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Discrete Mathematics 268 (2003) 207–216
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Composite planar coverings of graphs

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Received 20 November 2000; received in revised form 24 September 2001; accepted 26 November 2001

Abstract

We shall prove that a connected graph G is projective-planar if and only if it has a $2n$-fold planar connected covering obtained as a composition of an n-fold covering and a double covering for some $n \geqslant 1$ and show that every planar regular covering of a nonplanar graph is such a composite covering.

Keywords: Covering of graphs; Projective-planar graphs; Planar cover conjecture

0. Introduction

Our graphs are simple and finite. A graph $\tilde{G}$ is called an (*n-fold*) *covering* of a graph G with a *projection* $p: \tilde{G} \to G$ if there is an n-to-one surjection $p: V(\tilde{G}) \to V(G)$ which sends the neighbors of each vertex $v \in V(\tilde{G})$ bijectively to those of $p(v)$. In particular, if there is a subgroup A in the automorphism group $\mathrm{Aut}(\tilde{G})$ such that $p(u) = p(v)$ whenever $\tau(u) = v$ for some $\tau \in A$, then $\tilde{G}$ is called a *regular covering*. This group A is called the *covering transformation group* of $\tilde{G}$. It is easy to see that a 2-fold (or *double*) covering is necessarily a regular one.

A graph is said to be *projective-planar* if it can be embedded in the projective plane. Negami [10] has discussed the relationship between planar double coverings and embeddings of graphs in the projective plane, and established the following characterization of projective-planar graphs:

Theorem 1 (Negami [10]). *A connected graph is projective-planar if and only if it has a planar double covering.*

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0012-365X/03/$ - see front matter
PII: S0012-365X(02)00689-1

Furthermore, he has proved the following theorem, which extends Theorem 1, analyzing the connectivity and group actions of regular coverings.

Theorem 2 (Negami [11]). *A connected graph is projective-planar if and only if it has a planar regular covering.*

These theorems motivated him to propose the following conjecture. This is called "the 1–2–∞ conjecture" or "Negami's planar cover conjecture":

Conjecture 1 (Negami [11]). A connected graph is projective-planar if and only if it has a planar covering.

There have been many papers on studies around this conjecture, but the sufficiency is still open. At present, we have the following theorem, combining the results in [2,4,7,8,11,12]. Note that $K_{1,2,2,2}$ is isomorphic to the graph obtained from the octahedron by adding an extra vertex and edges joining it to all vertices of the octahedron.

Theorem 3 (Archdeacon [2]; Fellows [4]; Hliněný [7] and Negami [11,12]). *If* $K_{1,2,2,2}$ *has no planar covering, then Conjecture* 1 *is true.*

In this paper, we shall present a new aspect of coverings of graphs. An n-fold covering or a covering projection $p:\tilde{G}\to G$ is said to be (n_1,n_2)-*composite* if there are an n_1-fold covering $p_1:\tilde{G}\to G'$ and an n_2-fold covering $p_2:G'\to G$ with $p=p_2p_1$, and hence $n=n_1n_2$. In addition, if $n_1>1$ and $n_2>1$, then it is said to be *composite*. We often say that p or $\tilde{G}$ *factors through* G'.

The following two theorems are our main results in this paper.

Theorem 4. *A connected graph G is projective-planar if and only if it has an* $(n,2)$-*composite planar connected covering for some* $n\geqslant 1$.

Theorem 5. *Every planar connected regular covering of a nonplanar connected graph is* $(n,2)$-*composite for some* $n\geqslant 1$

Theorem 4 will be proved in Section 2 by graph minor arguments while Theorem 5 will be proved in Section 3, related to "faithful embeddings". The latter implies that the former extends Theorem 2. Since it is easy to construct an $(n,2)$-composite planar covering which is not regular. Theorem 4 is strictly stronger than Theorem 2.

