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TOTAL REFLECTIONS, PARTIAL PRODUCTS, AND HEREDITARY FACTORIZATIONS

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This paper shows that many, but not all, reflective subcategories of **Top** have a certain property, here called total reflectivity, hitherto studied in some special cases, such as for compactness. It is related to Pasynkov's partial topological products and to the stability of topological factorizations under pullback along open inclusions.

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Let $\mathscr{C}$ be a reflective subcategory of **Top**, the category of topological spaces and maps. For a space X, with reflection map $r_X : X \to X_{\mathscr{C}}$, we know that r_X is (uniquely) $\mathscr{C}$-extendable. Let $U \subseteq X_{\mathscr{C}}$ be open; can we say that $r_X| : r_X^{\leftarrow} U \to U$ is $\mathscr{C}$-extendable? "Yes" is the easy answer in the case $\mathscr{C} = \mathbf{Top}_0$ (the T_0-spaces), from the construction of r_X; with rather more work, we can say "yes" for Tychonoff X and $\mathscr{C}$ = {compact Hausdorff spaces} or $\mathscr{C}$ = {realcompact spaces}: c.f. 8G in [9].

Call $\mathscr{C}$ *totally reflective* when every $r_X| : r_X^{\leftarrow} U \to U$ is uniquely $\mathscr{C}$-extendable, for open $U \subseteq X_{\mathscr{C}}$. We give below a simple criterion for the total reflectivity of a reflective subcategory $\mathscr{C}$ of **Top**: that $\mathscr{C}$ should be closed under formation of partial products (Pasynkov [23]): and we show that 'uniquely' may be omitted from the definition. We then examine several reflective subcategories of **Top**, and find many which are totally reflective, but only one which is not. We also add to Pasynkov's list of topological properties preserved by partial product formation.

Recall that if U is an open subset of X, and Y is any space, the *partial product* $P(X, U, Y)$ of *X over U with fibre Y* has $(U \times Y) \oplus (X \setminus U)$ as underlying set, with the coarsest topology making the projections $\pi : P \to X$ and $\pi^{\leftarrow} U \to Y$ continuous. It is not generally either a subspace or a quotient space of the product $X \times Y$. Partial products were introduced for the construction of universal spaces in dimension theory.

Theorem 1. *Let $\mathscr{C}$ be reflective in* **Top**. *Then $\mathscr{C}$ is totally reflective iff pp-closed (i.e. X, Y in $\mathscr{C}$ imply $P(X, U, Y)$ is in $\mathscr{C}$, for open $U \subseteq X$).*

Proof. This follows from the following:

Theorem 2. *Let $\mathscr{C}$ be reflective in* **Top**, *$A \in$* **Top**. *Then the following are equivalent:*

(i) *every reflection map $r_X : X \to X_{\mathscr{C}}$ is hereditarily uniquely A-extendable;*

(ii) *every reflection map r_X is hereditarily A-extendable;*

(iii) *$\mathscr{C}$ is pp-closed for fibre A (i.e. $X \in \mathscr{C}$, open $U \subseteq X$, $\Rightarrow P(X, U, A) \in \mathscr{C}$).*

Proof. (ii)$\Rightarrow$(iii): suppose the reflection maps $r_X : X \to X_{\mathscr{C}}$ are all *hereditarily A-extendable* (i.e. [2] $r_X| : r_X^{\leftarrow} U \to U$ is A-extendable, for open $U \subseteq X_{\mathscr{C}}$. Let $C \in \mathscr{C}$, $U \subseteq C$ be open, $h : P = P(C, U, A) \to C$ be the projection from the partial product. Without loss of generality, we shall identify $h^{\leftarrow} U$ with the space $U \times A$. Let $f : P \to P_{\mathscr{C}}$ be the reflection map for P into $\mathscr{C}$; then $P_{\mathscr{C}}$ is in $\mathscr{C}$, and f is $\mathscr{C}$-extendable, so there is a factorization $h = gf : P \to P_{\mathscr{C}} \to C$. Let $V = g^{\leftarrow} U$, an open subset of $P_{\mathscr{C}}$; so $f| : h^{\leftarrow} U \to V$ is A-extendable by hypothesis. The projection $p : h^{\leftarrow} U \to A$ must therefore factor through $f|$, and there exists $s : g^{\leftarrow} U = V \to A$, with $sf| = p$. But then the maps $s : V \to A$ and $g : P_{\mathscr{C}} \to C$ induce a map $t : P_{\mathscr{C}} \to P$, by universality of partial products, with $ht = g$ and $t|_V = \langle g|, s\rangle$ from V to $U \times A$. Then $h(tf) = (ht)f = gf = h = h1$; and, on $h^{\leftarrow} U = U \times A$, $(tf)| = t|_V \circ f|_{h^{\leftarrow} U} = \langle g|, s\rangle \circ f| = \langle g|f|, sf|\rangle = \langle h|, p\rangle = 1$; by uniqueness of induced maps into partial products, $tf = 1$, P is a retract of $P_{\mathscr{C}}$ and so is in $\mathscr{C}$.

