On the integrability of a representation of $\mathfrak{sl}(2, \mathbb{R})$

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Received 15 November 2004; accepted 19 April 2007

Available online 20 July 2007

Communicated by R. Howe

Abstract

The Dunkl operators involve a multiplicity function $k$ as parameter [C.F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc. 311 (1989) 167–183]. For positive real values of this function, we consider on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ a representation $\omega_k$ of $\mathfrak{sl}(2, \mathbb{R})$ defined in terms of the Dunkl–Laplacian operator. By means of a beautiful theorem due to E. Nelson, we prove that $\omega_k$ exponentiates to a unique unitary representation of the universal covering group of $SL(2, \mathbb{R})$. The representation theory is used to derive an identity of Bochner type for the Dunkl transform.

Keywords: Dunkl operators; Integrable representations; Schrödinger model; Dunkl transform; Bochner identity

1. Introduction

The profound role of the representation theory of $SL(2, \mathbb{R})$ in harmonic analysis has received a good deal of attention and the literature is extensive (see e.g. [1,18,20,21,23,26,27,29]).

In a series of papers [3–5], B. Ørsted and the present author showed that there exists an infinitesimal representation $\omega$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$, that can be used to study several topics in the theory of Dunkl operators. This representation can be thought of as an analogue of the classical infinitesimal metaplectic representation of $\mathfrak{sl}(2, \mathbb{R})$. The varied topics investigated are: (1) the Hecke identity for the Dunkl transform, (2) Huygens’ principle for the wave equation for the Dunkl–Laplacian operator, and (3) a restriction theorem of Harish-Chandra type for the Dunkl transform. For the Fourier transform version of the Harish-Chandra type for the Dunkl transform. For the Fourier transform version of the Harish-Chandra type representation, the representation theory is used to derive an identity of Bochner type for the Dunkl transform.

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0022-1236/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jfa.2007.04.022
Chandra’s restriction theorem, we refer to [15]. The constant refrain behind all these melodies is the conjectured integrability of the infinitesimal representation $\omega$ (cf. [3]). In this paper we prove this conjecture. Our proof uses a famous result of E. Nelson [24]. Note that the integrability fact it is not obvious, since in infinite dimensions, the existence of a group representation is not guaranteed from the existence of a Lie algebra representation. We close the paper by using the exponentiated representation to enlarge the list of applications of the representation theory in Dunkl analysis. More precisely, we prove an identity of Bochner type for the Dunkl transform.

Let $\{h, e^+, e^-\}$ denote the standard basis of $\mathfrak{sl}(2, \mathbb{R})$, where

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e^+ = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad e^- = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},$$

so that

$$[e^+, h] = -2e^+, \quad [e^-, h] = 2e^-, \quad [e^+, e^-] = h.$$

Any triple $\{X, Y, Z\}$ of non-zero elements which satisfies (1.1) will be called a TDS (three-dimensional simple) triple.

Let $G \subset O(N)$ be a finite reflection group on $\mathbb{R}^N$ with root system $\mathcal{R}$, and choose a positive subsystem $\mathcal{R}^+$ in $\mathcal{R}$. Let $k : \mathcal{R} \to \mathbb{R}^+$, $\alpha \mapsto k_\alpha$ be a $G$-invariant multiplicity function, and let $\Delta_k$ be the Dunkl–Laplacian operator (see the next section for the definition). Define the modified Euler operator

$$\mathcal{J}_k := \frac{N}{2} + \gamma_k + \sum_{j=1}^N x_j \partial_j,$$

where $\gamma_k$ is a constant depending only on $k$ (see (2.8)). In [16], the author shows that

$$[\Delta_k, \|x\|^2] = 4 \mathcal{J}_k,$$

and therefore the operators

$$\mathcal{E} := i \frac{\|x\|^2}{2}, \quad \mathcal{F} := i \frac{\Delta_k}{2}, \quad \mathcal{H} := \mathcal{J}_k$$

form a TDS triple. Denote by $\omega_k$ the representation of $\mathfrak{sl}(2, \mathbb{R})$ on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ given by

$$\omega_k(h) = \mathcal{H}, \quad \omega_k(e^+) = \mathcal{E}, \quad \omega_k(e^-) = \mathcal{F}.$$

The main problem is to investigate the exponentiability of $\omega_k$.