Archdeacon and Richter [3] have shown that if there is an m-fold planar covering of a nonplanar graph, then m is even. So we have an over-expected question; is any $2n$-fold planar covering of a nonplanar graph $(n,2)$-composite?[1] The positive answer to this would imply Conjecture 1. In Section 1, we shall show a connection between

[1] Our subsequent work has already shown that the answer to this question is negative.

the different notions of coverings independently developed in topological graph theory and in topology to give a certain strategy to attack this question and also to clarify the notion of composite coverings.

1. Permutation voltage and subgraphs of π_1

In topological graph theory, the notion of "voltage graphs" has been developed to control coverings of graphs (see [6]). On the other hand, there is a general theory of "covering spaces" in topology. They are usually discussed in different fields, but they are equivalent to each other as far as we deal only with graphs. In this section, we shall show how they are related and discuss conditions for a covering of a graph to be composite.

First, we shall sketch a theory of covering spaces in topology for the special case both spaces are graphs. Let G and $\tilde{G}$ be graphs, regarded as topological spaces, assuming that G is connected. A *covering* $p: \tilde{G} \to G$ is a surjective continuous map with $p(V(\tilde{G})) = V(G)$ which induces a local homeomorphism at each point. Then this map p naturally induces a homomorphism $p_{\#}: \pi_1(\tilde{G}) \to \pi_1(G)$ between their fundamental groups.

The most important fact is that this homomorphism $p_{\#}$ is injective and that $\tilde{G}$ *corresponds to* (or *is associated with*) a subgroup $H = p_{\#}(\pi_1(\tilde{G}))$ in $\pi_1(G)$. The *fold number* (or the *covering index*) n of G coincides with the index of H in $\pi_1(G)$; $n = (\pi_1(G): H)$. Conversely, given a subgroup H in $\pi_1(G)$, there is a covering of G which corresponds to H. A closed walk W in G based at a fixed vertex x_0 can be lifted to a closed walk in $\tilde{G}$ if and only if the homotopy class $[W]$ belongs to H. This criterion suggests how to construct $\tilde{G}$ with H. (A covering $p: \tilde{G} \to G$ is regular if and only if H is normal in $\pi_1(G)$. In this case, the covering transformation group A is isomorphic to $\pi_1(G)/H$.)

Let $p_i: \tilde{G}_i \to G$ $(i = 1, 2)$ be two coverings of G associated with subgroups H_i in $\pi_1(G)$. They are said to be *equivalent* to each other if there is a homeomorphism $h: \tilde{G}_1 \to \tilde{G}_2$ with $p_1 = p_2 h$. According to the classification theorem of covering spaces, $\tilde{G}_1$ is equivalent to $\tilde{G}_2$ if and only if H_1 is *conjugate* to H_2 in $\pi_1(G)$, that is, there is an element $g \in \pi_1(G)$ with $gH_1g^{-1} = H_2$. This conjugation with g just corresponds to the re-choice of a base point for $\pi_1(\tilde{G}_i)$.

Considering the relationship between subgroup containment and composition of coverings, we can conclude easily that:

Lemma 6. *An n-fold covering $p: \tilde{G} \to G$ associated with a subgroup H in $\pi_1(G)$ is (n_1, n_2)-composite if and only if there is a subgroup H' of index n_2 in $\pi_1(G)$ which contains H as a subgroup of index n_1.*

Now we shall review a *permutation voltage graph*, which gives us a concrete way to construct coverings of a graph. Let G be a connected graph and let $\vec{E}(G)$ denote the set of directed edges uv and vu for $uv \in E(G)$. Let S_n denote the nth symmetry group. A *permutation voltage* (or a *voltage* simply here) is any assignment $\sigma: \vec{E}(G) \to S_n$

such that $\sigma(vu)=(\sigma(uv))^{-1}$ in S_n. Put $\sigma_{uv}=\sigma(uv)$ and call it a *voltage* of an edge uv. (Note that an ordinary *voltage graph* exhibits a regular covering and does not work for irregular coverings.)