(iii)$\Rightarrow$(i): suppose $\mathscr{C}$ is pp-closed for fibre A; let $X \in \mathscr{C}$, $r : X \to X_{\mathscr{C}}$ be the reflection map at X, $U \subseteq X_{\mathscr{C}}$ open, $s : r^{\leftarrow} U \to A$ a map to be extended through U. Let $P = P(X_{\mathscr{C}}, U, A)$; by hypothesis, P is in $\mathscr{C}$. Let $t : P \to X_{\mathscr{C}}$ be the projection map; wlog, $t^{\leftarrow} U = U \times A$. Consider the induced map $\langle r|, s\rangle : r^{\leftarrow} U \to U \times A$; by the universality of partial products, this induces $f : X \to P$ with $tf = r$, and $f| = \langle r|, s\rangle : r^{\leftarrow} U = f^{\leftarrow} t^{\leftarrow} U \to t^{\leftarrow} U = U \times A$. By the $\mathscr{C}$-extendability of r, there is $g : X_{\mathscr{C}} \to P$ with $gr = f$. Then $tgr = tf = r$; since r is a reflection map and $X_{\mathscr{C}} \in \mathscr{C}$, the map $r : X \to X_{\mathscr{C}}$ factors in just one way through r, hence $tg = 1$. So $tgU = U$, hence $g^{\leftarrow} U \subseteq t^{\leftarrow} U = U \times A$. With $p : U \times A \to A$ denoting the projection, $pg| : U \to A$ is the required extension of s through U, since $pg|r| = pf| = s$. Using the universality of partial products and of $\mathscr{C}$-reflections, it is easy to check that this extension is the unique extension of s through U. $\square$

Corollary 1. *Let $\mathscr{C}$ be reflective in* **Top**. *Then all the reflection maps are dense if and only if $\mathscr{C}$ is closed-hereditary.*

Proof. Let $A = \phi$; dense maps are the hereditarily ϕ-extendable ones, and a partial product with empty fibre is just a closed subspace. $\square$

Corollary 2. *Let $\mathscr{C}$ be a totally reflective subcategory of* **Top**, *other than $\mathscr{C}_1 = \{C : |C| = 1\}$. Then $\mathscr{C}$ is closed-hereditary and dense-reflective.*

Proof. If $\phi \notin \mathscr{C}$ and yet $\mathscr{C}$ contains a many-pointed space X, the unique map $\phi \to X$ has too many extensions through $\phi \to \phi_{\mathscr{C}}$. So for $\mathscr{C} \neq \mathscr{C}_1$, $\phi \in \mathscr{C}$ and we argue as in the previous corollary. □

Corollary 3. *The following are totally reflective subcategories of* **Top**: $\mathbf{Top}_0$, $\mathbf{Top}_1$, **Haus**, **Reg**, **Tych**, **Zero**, $\mathbf{Zero}_0$, **CompHaus**, **Bool**, **Sober**.

Proof. Pasynkov [23] shows that almost all of these are partial-product closed; note that **Zero** = {zero-dimensional spaces}; $\mathbf{Zero}_0 = \mathbf{Zero} \cap \mathbf{Top}_0$; $\mathbf{Bool} = \mathbf{Zero} \cap$ **CompHaus**. The exception is **Sober**; here, a direct proof that **Sober** is *pp*-closed may be made along the lines of the usual proof that **Sober** is closed under ordinary products. (For an easier proof, see our next Theorem). □

Example. Kennison [18] gives an example of a non-trivial reflective subcategory of **Top** for which not all reflection maps are dense. By Corollary 2, this is not totally reflective in **Top**.

We conjecture that every epireflective subcategory of **Top**, of $\mathbf{Top}_0$, or of **Haus** is totally reflective. Towards this, we have

Theorem 3. *Let $\mathscr{A} \subseteq \mathscr{D}$, where $\mathscr{D} =$* **Top**, $\mathbf{Top}_0$, *or* **Haus**; *and let $\mathscr{C}$ be the epireflective hull of $\mathscr{A}$ in $\mathscr{D}$. Then the following conditions are equivalent*:

(i) *for $A_1, \ldots, A_n$, A in $\mathscr{A}$, and open $U \subseteq A_1 \times \cdots \times A_n$, the partial product $P(A_1 \times \cdots \times A_n, U, A)$ is in $\mathscr{C}$*;

(ii) *$\mathscr{C}$ is pp-closed for fibres in $\mathscr{A}$*;

(iii) *every $\mathscr{C}$-reflection map is hereditarily $\mathscr{A}$-extendable*;

(iv) *every $\mathscr{C}$-reflection map is hereditarily $\mathscr{C}$-extendable*;

