(2t)}(\text{tr}(\Sigma^- W_1 D^2 W'_1 + \Omega))}{t!} \left( \frac{1}{2k} \right)^\kappa (dD),
\]
\[
\times \frac{C_\kappa(\Omega\Sigma^{-1}W_1D^2W'_1)}{(\frac{1}{2}k)_\kappa} (W'_1dW_1)(dD).
\]

(12)

Now expanding \( h^{2f}(v) \) in power series and \( (\text{tr} B) \) in zonal polynomials we have (see [25, Eq. (17)]),

\[
h^{(2f)}(\text{tr}(\Sigma^{-1}W_1D^2W'_1 + \Omega)) = \sum_{l=0}^{\infty} \sum_{\theta} \frac{h^{2f+l}(\text{tr} \Omega) C_{\theta}(\text{tr} \Sigma^{-1}W_1D^2W'_1)}{l!},
\]

(13)

where \( C_\kappa(A) \) are the zonal polynomials of \( A \) corresponding to the partition \( \phi = (l_1, \ldots, l_\beta) \) of \( l \), with \( \sum_1^\beta l_i = l \). Substituting (13) in (12) and considering only the integral over \( W_1 \), denoting its by \( J \), we have

\[
J = \int_{W_1 \in V_{q,m}} C_\kappa(\Omega\Sigma^{-1}W_1D^2W'_1) C_\theta(\Sigma^{-1}W_1D^2W'_1)(W'_1dW_1)
\]

\[
= \frac{2^q \pi^{km/2}}{\Gamma_q \left( \frac{1}{2}m \right)} \sum_{\phi \in \theta, \kappa} C_{\phi, \kappa}^{\theta, \kappa} (\Sigma^{-1}, \Omega\Sigma^{-1}) C_{\phi, \kappa}^{\theta, \kappa} (D^2, D^2)
\]

\[
= \frac{2^q \pi^{km/2}}{\Gamma_q \left( \frac{1}{2}m \right)} \sum_{\phi \in \theta, \kappa} A_{\phi, \kappa}^{\theta, \kappa} C_{\phi}(D^2) C_{\phi, \kappa}^{\theta, \kappa} (\Sigma^{-1}, \Omega\Sigma^{-1})
\]

(14)

It is follows from Eqs. (4.13) and (5.1) in [6] (see also [4]). Finally, the proof follows substituting (14) in (12).

Corollary 10. (i) The central joint density of \((D, W_1)\) is given by

\[
d_{FD,W_1}(D, W_1) = \frac{2^{-q} \pi^{q(k+m)/2} \Gamma_q \left( \frac{1}{2}m \right) 2^{k+m-2q} \prod_{i<j} D_{ii}^2 - D_{jj}^2}{\Gamma_q \left( \frac{1}{2}k \right) \left( \prod_{i=1}^r \frac{1}{2}k \right) \prod_{i=1}^r \frac{1}{2}k}/\frac{h(\text{tr} \Sigma^{-1}W_1D^2W'_1)(dD)(dW_1)}{/afii9837}{/afii9814}.
\]

(ii) The central density of \(D\) is given by

\[
d_{FD}(D) = \frac{2^q \pi^{q(k+m)/2} \prod_{i=1}^q D_{ii}^{k+m-2q} \prod_{i<j}^q (D_{ii}^2 - D_{jj}^2)}{\Gamma_q \left( \frac{1}{2}k \right) \Gamma_q \left( \frac{1}{2}m \right) \left( \prod_{i=1}^r \frac{1}{2}k \right) \prod_{i=1}^r \frac{1}{2}k} \times \sum_{t=0}^{\infty} \sum_{\kappa} \frac{h^t(0) C_{\kappa}(\Sigma^{-1}) C_{\kappa}(D^2)}{t! C_{\kappa}(I_m)} (dD).
\]

Díaz-García et al. [10], under normal theory, find the non-central density of \(D^2\), when \( q = \min(k, m)\). Also, Díaz-García et al. [11] show that the density of \(D/||D||\) in the central case, is invariant under all the elliptically contoured distributions.
4. Some applications

Now two particular cases of elliptically contoured distributions are considered, the matrix variate normal distribution, and the class of matrix variate symmetric Pearson type VII distributions, Gupta and Varga [21, pp. 75–76], for which the density of \( R \) is found. The densities of \( T, (D, W_1) \) and \( D \) are obtained in a similar form.