Given a permutation voltage $\sigma:\vec{E}(G)\to S_n$, we can construct an n-fold covering $p:\tilde{G}\to G$, as follows. Let $N=\{1,\dots,n\}$ and put $V(\tilde{G})=V(G)\times N$. Join two vertices (u,i) and (v,j) with an edge whenever $uv\in\vec{E}(G)$ and $\sigma_{uv}(i)=j$. Then the projection $p:V(\tilde{G})\to V(G)$ can be defined by $p((u,i))=u$. This covering is often called the covering *derived from* σ.

To modify a permutation voltage to be more algebraic, we extend it for closed walks based at a fixed vertex x_0. Let $W=x_0u_1\dots u_mx_0$ be a closed walk in G and define its voltage as the product $\sigma_W=\sigma_{x_0u_1}\sigma_{u_1u_2}\cdots\sigma_{u_mx_0}$ in S_n. Since two homotopic closed walks based at x_0 have the same voltage, this defines a homomorphism $\sigma:\pi_1(G)\to S_n$. We call this σ the *permutation voltage* for $\tilde{G}$.

Here, we shall show the relationship between the permutation voltage σ and the subgroup H for a given covering $p:\tilde{G}\to G$. To recognize it, we should consider what $\sigma(H)$ is. Let S_X denote the group of all permutations over a set X. For example, $S_n=S_{\{1,\dots,n\}}=S_N$ in particular. Put $p^{-1}(x_0)=\{x_1,\dots,x_n\}$ and choose one of them, say x_1. Recall that $\pi_1(\tilde{G})$ is the group of closed walks based at x_1 and it projects bijectively to H. This implies that a closed walk W based at x_0 belongs to H if and only if $\sigma_W(1)=1\in N$, and hence we have:

Lemma 7. *H is conjugate to $\sigma^{-1}(S_{N-\{1\}})$ in $\pi_1(G)$.*

To define a permutation voltage σ from a given subgroup H in $\pi_1(G)$, we shall consider the coset decomposition of $\pi_1(G)$ for H:

$$\pi_1(G)=H\cup g_2H\cup\cdots\cup g_nH,$$

where $g_1,\dots,g_n\in\pi_1(G)$ are representatives of these cosets with $g_1=1$. Let g be any element of $\pi_1(G)$. Then gg_iH must be one of g_iH's, say g_jH. Define $\sigma_g:N\to N$ by this correspondence $\sigma_g(i)=j$. This σ_g must be a permutation over N and can be regarded as an element in $S_n=S_N$. Thus, we have a homomorphism $\sigma:\pi_1(G)\to S_n$ with $\sigma(g)=\sigma_g$. This is nothing but our permutation voltage and we have for this σ so defined:

$$H=\sigma^{-1}(S_{N-\{1\}}),\quad \sigma(H)=\sigma(\pi_1(G))\cap S_{N-\{1\}}.$$

The permutation voltage σ over $\vec{E}(G)$ can be defined as follows. Let T be a spanning tree in G with a root x_0. Then $p^{-1}(T)$ consists of a disjoint union of trees, isomorphic to T. Enumerate them as $T_1,\dots,T_n$ with $x_i\in V(T_i)$. First, we set $\sigma(uv)=\mathrm{id}_N$ for each edge $uv\in E(T)$, where id_N stands for the identity in S_n. Let uv be any edge not in $E(T)$ and put $p^{-1}(u)=\{u_1,\dots,u_n\}$ and $p^{-1}(v)=\{v_1,\dots,v_n\}$ with $u_i,v_i\in V(T_i)$. Corresponding to uv, there are n edges in $\tilde{G}$ which join $\{u_1,\dots,u_n\}$ bijectively to $\{v_1,\dots,v_n\}$. This bijection defines a voltage $\sigma(uv)\in S_n$.