(v) *$\mathscr{C}$ is pp-closed.*

Proof. From Theorems one and two, the implications (ii)$\Leftrightarrow$(iii), (iv)$\Leftrightarrow$(v)$\Rightarrow$(i) are trivial. For (iii)$\Rightarrow$(iv), let $r_X : X \to X_{\mathscr{C}}$ be a reflection map, $U \subseteq X_{\mathscr{C}}$ be open, $s : r_X^{\leftarrow} U \to C$, for C in $\mathscr{C}$, be a map to be extended through U. Then C is a subspace (resp. b-closed subspace [25], closed subspace) of a product of spaces A_i from $\mathscr{A}$, and it is then easy to complete the diagram

$$\begin{array}{ccc} r_X^{\leftarrow} U & \xrightarrow{\ r_X| \ } & U \\ {\scriptstyle s}\downarrow & & \vdots \\ C & \longrightarrow & \Pi A_i \end{array}$$

using (iii). If $\mathscr{D} =$ **Top**, then $r_X^{\leftarrow} U \to U$ is a surjection; if $\mathscr{D} = \mathbf{Top}_0$, then it is b-dense [25]; and if $\mathscr{D} =$ **Haus**, then it is dense; in any case, we can find the required diagonal $U \to C$.

As for (i)$\Rightarrow$(ii), consider the following lemmas, whose proofs are straightforward:

Lemma 1. *If B is a ((b-) closed) subspace of $\mathscr{C}$, and $V \subseteq C$ is open, then $P(B, V \cap B, A)$ is a ((b-) closed) subspace of $P(C, V, A)$.*

Lemma 2. *If $U = \bigcup U_i \subseteq B$, where all U_i are open, then $P(B, U, A)$ is a subspace of $\Pi P(B, U_i, A)$; and if A, B are in* **Top**$_0$ *(resp.,* **Haus**), *then it is a b-closed (resp., closed) subspace.*

Lemma 3. *If $V \subseteq Y$ is open, then $P(X \times Y, X \times V, A)$ and $X \times P(Y, V, A)$ are isomorphic.*

Now let $C \in \mathscr{C}$, $A \in \mathscr{A}$, $U \subseteq C$ be open. We can consider C as a ((b-) closed) subspace of a product ΠA_i; by Lemma 1, we need only consider the case that it is all of ΠA_i. By Lemma 2, we may assume U is a basic open set of the product. By Lemma 3, we can neglect all but the factors A_i where $\pi_i U \neq A_i$; and by our hypothesis (i) we may deduce that the partial product is in $\mathscr{C}$; hence (ii). □

Corollary 1. *Each of the following reflective subcategories of* **Top** *is totally reflective: {realcompact spaces}, {$\mathbb{N}$-compact spaces},* **Sober**.

Proof. For the realcompact spaces, take $\mathscr{A} = \{\mathbb{R}\}$; note that any partial product $P(\mathbb{R}^n, U, \mathbb{R})$ is a continuous image of the full product $\mathbb{R}^{n+1}$, so is Lindelof; being also Tychonoff [23], it must be realcompact. Alternatively, we could use the total reflectivity of **Tych** and the argument in our first paragraph.

For the $\mathbb{N}$-compact spaces, just note that $P(\mathbb{N}^n, U, \mathbb{N}) \cong \mathbb{N}$.

For sober spaces, let D be the Sierpinski dyad; then $P(D^n, U, D)$ is a finite T_0-space, hence is sober. □

Corollary 2. *Each of the following reflective subcategories of* **Top** *is totally reflective: {k-compact spaces}, {k-ultracompact spaces}, {zero-dimensionally k-compact spaces}.*

Proof. For definitions, see for example [13]. W.l.o.g., k is uncountable. For the k-compact spaces, note two cases: first, let k be a successor cardinal $l+1$. Then $P_k = I^l \backslash \{\text{a corner}\}$ is covered by l compact sets, and so therefore is $P(P_k^n, U, P_k)$, which then is k-compact. Second, if k is a limit cardinal, take $P_k = \Pi(P_m : m \in M)$, where M is cofinal in the set of infinite cardinals less than k, and without loss of generality all m in M are successor cardinals. But then $P(P_{m_1} \times \cdots \times P_{m_n}, U, P_m)$ is k-compact, since $\max\{m_1, \ldots, m_n, m\}$ is less than k. So any partial product $P(P_k \times \cdots \times P_k, U, P_k)$ is k-compact, by the argument in the proof of (i)$\Rightarrow$(ii) of the Theorem. But the space P_k generates the simple class of k-compact spaces in **Haus**; hence this class is totally reflective.

For the k-ultracompact spaces, use the Frolik characterization of these spaces as the Tychonoff perfect images of k-compact spaces, and Pasynkov's [23]

Lemma. *Let $f: X_0 \to X$, $g: Y_0 \to Y$ be perfect, onto, then the induced map $P(X_0, f^{\leftarrow} U, Y_0) \to P(X, U, Y)$ is perfect, onto, for open $U \subseteq X$.*

For the zero-dimensionally k-compact spaces, note that each of their partial products is zero-dimensional, so embedded in the Boolean reflection ζP; and is a continuous image of A^{n+1} (where A is the generator $2^l \setminus \{\text{corner}\}$, assuming k is a successor cardinal $l+1$), so has a cover by less than k compact sets, whence P is a co-G_k-set in ζP, therefore zero-dimensionally k-compact [15]. When k is not a successor, argue as for the k-compact case. □

Corollary 3. *{topologically complete spaces} is totally reflective in* **Top**.