**Corollary 11.** Let \( X \in \mathcal{L}_{m,k}(q) \) such that \( X \sim \mathcal{E}^{k,r}_{k \times m}(\mu_x, \Sigma, I_k, h) \), with \( h \) expanding in power series. Then,

(i) If \( X \) has a matrix variate normal distribution, the density of \( R \) is

\[
d_{FR}(R) = \frac{2^q(2q-kr)/2 \pi^{k(q-r)/2} |L|^{k-q} \prod_{i<j} (l_{ii} + l_{jj})}{\Gamma_q[\frac{1}{2}k] \left( \prod_{i=1}^r \lambda_i^{k/2} \right) \times \text{etr}( -\frac{1}{2} (\Sigma^{-1} - \Omega))_0 F_1 \left( \frac{1}{2}k; \frac{1}{2} \Omega \Sigma^{-1} R^2 \right) (dR),}
\]

where \( \Omega \) is a hypergeometric function of matrix argument, [25,29, p. 258].

(ii) If \( X \) has a matrix variate symmetric Pearson type VII distribution, the density of \( R \) is

\[
d_{FR}(R) = \frac{2^q \pi^{k(q-r)/2} \Gamma[b] |L|^{k-q} \prod_{i<j} (l_{ii} + l_{jj})} {\Gamma_q[\frac{1}{2}k] a^{kr/2} \Gamma[\frac{1}{2}(2b - kr)] \left( \prod_{i=1}^r \lambda_i^{k/2} \right) \times \sum_{t=0}^{\infty} \frac{(b)_t (1 + v/a)^{(2t)}}{t! (\frac{1}{2}k)^{t}}} C \left( \frac{1}{a^2} \Omega \Sigma^{-1} R^2 \right) (dR),
\]

where \( a, b \in \mathbb{R} \), with \( a > 0, b > km/2 \) and \( \text{tr}(\Sigma^{-1} R^2 + \Omega) < a \).

**Proof.** The proof follows from Theorem 5, observing that:

(i) For the normal case

\[
h(v) = \frac{1}{(2\pi)^{kr/2}} \exp(-\frac{1}{2}v),
\]

therefore

\[
h^{(2t)}(v) = \frac{1}{2^{2t+kr/2} \pi^{kr/2}} \exp(-\frac{1}{2}v),
\]

where the corresponding power series converges for all \( v \in \mathbb{R} \).

(ii) For the Pearson type VII case

\[
h(v) = \frac{\Gamma[b]}{(\pi a)^{kr/2} \Gamma[b - kr/2]} (1 + v/a)^{-b},
\]

then

\[
h^{(2t)}(v) = \frac{\Gamma[b]}{(\pi a)^{kr/2} \Gamma[b - kr/2]} \frac{(b)_t}{a^{2t}} (1 + v/a)^{-(b+2t)}.
\]
where the corresponding power series converges if \( v < a \),
the results are obtained. □

5. About Wishart and pseudo-Wishart distributions

Let \( Y \sim \mathcal{E}^{k,r}_{N \times m}(\mu, \Sigma, \Theta, h) \), we want to find the distribution of the matrix Wishart or the generalized pseudo-Wishart \( S = Y'\Theta^{-}Y \), where \( \Theta^{-} \) is a generalized inverse of \( \Theta \).

The density of \( S \) could be found through those of \( T, R, (N, \Omega) \) and \( (D, W_{1}) \), with the help of Theorems 13(1), 16, 15(1) and 3(1) in [8], respectively. Other approach would be by considering the density of \( S, S = Y'\Theta^{-}Y = X'X \), where \( X = QY \) such that \( Q \in \mathcal{N}_{N,k}(k) \) with \( \Theta = Q'R \), and using Theorems 13(2), 18(2), 15(2) and 3(2) in [8], respectively.