The above formulation will suggest some hint to decide whether or not a given covering $\tilde{G}$ of a graph G is composite. For example, consider the *normalizer* $N(H)$

of H in $\pi_1(G)$, that is, the maximal subgroup in $\pi_1(G)$ which contains H as a normal subgroup. If $N(H)=\pi_1(G)$, then H is normal in $\pi_1(G)$ and hence $\tilde{G}$ is a regular covering. By Theorem 5, especially if it is planar but G is not, then it is composite. If $N(H)$ coincides with neither $\pi_1(G)$ nor H, then $\tilde{G}$ factors through the covering of G associated with $N(H)$ and hence it is composite, again. If $N(H)=H$, then we can say nothing with only such an abstract argument. When can we exclude the third case, assuming the planarity of $\tilde{G}$?

As another hint, consider the subgroup $\sigma(H)$ in S_n, which is a more concrete object than a subgroup in a free group. (Note that $\pi_1(G)$ is a free group of rank $\beta(G)=|E(G)|-|V(G)|+1$ for every connected graph G.) According to our argument after Lemma 7, we have $\ker\sigma\subset H$. This implies that $(\pi_1(G):H)=(\sigma(\pi_1(G)):\sigma(H))=n$. Thus, we have:

Lemma 8. *An n-fold covering $\tilde{G}$ associated with a subgraph H in $\pi_1(G)$ is composite if and only if there is a subgraph H' in S_n such that $\sigma(H)\subsetneqq H'\subsetneqq\sigma(\pi_1(G))$.*

2. Projective-planarity with coverings

In this section, we shall prove Theorem 4. As well as for Conjecture 1, we need to analyze the coverings of $K_{1,2,2,2}$. First we shall prepare the following lemma to decide whether or not $K_{1,2,2,2}$ has an $(n,2)$-composite planar covering. We can find similar lemmas in [1,12]. Since our proof proceeds similarly, we shall only sketch it. Note that the same conditions in the lemma imply that G is not projective-planar.

Lemma 9. *If a connected graph G satisfies the following three conditions* (i), (ii) *and* (iii), *then G has no planar covering*:

(i) *There exist two disjoint subgraphs F_1 and F_2 of G each of which is isomorphic to either K_4 or $K_{2,3}$.*
(ii) *Each vertex of F_i is adjacent to a vertex in $G-V(F_i)$ for $i=1,2$.*
(iii) *Both $G-V(F_1)$ and $G-V(F_2)$ are connected.*

Proof. Suppose that G has a planar covering $p:\tilde{G}\to G$ and embed it on the plane. Then we can choose a component F of either $p^{-1}(F_1)$ or $p^{-1}(F_2)$, say the former, so that every inner face of F contains no component of $p^{-1}(F_2)$. This implies that each vertex of F is adjacent to a vertex of $\tilde{G}-V(F)$ which lies in the outer face of F, and hence F would be outer planar. However, this is impossible; it is easy to see that any covering of K_4 and of $K_{2,3}$ is not outer planar. □

The following lemma implies that $K_{1,2,2,2}$ has no $(n,2)$-composite planar connected covering for any $n\geqslant 1$.

Lemma 10. *Every connected double covering of $K_{1,2,2,2}$ has no planar covering.*

Proof. The graph $K_{1,2,2,2}$ can be regarded as the join of a graph T isomorphic to $K_{2,2,2}$ with an extra vertex x. Let $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ be the six vertices of T labeled so that two vertices are adjacent only when they have different alphabets. Then there are eight triangles $a_i b_j c_k$ $(i, j, k \in \{1, 2\})$ and T can be embedded on the sphere so that it forms the octahedron with faces $a_i b_j c_k$.

Consider subgraphs in $K_{1,2,2,2}$ isomorphic to either K_4 or $K_{2,3}$ and categorize them into the following three:

1. Each of the subgraphs induced by $\{a_i, b_j, c_k, x\}$ is isomorphic to K_4.
2. The union of any cycle $u_1u_2u_3u_4$ of length 4 in T with a path u_1xu_3 forms a subgraph isomorphic to $K_{2,3}$.
3. The union of three paths $a_1b_1a_2$, $a_1b_2a_2$, $a_1c_ka_2$ forms a subgraph isomorphic to $K_{2,3}$. Any permutation over $\{a, b, c\}$ generates this type of a subgraph.