Proof. These are generated (in **Haus**) by the metric spaces; we need only verify the

Lemma. *If X, Y are metric, and $U \subseteq X$ is open, then $P(X, U, Y)$ is metric.*

This follows from standard metrization theory, and the normality of P shown by the

Lemma. *If X, Y and $X \times Y$ are paracompact Hausdorff, and $U \subseteq X$ open, the $P(X, U, Y)$ is also paracompact Hausdorff.*

The proof of this is similar to that on p. 179 of [23], where Pasynkov does it for discrete Y. □

More generally, the same sort of argument proves

Corollary 4. *{m-complete spaces} is totally reflective in* **Top**, *(where m is an infinite cardinal, and the class is that of Tychonoff spaces having a complete uniformity of coverings with cardinality less than or equal to m* [4]). □

Many interesting subcategories of **Top**, therefore, are totally reflective. If an epireflective subcategory is totally reflective, we call it *totally epireflective*; and all such notions can be relativized for subcategories $\mathcal{T}$ of **Top**, e.g. *totally reflective in $\mathcal{T}$.*

Theorem 4. *If $\mathcal{S}$ is totally reflective in* **Top**, *then the category $\mathcal{T}$ of subspaces of members of $\mathcal{S}$ is totally epireflective in* **Top** *and $\mathcal{S}$ is totally epireflective in $\mathcal{T}$.*

Proof. Trivial: cf. [12]. □

The following Theorem, and its consequence (Theorem 12), inspired the idea of 'totally reflective':

Theorem 5. *Let $\mathcal{S}$ be totally reflective in* **Top**, *and open-hereditary. Then pushout squares in $\mathcal{S}$ are preserved by pullbacks along open inclusions.*

N.B. This was proved for $\mathcal{S}=$ **Top** in [2], where it plays the main role in our construction of hereditary factorization systems in **Top**.

Proof. Let

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{array}$$

be a pushout in $\mathcal{S}$; since $\mathcal{S}$ is reflective, this can be considered as a pushout

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & E \end{array}$$

in **Top**, followed by the $\mathcal{S}$-reflection map $r: E \to D$. Let $U \subseteq D$ be open; then $r^{\leftarrow} U \to U$ is also an $\mathcal{S}$-reflection map, and by the quoted result from [2], the pushout in **Top** may be pulled back along $r^{\leftarrow} U$, yielding a pushout in **Top**; composing this with $r^{\leftarrow} U \to U$, we get the required pushout in $\mathcal{S}$. □

Now we apply some of these ideas to the study of hereditary factorization systems. By a *factorization system* we mean a pair $(\mathcal{Q}, \mathcal{P})$ of classes of morphisms in some category $\mathcal{C}$, each being composition-closed, containing all isomorphisms, and admitting functorial (Q, P)-factorizations: for equivalent definitions and theory, see Bousfield [1], Dyckhoff [5], or Freyd, Kelly [8]. These are not quite as in the study of topological categories [13], where $\mathcal{P}$ is usually a conglomerate of sources and $\mathcal{Q}$ consists only of epimorphisms: in **Top**, for example, there are factorization systems [6], [7], [19] where the left factor $\mathcal{Q}$ does not consist just of epimorphisms. We say the factorization system $(\mathcal{Q}, \mathcal{P})$ is *hereditary* [2] when $\mathcal{Q} = h(\mathcal{Q})$, with

$$h(\mathcal{Q}) = \{f: X \to Y \text{ with } f^{\leftarrow} U \to U \text{ in } \mathcal{Q} \text{ for all open } U \subseteq Y\}.$$

Proposition. [2] *If $(\mathcal{Q}, \mathcal{P})$ is a factorization system in* **Top** *and $\mathcal{Q}$ consists of onto maps, then there is a class $\mathcal{P}'$ so that $(h(\mathcal{Q}), \mathcal{P}')$ is a hereditary factorization system.* □

We are interested in these for two reasons; first, hereditariness is a property of some useful factorization systems [2], [7], [19]; second, factorizations in topos theory have a corresponding property; cf. [17].

It will be useful to have a characterization of the hereditary factorization systems. Let D be the Sierpinski dyad: $D=\{0,1\}$ with $\{1\}$ open, not closed. Let X^* denote $P(D,\{1\},X)$; equivalently, X^* is X with an extra closed point, which we shall denote by x_0. Some would call X^* the *Sierpinski cone* on X. A map $f:X\to Y$ induces a map $f^*:X^*\to Y^*$ with $f^*|_X=f$, $f^*(x_0)=y_0$. Note that any partial product $P(X,U,Y)$ can be considered as a pullback from the maps $\chi_Y: Y^*\to D$, $\chi_U: X\to D$. Let $\mathcal{P}$ be a class of maps; we say $\mathcal{P}$ is $*$*-closed* if and only if $p\in\mathcal{P}\Rightarrow p^*\in\mathcal{P}$.