Let $\vartheta_k$ be the weight function on $\mathbb{R}^N$ defined by $\vartheta_k(x) = \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha}$. By (1.3), the operator $\omega_k(X)$, for all $X \in \mathfrak{sl}(2, \mathbb{R})$, is skew-symmetric in the space $L^2(\mathbb{R}^N, \vartheta_k(x) \, dx)$. This is a consequence of the fact that $\mathcal{J}^*_k = -\mathcal{J}_k$ in $L^2(\mathbb{R}^N, \vartheta_k(x) \, dx)$, and the well-known symmetry property of $\Delta_k$. Moreover, we show that $\omega_k$ satisfies Nelson’s criterion for a skew-symmetric representation of a Lie algebra to be integrable to a unitary representation of the corresponding simply connected Lie group [24]. Thus, $\omega_k$ exponentiates to a unique unitary representation $\Omega_k$. 

of the universal covering \( \widetilde{SL}(2, \mathbb{R}) \) of \( SL(2, \mathbb{R}) \), for every positive real-valued \( k \). In particular, the representation \( \Omega_k \) descends to \( SL(2, \mathbb{R}) \) if and only if \( \gamma_k + \frac{N}{2} \in \mathbb{N} \), and to the metaplectic group \( Mp(2, \mathbb{R}) \) if and only if \( \gamma_k + \frac{N}{2} \in \mathbb{N} \).

As an application, we prove an identity of Bochner type for the Dunkl transform (announced, without a proof, in the expository paper [5]). The identity asserts that, if \( \gamma_k > N/2 \), then the Dunkl transform of the product of \( p \) with a radial function on \( \mathbb{R}^N \) is the product of \( p \) with a Hankel transform of the radial factor. The representation theory was used earlier by R. Howe to give a new proof for the classical Bochner identity, which corresponds to \( k = 0 \) and \( G = O(N) \) (cf. [17,18]). The Bochner identity for the Dunkl transform generalizes the Hecke-type formula proved by Dunkl in [12], and later in [3] using a representation theory approach.

2. Notations and background

Let \( \langle \cdot, \cdot \rangle \) be the standard Euclidean scalar product in \( \mathbb{R}^N \) as well as its bilinear extension to \( \mathbb{C}^N \times \mathbb{C}^N \). For \( x \in \mathbb{R}^N \), denote by \( \|x\| = \langle x, x \rangle^{1/2} \). Let \( S(\mathbb{R}^N) \) be the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^N \) equipped with the usual Fréchet space topology.

For \( \alpha \in \mathbb{R}^N \setminus \{0\} \), let \( r_\alpha \) be the reflection in the hyperplane \( \langle \alpha \rangle \) orthogonal to \( \alpha \)

\[
r_\alpha(x) := x - 2 \frac{\langle \alpha, x \rangle}{\|\alpha\|^2} \alpha, \quad x \in \mathbb{R}^N.
\]

A root system is a finite spanning set \( \mathcal{R} \subset \mathbb{R}^N \) of non-zero vectors such that, for every \( \alpha \in \mathcal{R} \), \( r_\alpha \) preserves \( \mathcal{R} \). We shall always assume that \( \mathcal{R} \) is reduced, i.e. \( \mathcal{R} \cap \mathbb{R}^N = \{ \pm \alpha \} \), for all \( \alpha \in \mathcal{R} \). Each root system can be written as a disjoint union \( \mathcal{R} = \mathcal{R}^+ \cup (-\mathcal{R}^+) \), where \( \mathcal{R}^+ \) and \( (-\mathcal{R}^+) \) are separated by a hyperplane through the origin. The subgroup \( G \subset O(N) \) generated by the reflections \( \{r_\alpha \mid \alpha \in \mathcal{R} \} \) is called the finite reflection group associated with \( \mathcal{R} \). Henceforth, we shall normalize \( \mathcal{R} \) so that \( \langle \alpha, \alpha \rangle = 2 \) for all \( \alpha \in \mathcal{R} \). This simplifies formulas, without loss of generality for our purposes. We refer to [19] for more details on the theory of root systems and reflection groups.

A multiplicity function on \( \mathcal{R} \) is a \( G \)-invariant function \( k : \mathcal{R} \to \mathbb{C} \). Setting \( k_\alpha := k(\alpha) \) for \( \alpha \in \mathcal{R} \), we have \( k_{h\alpha} = k_\alpha \) for all \( h \in G \). The \( \mathbb{C} \)-vector space of multiplicity functions on \( \mathcal{R} \) is denoted by \( \mathcal{K} \). If \( m := \mathfrak{z}[G\text{-orbits in } \mathcal{R}] \), then \( \mathcal{K} \cong \mathbb{C}^m \).