Let $p: \tilde{K} \to K_{1,2,2,2}$ be a connected double covering and let F be a subgraph in $K_{1,2,2,2}$ of one of the above three types. Suppose that F can be lifted isomorphically to $\tilde{K}$, that is, $p^{-1}(F)$ consists of two components, say F_1 and F_2, and each of them is isomorphic to F. It is clear that conditions (i) and (ii) in Lemma 9 hold for these F_1 and F_2. We shall examine the three cases in turn to show that condition (iii) of Lemma 9 holds.

First suppose that F is of the first type. Let J be any component of $p^{-1}(T - V(F))$. If J is joined to only one of F_1 and F_2 with edges, say F_1, then J must be a cycle of length 3 obtained as a lift of $a_{3-i}b_{3-j}c_{3-k}$ and $J \cup F_1$ induces one component of $\tilde{K}$, isomorphic to $K_{1,2,2,2}$. This implies that $\tilde{K}$ is disconnected, which is contrary to our assumption of $\tilde{K}$. Otherwise, all components of $p^{-1}(T - V(F))$ are joined to F_2 with edges and they form a connected subgraph $\tilde{K} - V(F_1)$ with F_2. Thus, condition (iii) holds in this case and hence $\tilde{K}$ has no planar covering by Lemma 9.

Suppose that F is of the second type. Similarly to the previous case, let J be any component of $p^{-1}(T - V(F))$ and suppose that J is joined to only F_1 with edges. First assume that F contains the cycle $C = a_1b_1a_2c_1$ as $u_1u_2u_3u_4$ and the path a_1xa_2 as u_1xu_3. Then J is a lift of an edge b_2c_2 and the both ends of J are adjacent to all vertices of the lift $\tilde{C}$ of C in F_1. Let $\tilde{a}_1$ and $\tilde{x}$ be the lifts of a_1 and x in F_1, respectively. Then $J \cup \{\tilde{a}_1, \tilde{x}\}$ induces a subgraph in $\tilde{K}$ which projects isomorphically to a subgraph of the first type. Thus, we can assume that J is joined to F_2 in this case and hence $\tilde{K} - V(F_1)$ is connected. This implies that $\tilde{K}$ has no planar covering by Lemma 9.

In the remaining cases with F of the second type, we can find those subgraphs already discussed in the previous cases, as follows. If F consists of the above C and the path b_1xc_1, then either the subgraph induced by F_1 contains a subgraph isomorphic to K_4, or there is a path in $\tilde{K}$ projecting to a_1xa_2. On the other hand, if F consists of the cycle $C' = a_1b_1a_2b_2$ and the path a_1xa_2, then J consists of a single vertex which projects to c_1 or c_2, say c_1, and the vertex is adjacent to all vertices of the lift of C'. In this case, there is a subgraph in $\tilde{K}$ which projects to $C \cup a_1xa_2$.

Finally, suppose that F is of the third type. Then $p^{-1}(T - V(F))$ has two components and each of which consists only of an edge projecting to $c_{3-k}x$. If one of the components is joined only to F_1 with edges, then we can find a subgraph in $\tilde{K}$ isomorphic to K_4 and conclude that $\tilde{K}$ has no planar covering, as well as in the previous case. Otherwise, condition (iii) in Lemma 9 holds and $\tilde{K}$ has no planar covering, again.

To complete the proof, it suffices to show that every double covering of $K_{1,2,2,2}$ has a subgraph isomorphic to K_4 or $K_{2,3}$ which can be lifted isomorphically. To describe a possible double covering, we use a *voltage assignment* to $E(K_{1,2,2,2})$ with $\boldsymbol{Z}_2 = \{\bar{0}, \bar{1}\}$. This is equivalent to a permutation voltage $\sigma : \vec{E}(K_{1,2,2,2}) \to S_2$ since S_2 is isomorphic to the additive group $\boldsymbol{Z}_2$. We may assume that each edge incident to x has voltage $\bar{0}$ since they form a spanning tree of $K_{1,2,2,2}$ and consider only the voltages over edges of T. The *voltage* of a path or a cycle is defined as the summation of the voltages along it. Any path can be lifted isomorphically while a cycle can be lifted as a cycle of the same length if and only if its voltage is $\bar{0}$.