A related notion, the *amalgamation* of a pair (e,f), where e is an open embedding $U\subseteq X$, $f:Z\to U$ a map, was introduced in [7]: it is the obvious map $Z\oplus(X\setminus U)\to X$, where the domain has the coarsest topology making this continuous and inducing the right topology on Z. When f is a projection $U\times Y\to U$, we get the partial product. It can be written as a pullback:

$$\begin{array}{ccc} X\times_{U^*}Z^* & \longrightarrow & Z^* \\ \downarrow & & \downarrow{\scriptstyle f^*} \\ X & \longrightarrow & U^* \end{array}$$

where $X\to U^*$ shrinks $X\setminus U$ to the point u_0. We let $A(e,f)$ denote the amalgamation of the pair (e,f).

Theorem 6. *Let $(\mathcal{Q},\mathcal{P})$ be a factorization system on* **Top**. *Then the following are equivalent*:

(i) *it is hereditary*;

(ii) *$\mathcal{P}$ is $*$-closed*;

(iii) *$\mathcal{P}$ is amalgamation-closed (i.e. $f\in\mathcal{P}\Rightarrow A(e,f)\in\mathcal{P}$).*

Proof. Recall that if $(\mathcal{Q},\mathcal{P})$ is a factorization system, and if $pf=pg$ and $fq=gq$ for some p in $\mathcal{P}$, q in $\mathcal{Q}$, then $f=g$. Now suppose (i): we prove (ii). Let $p:X\to Y\in\mathcal{P}$, consider the $(\mathcal{Q},\mathcal{P})$-factorization $X^*\to Z'\to Y^*$ of p^*. Since $Y\subseteq Y^*$ is open, pull this back to a factorization $X\to Z\to Y$ of p. Since $(\mathcal{Q},\mathcal{P})$ is hereditary, this is a $(\mathcal{Q},\mathcal{P})$-factorization, so the map $X\to Z$ is an isomorphism. We have to show $X^*\to Z'$ is an isomorphism; define $h:Z'\to X^*$ by $h|_Z=Z\to X$, $h(Z'\setminus Z)=x_0$. Then $X^*\to Z'\to X^*$ is the identity 1 on X^*; and we have the equations

$$X^*\to Z'\to X^*\to Z'=X^*\xrightarrow{1}X^*\to Z'=X^*\to Z'\xrightarrow{1}Z'$$

and

$$Z'\to X^*\to Z'\to Y^*=Z'\to Y^*=Z'\xrightarrow{1}Z'\to Y^*.$$

Now, $X^*\to Z'$ is in $\mathcal{Q}$, $Z'\to Y^*$ is in $\mathcal{P}$; so $Z'\to X^*\to Z'=1_{Z'}$, and we deduce that $X^*\to Z'$ is an isomorphism; hence $X^*\to Y^*$ is in $\mathcal{P}$, i.e. $\mathcal{P}$ is $*$-closed, hence (ii).

(ii)$\Rightarrow$(iii), from the pull-back description of amalgamations, bearing in mind that $\mathscr{P}$ is closed under pullbacks.

(iii)$\Rightarrow$(i), let $q: X \to Y$ be in $\mathscr{Q}$, $U \subseteq Y$ open. Consider the $(\mathscr{Q}, \mathscr{P})$-factorization $q^{\leftarrow}U \to Z \to U$ of $t: q^{\leftarrow}U \to U$; $t = p'q'$ say. We have to show p' is an isomorphism. Let $p: A \to Y$ be the amalgamation of the pair $(e: U \subseteq Y, p')$. Then there is an induced factorization $q = pa: X \to A \to Y$ of q, with $p \in \mathscr{P}$ and $q' = a|_{a^{\leftarrow}Z}$. Since $q: X \to Y$ is in $\mathscr{Q}$, and $p: A \to Y$ is in $\mathscr{P}$, there is a map $r: Y \to A$ with $pr = 1_Y$, $rq = a$. Let $s = r|_U$: then $st = q'$, $p's = 1_U$. Now $(sp')q' = s(p'q') = st = q' = 1q'$; and $p'(sp') = (p's)p' = 1_U p' = p'1$. But $p' \in \mathscr{P}$, $q' \in \mathscr{Q}$; so $sp' = 1$, hence p' is an isomorphism, and $t: q^{\leftarrow}U \to U$ is in $\mathscr{Q}$. Hence $(\mathscr{Q}, \mathscr{P})$ is hereditary, i.e. (i). □

The argument herein is the basis of the proof [7] that the (improper, proper)-factorization system in **Top** [6] is hereditary; similarly, the (?, light)-factorization system [3] is hereditary. For another example, let $f: X \to Y$ be called an *I-map* if and only if (i) its fibres are T_0-spaces, and (ii) if $x \in$ open $U \subseteq X$, then for some open nbd V of fx in Y there is a map $f_1: f^{\leftarrow}V \to I$ with $f_1 x = 1$, $f_1 = 0$ outside U, where I denotes the unit interval $[0, 1]$. The class of I-maps is easily shown to be closed under pullbacks and products; less easily, we may show it to be composition-closed, so by [3] it is the right factor of a factorization system on **Top**. By our theorem above, using (ii), this factorization system is hereditary. Note that if Y is Tychonoff and $f: X \to Y$ is a map, then f is an I-map if and only if X is Tychonoff.