However, note that depending on the factorization given to \( Y, X \), there exist four measures \((dS)\) (between many others) and therefore, four expressions for the density of \( S \). The general density form for any of those factorization will be,

\[
dF_{S}(S) = \frac{\pi^{k/2}\omega_{S}^{(k-m-1)/2}}{\Gamma_{q}(\frac{k}{2})} \sum_{i=0}^{\infty} \frac{h^{(2i)}(\text{tr}(\Sigma^{-}S + \Omega)) C_{k}(\Omega \Sigma^{-}S)}{t!} \left( \frac{1}{2}k \right)_{k} (dS),
\]

where \( A^{-} \) is a symmetric generalized inverse of \( A, \Omega = \Sigma^{-}\mu'\Theta^{-}\mu, C_{k}(B) \) are the zonal polynomials of \( B \) corresponding to the partition \( \kappa = (t_{1}, \ldots, t_{l}) \) of \( t \), with \( \sum_{1}^{l} t_{i} = t \) and \( h^{(j)}(\kappa) \) is the \( j \)th derivative of \( h \) with respect to \( v \).

The matrix \( \Psi \) and the volumen \((dS)\), are defined according to each factorization and excluding the polar decomposition, the density \( S \) can be found for all the cases, i.e. when \( N \geq m \) (Wishart distribution), \( N < m \) (pseudo-Wishart distribution) and \( q = \min(k, r) \) (singular and non-singular cases), with \( k \geq r \) or \( k < r \). In particular, we have:

(i) **QR decomposition, \( X = H_{1}T \).** In this case, the matrix \( \Psi \) is defined by \( S_{11} \) where, if

\[
T = \begin{pmatrix} T_{1} & T_{2} \\ q \times q & q \times m - q \end{pmatrix}
\]

then,

\[
S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}_{q \times q \times m} = \begin{pmatrix} T_{1}' & T_{2}' \\ T_{1}' & T_{2}' \end{pmatrix}_{q \times q \times m - q} = \begin{pmatrix} T_{1}'T_{1} & T_{1}'T_{2} \\ T_{2}'T_{1} & T_{2}'T_{2} \end{pmatrix}_{q \times q \times m - q}
\]

and \( |S_{11}| = |T_{1}'T_{1}| = |T_{1}'|^{2} = \prod_{i=1}^{q} t_{i}^{2} \). In this way, the volumen with respect to which the density of \( S = X'X = T'T \) exist, is given by

\[
(dS) = 2^{q} \prod_{i=1}^{q} t_{i}^{m-1} (dT).
\]

(ii) **Polar decomposition, \( X = P_{1}R \).** Under this decomposition, the density of \( S = X'X = R^2 \) can be established only when \( N \geq m \) (Wishart distribution) and \( q = k \geq r \) (singular case). Here, the matrix \( \Psi \) is defined by \( C = L^{2} \), with \( R = J'_{1}LJ_{1} \) and the volumen \((dS)\)
is defined as
\[(dS) = 2^q |L|^{m-q+1} \prod_{i<j}^q (l_{ii} + l_{jj}) (dR) = |L|^{m-q} \prod_{i \leq j}^q (l_{ii} + l_{jj}) (dR).\]

(iii) **QDR decomposition**, \(X = H_1 CG\). \(\Psi\) would be defined as \(O = C^2\), and the corresponding volume for which the density of \(S = X'X = G'OG\) will exist, is given by
\[(dS) = \prod_{i=1}^q a_{ii}^{m-i} (dG)(dO).\]

(iv) **SV decomposition**, \(X = V_1 DW_1\). Here, \(S = X'X = W_1D^2W_1', \Psi = A = D^2\), and the volume \(dS\) is defined as
\[(dS) = 2^{-q} |A|^{m-q} \prod_{i \leq j}^q (A_{ii} - A_{jj}) (dA)(W_1'dW_1).\]

This case has been worked in detail by Díaz-García and Gutiérrez-Jáimez [12], for the elliptical case and in [10] for the normal distribution case.

**References**