Consider the five paths of length 2 between a_1 and a_2 in $K_{1,2,2,2}$, given as $a_1b_ia_2$, $a_1c_ia_2$ $(i = 1,2)$ and a_1xa_2. Then at least three of them must get the same voltage, either all $\bar{0}$ or all $\bar{1}$. In the former case, one of the three paths may be assumed to be a_1xa_2 and they form a subgraph of the second type, which is isomorphic to $K_{2,3}$. Since all cycles in the subgraph has voltage $\bar{0}$, it can be lifted isomorphically to the double covering of $K_{1,2,2,2}$ derived by the voltage. In the latter case, the three paths form a subgraph of the third type and all cycles in the subgraph has voltage $\bar{1} + \bar{1} = \bar{0}$. Thus, it can be lifted isomorphically, too. □

The argument in our proof of Theorem 4 proceeds very similarly to the proof of Theorem 3 and will be applied to similar theorems in further studies. So we shall prepare an abstract formulation for its general use, as follows.

A graph H is called a *minor* of a graph G if H can be obtained from G by contracting and deleting some edges. Let G_Y be a graph with a vertex v of degree 3 and let v_1, v_2 and v_3 be the three neighbors of v. A *Y-Δ transformation* is to add three new edges v_1v_2, v_2v_3 and v_3v_1 after deleting v. Let G_Δ denote a graph obtained from G_Y by a Y-Δ transformation. Let $\mathscr{P}$ be a property or a class of connected graphs closed under taking connected minors and under Y-Δ transformations. (We say that G *has the property* $\mathscr{P}$ if G belongs to $\mathscr{P}$.) In addition, if every graph belonging to $\mathscr{P}$ has a planar covering, then $\mathscr{P}$ is said to be *projective-planar-like*.

Lemma 11. *Every connected graph with a projective-planar-like property $\mathscr{P}$ is projective-planar if $K_{1,2,2,2}$ does not have the property $\mathscr{P}$.*

Proof. We must show that if a connected graph G is not projective-planar, then G does not have the property $\mathscr{P}$. Since $\mathscr{P}$ is closed under taking connected minors, it suffices to show that every minor-minimal graph among those graphs that are not projective-planar does not have the property $\mathscr{P}$. Such minor-minimal graphs have been already identified in [1,5]; they are 35 in number and three of them are disconnected. We do not need those disconnected ones.

Furthermore, it has been known that the 32 minor-minimal graphs can be classified into 11 families, up to Y-Δ transformations, and that every member in 10 families not including $K_{1,2,2,2}$ does not have any planar covering. Each member in the exceptional family can be deformed into $K_{1,2,2,2}$ by Y-Δ transformations. Thus, the last condition of a projective-planar-like property implies that those minor-minimal graphs in the 10 families do not have the property $\mathscr{P}$. Since $\mathscr{P}$ is closed under Y-Δ transformations, it remains to show that $K_{1,2,2,2}$ does not have the property $\mathscr{P}$. The assumption of the lemma however guarantees this. This completes the proof. □

For example, the property of having a planar covering is a trivial projective-planar-like property. Thus, if we take it as $\mathscr{P}$, then Theorem 3 follows from Lemma 11. Theorem 4 is just a corollary of this lemma, too.

Proof of Theorem 4. The necessity follows from Theorem 1 with $n=1$, so it suffices to show the sufficiency. Let $\mathscr{P}_n$ be the class of all connected graphs that have $(n,2)$-composite planar connected coverings. It is clear that $\mathscr{P}_n$ is projective-planar-like. By Lemma 10, $K_{1,2,2,2}$ does not have the property $\mathscr{P}_n$ for any $n \geqslant 1$. Thus, the theorem follows immediately from Lemma 11. □

3. Regular planar coverings

Let G be a graph and F^2 a closed surface. An embedding $f: G \to F^2$ is said to be *faithful* if there is a homeomorphism $h: F^2 \to F^2$ with $hf = f\tau$ for any automorphism $\tau: G \to G$. The notion of faithful embeddings was first introduced in [9] and the author pointed out there that any embedding of a 3-connected planar graph in the sphere is faithful, which is just a consequence of the uniqueness of its dual, proved by Whitney [13]. This fact has played an essential role in the proof of Theorem 2 in [11].