There is a simple relationship between the totally epireflective subcategories of **Top** and some of the hereditary factorization systems:

Theorem 7. *Let $\mathscr{C} \subseteq$ **Top**. Then $\mathscr{C}$ is totally epireflective if and only if there is a hereditary factorization system $(\mathscr{Q}, \mathscr{P})$, with $\mathscr{Q} \subseteq$ **Epi**, such that a space X is in $\mathscr{C}$ if and only if the terminal map $t_X: X \to T$ is in $\mathscr{P}$. (T denotes the one-point space.)*

Proof. $\Rightarrow$: Let $\mathscr{Q}$ be the class of hereditarily $\mathscr{C}$-extendable onto maps; a left factor by [2], let $\mathscr{P}$ be the corresponding right factor. Let $X \in \mathscr{C}$, $X \to Y \to T$ the $(\mathscr{Q}, \mathscr{P})$-factorization of t_X. Since $X \to Y$ is $\mathscr{C}$-extendable, X is a retract of Y; since $X \to Y$ is onto, $X \cong Y$. Hence $X \to T$ is in $\mathscr{P}$. Conversely, let X be a space with $\mathscr{C}$-reflection $X \to X_{\mathscr{C}}$, and let $t_X: X \to T$ be in $\mathscr{P}$. Since $\mathscr{C}$ is totally reflective, $X \to X_{\mathscr{C}}$ is in $\mathscr{Q}$, and the diagram

$$\begin{array}{ccc} X & \longrightarrow & X_{\mathscr{C}} \\ \| & & \downarrow \\ X & \longrightarrow & T \end{array}$$

has a diagonal, making X a retract of $X_{\mathscr{C}}$, and so X is in $\mathscr{C}$.

$\Leftarrow$: Conversely, let X be a space, $X \to Y \to T$ the $(\mathscr{Q}, \mathscr{P})$-factorization of t_X. Then $Y \in \mathscr{C}$, and $X \to Y$ is hereditarily uniquely $\mathscr{C}$-extendable; so $\mathscr{C}$ is totally reflective in **Top**. □

Note. The result is well known [14] with omission of 'hereditary' and 'totally'.

What hereditary factorization systems are there in **Top**? More precisely, take for example two subclasses $\mathscr{A}$, $\mathscr{B}$ of **Top**; then the classes $\mathcal{O}_{\mathscr{A}}$, $\mathcal{O}_{\mathscr{B}}$ of $\mathscr{A}$ (resp., $\mathscr{B}$)-extendable onto maps in **Top** are left factors of factorization systems, which coincide if and only if $\mathscr{A}$, $\mathscr{B}$ have the same epireflective hull in **Top** [14]. What about $h(\mathcal{O}_{\mathscr{A}})$, $h(\mathcal{O}_{\mathscr{B}})$? Let $\hat{\mathscr{A}}$ denote the smallest *pp*-closed epireflective subcategory of **Top** containing $\mathscr{A}$: the *totally epireflective hull* of $\mathscr{A}$ in **Top**. Easily, $h(\mathcal{O}_{\mathscr{A}}) = h(\mathcal{O}_{\hat{\mathscr{A}}})$: in fact,

Theorem 8. *$h(\mathcal{O}_{\mathscr{A}}) = h(\mathcal{O}_{\mathscr{B}})$ if and only if $\hat{\mathscr{A}} = \hat{\mathscr{B}}$.*

Proof. We begin by showing $h(\mathcal{O}_{\mathscr{A}}) \subseteq h(\mathcal{O}_{\hat{\mathscr{A}}})$. Let $\mathscr{C}$ be the class of spaces C for which every hereditarily $\mathscr{A}$-extendable onto map is C-extendable. Clearly $\mathscr{A} \subseteq \mathscr{C}$, and $\mathscr{C}$ is subspace and product closed, i.e. is epireflective in **Top**. We show $\mathscr{C}$ is *pp*-closed. Let $f : X \to Y \in h(\mathcal{O}_{\mathscr{A}})$, $P = P(C, U, C')$, with $C, C' \in \mathscr{C}$, $U \subseteq C$ open. Consider a map $X \to P$. Then $X \to P \to C$ factors through Y (by hypothesis defining $\mathscr{C}$), i.e. it factors $X \to Y \xrightarrow{g} C$, and by restriction there is a commutative diagram

$$\begin{array}{ccc} f^{\leftarrow}g^{\leftarrow}U & \longrightarrow & g^{\leftarrow}U \\ \downarrow & & \downarrow \\ U \times C' & \longrightarrow & U \end{array}$$