Furthermore, the author has established the following theorem on the connectivity of regular coverings in [11]. Let $G_0, \dots, G_{n-1}$ be n disjoint copies of a connected graph G' and choose two vertices v' and v'' of G'. Let v'_i and v''_i be the vertices of G_i corresponding to v' and v'', respectively. Identify v'_i with v''_{i+1} for $i \equiv 0, \dots, n-1 \bmod n$. The resulting graph $\tilde{G} = G_0 \cup \cdots \cup G_{n-1}$ is called a *cyclic chain*. Clearly, $\tilde{G}$ is a regular covering of the graph G obtained from G' with v' and v'' identified. The cyclic group $\boldsymbol{Z}_n$ of order n acts on $\tilde{G}$ so that it shifts G_i to G_{i+1}.

Theorem 12 (Negami [11]). *Every connected regular covering of a* 3*-connected graph is either* 3*-connected or a cyclic chain.*

Following carefully the whole arguments in [11], including the proofs of the above and Theorem 2, we can conclude another useful fact for graphs with lower connectivity, as shown below. Here we shall split the fact into two lemmas, purely combinatorial and topological, for the convenience of studies in future.

In general, if a connected graph G splits into two connected subgraphs G' and F such that $G' \cap F = \{u_1, \dots, u_k\} \subset V(G)$ and that both G' and F contain vertices other than $u_1, \dots, u_k$, then we call $(F, \{u_1, \dots, u_k\})$ a *k-fragment* with a *k-cut* $\{u_1, \dots, u_k\}$. Let $\tilde{G}$ be a regular covering of a connected graph G with the covering transformation group A. A k-fragment (F, U) is said to be *equivariant* under A if either $F = \tau(F)$ or $F \cap \tau(F) \subset U$ for any element $\tau \in A$.

Lemma 13. *Let $p: \tilde{G} \to G$ be a connected regular covering of a connected graph G with a covering transformation group A. If $\tilde{G}$ is neither 3-connected nor a cyclic chain, then there is either a 1-fragment $(\tilde{F}, \{u\})$ of $\tilde{G}$ which projects into G isomorphically or a 2-fragment $(\tilde{F}, \{u, v\})$ equivariant under A with one of the following three conditions*:

(i) *$p(u) \neq p(v)$ and $\tilde{F}$ projects to F isomorphically.*
(ii) *$p(u) = p(v)$ and $\tilde{F} - \{u, v\}$ projects to $F - p(u)$ isomorphically.*
(iii) *$p(u) = p(v)$ and $\tilde{F}$ is a 2-fold covering of F.*

where we set $F = p(\tilde{F})$.

Let G_F and $\tilde{G}_F$ be the graphs obtained from G and $\tilde{G}$, respectively, by replacing F with an edge $p(u)p(v)$ and $\tau(\tilde{F})$ with an edge $\tau(u)\tau(v)$ for each $\tau \in A$ in Case (i). Put $G_F = G - V(F - p(u))$ and $\tilde{G}_F = \tilde{G} - \bigcup_{\tau \in A} \tau(V(\tilde{F} - \{u, v\}))$ in the other cases, including the case of a 1-fragment, say Case (iv). It is clear that the projection p induces naturally a regular covering $p_F: \tilde{G}_F \to G_F$ and that its covering transformation group A_F is isomorphic to A.