But also, $f^{\leftarrow}g^{\leftarrow}U \to g^{\leftarrow}U$ is hereditarily $\mathscr{C}$-extendable, so we can complete the diagram

$$\begin{array}{ccc} f^{\leftarrow}g^{\leftarrow}U & \longrightarrow & g^{\leftarrow}U \\ \downarrow & & \vdots \\ U \times C' & \longrightarrow & C' \end{array}$$

hence there is an induced map $g^{\leftarrow}U \to U \times C'$ in each of these diagrams. But then the maps $g : Y \to C$, $g^{\leftarrow}U \to U \times C'$ induce a map $Y \to P$, easily seen to be a lifting of $X \to P$ through f. So $\mathscr{P} \in \mathscr{C}$, and $\mathscr{C}$ is *pp*-closed.

The rest is easy: from $h(\mathcal{O}_{\mathscr{A}}) = h(\mathcal{O}_{\hat{\mathscr{A}}})$, we get $\hat{\mathscr{A}} = \hat{\mathscr{B}} \Rightarrow h(\mathcal{O}_{\mathscr{A}}) = h(\mathcal{O}_{\mathscr{B}})$; conversely, if $\mathscr{A} = \hat{\mathscr{A}}$, $\mathscr{B} = \hat{\mathscr{B}}$, $h(\mathcal{O}_{\mathscr{A}}) = h(\mathcal{O}_{\mathscr{B}})$ and $A \in \mathscr{A} \backslash \mathscr{B}$, we show the $\mathscr{B}$-reflection map for A is hereditarily $\mathscr{B}$-extendable, therefore in $h(\mathcal{O}_{\mathscr{A}})$, so is $\mathscr{A}$-extendable; so A is a retract of its $\mathscr{B}$-reflection, therefore in $\mathscr{B}$ after all. $\square$

It follows that **Top** has a plentiful supply of hereditary factorization systems: in fact, there is a bijection between the totally epireflective subcategories of **Top** and the factorization systems $(\mathscr{Q}, \mathscr{P})$ on **Top** for which $\mathscr{Q}$ is the class of hereditarily

$\mathcal{A}$-extendable epimorphisms for some subcategory $\mathcal{A}$ of **Top**. There is a similar result for $h(\mathcal{Q}_{\mathcal{A}})$, the hereditarily $\mathcal{A}$-extendable quotients, using totally quotient-reflective hulls.

We remarked earlier that our factorization systems may have non-epimorphic left factors; let us therefore find some that are also hereditary. First, recall the standard definition of a separated map [6], and note that these maps form a right factor, as in [3]. Let {*joint* maps} be the corresponding left factor.

Lemma. *(joint maps, separated maps) is a hereditary factorization system on* **Top**.

Proof. s separated implies s^* is separated. □

Theorem 9. *Let $\mathcal{Q}$ be a class of dense maps in* **Top**, *so that all joint maps are in $\mathcal{Q}$; and suppose $\mathcal{Q}$ is composition-closed, pushout-closed, and coproduct-closed. Then $\mathcal{Q}$ is a left factor.*

Proof. From the pushout- and coproduct-closure, $\mathcal{Q}$ is closed under multiple pushouts. Let $f: X \to Y$, and consider the class of factorizations $f = s_i q_i$ with $q_i \in \mathcal{Q}$, s_i a separated map. For example, the (joint, separated)-factorization, so the class is non-empty; as in [6], since maps in $\mathcal{Q}$ are dense, there is a representative subset indexed by a set I. Let q_0 be the co-intersection of $(q_i\colon i \in I)$, and t_0 the induced map with $f = t_0 q_0$. Apply the (joint, separated)-factorization to t_0, obtaining $t_0 = pj$; then $q = jq_0$ is in $\mathcal{Q}$, and $f = pq$. Now, it is straightforward to check that p satisfies the usual diagonal condition w.r.t. $\mathcal{Q}$. □

Corollary 1. *Let $\mathcal{A} \subseteq$ **Haus**; then $\mathcal{D}_{\mathcal{A}} = \{$dense, $\mathcal{A}$-extendable maps$\}$ is a left factor.*

Proof. All joint maps are $\mathcal{A}$-extendable and dense (in fact, quotient); and $\mathcal{D}_{\mathcal{A}}$ is composition-closed, pushout-closed, and coproduct-closed. □

Note. This is not quite the usual factorization system in **Haus**, having as left factor the class of dense, $\mathcal{A}$-extendable maps of Hausdorff spaces; its left factor includes maps of non-Hausdorff spaces.

Corollary 2. *Let $\mathcal{A} \subseteq$ **Haus**; then $h(\mathcal{D}_{\mathcal{A}})$, the class of hereditarily $\mathcal{A}$-extendable dense maps, is a left factor.*

Proof. Like Corollary 1, bearing in mind that joint maps are hereditarily $\mathcal{A}$-extendable and that pushouts in **Top** are preserved [2] by pullback along open inclusions. □

As in Theorem 7, we could show that for $\mathcal{A}, \mathcal{B} \subseteq$ **Haus**, we have $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\mathcal{B}}$ if and only if $\mathcal{A}, \mathcal{B}$ have the same epireflective hull in **Haus**; and $h(\mathcal{D}_{\mathcal{A}}) = h(\mathcal{D}_{\mathcal{B}})$ if and only if $\mathcal{A}, \mathcal{B}$ have the same totally epireflective hull in **Haus**.

In place of **Haus**, we can use $\mathbf{Top}_0$; replace the separated maps by the T_0-fibred ones, the joint maps by the indiscrete fibred quotients, dense maps by the b-dense [25] (= very dense [10]) ones: then there is a similar proof of

Theorem 10. *Let $\mathcal{Q}$ be a class of very dense maps in* **Top**, *containing all the indiscrete-fibred quotients, and closed under composition, pushouts and coproducts. Then it is a left factor.* □

We now have a multidimensional array of factorization systems in **Top**. Herrlich et al. [14], and Melton [20], showed that almost none of the left factors $\mathcal{C}_{\mathcal{A}}$, $\mathcal{Q}_{\mathcal{A}}$ are in hereditary factorization systems; similar results are true for $\mathcal{D}_{\mathcal{A}}$ and for $\mathcal{V}_{\mathcal{A}}$ (the very dense $\mathcal{A}$-extendable maps), where $\mathcal{A} \subseteq$ **Haus** (resp., $\mathbf{Top}_0$). But it is difficult to identify the corresponding right factors. Here are two new cases where we can:

Theorem 11. *Let $\mathcal{T}$ = Tychonoff spaces, $\mathcal{K}$ =* **CompHaus**; *then*
(i) *$(h(\mathcal{C}_{\mathcal{T}}), \{I\text{-maps}\})$ is a factorization system on* **Top**;
(ii) *$(h(\mathcal{D}_{\mathcal{K}}), \{\text{proper } I\text{-maps}\})$ is a factorization system on* **Top**.

Proof. First, it is easy to check that a surjective, hereditarily I-extendable I-map is a homeomorphism. Second, the I-maps form a right factor, as noted above, and the maps in the left factor are all onto, I-extendable, and hence Tychonoff-extendable; in fact, hereditarily so, by Theorem 6. Hence (i). Similarly for (ii). □

Corollary. *Not every map in $h(\mathcal{D}_{\mathcal{K}})$ is improper* [6].

Proof. By [11], there is a proper map with a Tychonoff codomain, but non-Tychonoff domain; such a map cannot be an I-map. □

Finally, our main application of the idea of 'total reflectivity':

Theorem 12. *Let $\mathcal{C}$ be a totally reflective subcategory of* **Top**. *Suppose also*
(i) *$\mathcal{C}$ is co-complete, co-well-powered, and open-hereditary;*
(ii) *$\mathcal{Q}$, a subclass of* **Epi**$(\mathcal{C})$, *is a left factor in $\mathcal{C}$.*
Then $h(\mathcal{Q})$ is also a left factor in $\mathcal{C}$.

Proof. Easily, $h(\mathcal{Q})$ is a class of epimorphisms of $\mathcal{C}$, which is composition-closed and contains all isomorphisms. Consider a pushout in $\mathcal{C}$, whose top arrow is in $h(\mathcal{Q})$; let U be an open subset of the codomain of the bottom arrow. Pulling back along U, we get (by (i)) a diagram in $\mathcal{C}$; by Theorem 5, this is a pushout in $\mathcal{C}$; the top arrow is in $\mathcal{Q}$, which is pushout-closed (being a left factor), so the bottom arrow is also in $\mathcal{Q}$. Hence $h(\mathcal{Q})$ is pushout-closed. Similar argument, again using the total reflectivity of $\mathcal{C}$ and the coproduct-closure of $\mathcal{Q}$, shows that $h(\mathcal{Q})$ is coproduct-closed, hence closed under multi-pushouts. By Lemma 16 of [24], $h(\mathcal{Q})$ is a left factor in $\mathcal{C}$. □

Corollary 1. [21]. *Let* $\mathcal{Q} \subseteq \mathbf{Epi}(\mathbf{Top}_0)$ *be a left factor in* $\mathbf{Top}_0$. *Then* $h(\mathcal{Q})$ *is also a left factor in* $\mathbf{Top}_0$.

Corollary 2. *Let* $\mathcal{Q} \subseteq \mathbf{Epi}(\mathbf{Haus})$ *be a left factor in* **Haus**. *Then* $h(\mathcal{Q})$ *is also a left factor in* **Haus**.

Corollary 3. *The same, with* **Haus** *replaced by* **Sober**.

This last result now permits the construction of hereditary factorization systems in **Sober**; this should give some insight into the factorizations in the categories **Loc** of locales or **Topoi** of toposes [16].

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