Lemma 14. *With the same notation as above, if G is nonplanar and $\tilde{G}$ is planar, then G_F is nonplanar and $\tilde{G}_F$ is planar.*

Proof of Theorem 5. Let G be a nonplanar connected graph and $\tilde{G}$ a $2n$-fold planar connected regular covering with a covering transformation group A. First suppose that $\tilde{G}$ is 3-connected. Embed $\tilde{G}$ on the sphere S^2. Since the embedding is faithful, each element τ in A can be regarded as an auto-homeomorphism over S^2. We may assume that the group A acts on S^2 so as to realize the symmetry of $\tilde{G}$. Any fixed point of this action lies in a face of $\tilde{G}$ embedded on S^2.

Let A_0 be the set of orientation-preserving homeomorphisms $\tau \in A$. It is clear that A_0 is a subgroup in A of index at most 2. Consider the quotient space S^2/A_0 by the action of A_0. Then S^2/A_0 must be an orientable surface. By calculation of Euler characteristic, we can conclude that S^2/A_0 is homeomorphic to the sphere and $\tilde{G}/A_0$ is embedded there. (See the proof of Theorem 2 in [11] for the details.) If $A = A_0$, then $G = \tilde{G}/A = \tilde{G}/A_0$ would be embeddable in the sphere, contrary to the nonplanarity of G. Thus, A_0 has index 2 in A. In this case, $\tilde{G}$ is an n-fold covering of $\tilde{G}/A_0$ while $\tilde{G}/A_0$ is a 2-fold covering of G with the covering transformation group $A/A_0 \cong \boldsymbol{Z}_2$. Therefore, $\tilde{G}$ is $(n, 2)$-composite since it factors through $\tilde{G}/A_0$.

Now we shall proceed to the general case, using induction on the order of G. The initial step of induction is the case when $\tilde{G}$ is 3-connected. Suppose that $\tilde{G}$ is not 3-connected. If $\tilde{G}$ were a cyclic chain, then shrinking all parts G_i but one in $\tilde{G}$ embedded on the plane yields a planar embedding of G, contrary to the nonplanarity of G. Thus, there is a 1- or 2-fragment of $\tilde{G}$ equivariant under A, described in Lemma 13. Use the same notation as in the lemma. By the induction hypothesis, $\tilde{G}_F$ is $(n,2)$-composite and there are an n-fold covering $p'_F : \tilde{G}_F \to G'_F$ and a 2-fold covering $p''_F : G'_F \to G_F$ with $p_F = p''_F p'_F$.

In each case of (i) to (iv), we can construct a 2-fold covering $p' : G' \to G$ which $\tilde{G}$ factors through, as follows:

(i) Replace each of two lifts of the edge $p(u)p(v)$ in G'_F with a copy of $(F, \{p(u), p(v)\})$.

(ii) Put $w = p(u) = p(v)$. Then $(\tilde{F}, \{u, v\})$ projects to a 1-fragment $(F, \{w\})$ in G. If $p'_F(u) \neq p'_F(v)$, then attach two copies of $(\tilde{F}, \{u, v\})$, say $(F_1, \{u_1, v_1\})$ and $(F_2, \{u_2, v_2\})$, to G'_F so that $u_1 = v_2 = p'_F(u)$ and $v_1 = u_2 = p'_F(v)$. If $p'_F(u) = p'_F(v)$, then there is another vertex w' in G'_F which projects to w in G_F. Attach two copies of $(F, \{w\})$, say $(F_1, \{w_1\})$ and $(F_2, \{w_2\})$, to G'_F so that $w_1 = p'_F(u)$ and $w_2 = w'$.

(iii) The same argument as in the previous case works formally with the same symbols.

(iv) Let u'_1 and u'_2 be the two vertices of G'_F which project to $p(u)$. Attach two disjoint copies of $(\tilde{F}, \{u\})$, say $(F_1, \{u_1\})$ and $(F_2, \{u_2\})$, to G'_F so that $u_1 = u'_1$ and $u_2 = u'_2$.

Therefore, $\tilde{G}$ is $(n,2)$-composite and the induction completes. □

Acknowledgements

The author would like to express his thanks to one of the referees who gave him a good idea to improve the last half of the proof of Lemma 10.

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