For \( \xi \in \mathbb{C}^N \) and \( k \in \mathcal{K} \), in [10], C. Dunkl defined a family of first order differential-difference operators \( T_\xi(k) \) that play the role of the usual partial differentiation. Dunkl’s operators are defined by

\[
T_\xi(k) f(x) := \partial_\xi f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle}, \quad f \in \mathcal{C}^1(\mathbb{R}^N). \tag{2.1}
\]

Here \( \partial_\xi \) denotes the directional derivative corresponding to \( \xi \). This definition is independent of the choice of the positive subsystem \( \mathcal{R}^+ \). The operators \( T_\xi(k) \) are homogeneous of degree \((-1)\). Moreover, by the \( G \)-invariance of the multiplicity function \( k \), the Dunkl operators satisfy

\[
h \circ T_\xi(k) \circ h^{-1} = T_{h\xi}(k), \quad \forall h \in G, \tag{2.2}
\]
where \( h \cdot f(x) = f(h^{-1} \cdot x) \). Remarkably enough, the Dunkl operators mutually commute, i.e.

\[
T_{\xi}(k)T_{\eta}(k) = T_{\eta}(k)T_{\xi}(k), \quad \forall \xi, \eta \in \mathbb{R}^N.
\]

Further, if \( f \) and \( g \) are in \( C^1(\mathbb{R}^N) \), and at least one of them is \( G \)-invariant, then

\[
T_{\xi}(k)[fg] = gT_{\xi}(k)f + fT_{\xi}(k)g.
\]  

(2.3)

We refer to [10,13] for more details on the theory of Dunkl’s operators.

The counterpart of the usual Laplacian is the Dunkl–Laplacian operator defined by

\[
\Delta_k := \sum_{j=1}^{N} T_{\xi_j}(k)^2,
\]  

(2.4)

where \( \{\xi_1, \ldots, \xi_N\} \) is an arbitrary orthonormal basis of \( (\mathbb{R}^N, \langle \cdot , \cdot \rangle) \). By the normalization \( \langle \alpha, \alpha \rangle = 2 \), we can rewrite \( \Delta_k \) as

\[
\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathcal{R}^+} k_{\alpha} \left\{ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(r_{\alpha}x)}{(\langle \alpha, x \rangle)^2} \right\},
\]

where \( \Delta \) and \( \nabla \) denote the usual Laplacian and gradient operators, respectively (cf. [10]). It follows from (2.2) and (2.4) that \( \Delta_k \) commutes with \( G \),

\[
h \circ \Delta_k \circ h^{-1} = \Delta_k, \quad \forall h \in G.
\]  

(2.5)

**Remark 2.1.** For the \( j \)th basis vector \( \xi_j \), we will use the abbreviation \( T_{\xi_j}(k) = T_j(k) \).

Henceforth, \( \mathcal{X}^+ \) denotes the set of multiplicity functions \( k = (k_{\alpha})_{\alpha \in \mathcal{R}} \) such that \( k_{\alpha} \in \mathbb{R}^+ \) for all \( \alpha \in \mathcal{R} \). For \( k \in \mathcal{X}^+ \), there exists a generalization of the usual exponential kernel \( e^{\langle \cdot, \cdot \rangle} \) by means of the Dunkl system of differential equations.

**Theorem 2.2.** Assume that \( k \in \mathcal{X}^+ \).

(i) (Cf. [11,25].) There exists a unique holomorphic function \( E_k \) on \( \mathbb{C}^N \times \mathbb{C}^N \) characterized by

\[
T_{\xi}(k)E_k(z, w) = \langle \xi, w \rangle E_k(z, w), \quad \forall \xi \in \mathbb{C}^N,
\]

\[
E_k(0, w) = 1.
\]  

(2.6)

Further, the Dunkl kernel \( E_k \) is symmetric in its arguments and \( E_k(hz, w) = E_k(z, h^{-1}w) \) for \( h \in G \) and \( z, w \in \mathbb{C}^N \).

(ii) (Cf. [8].) For \( x \in \mathbb{R}^N \) and \( w \in \mathbb{C}^N \), we have

\[
|E_k(x, w)| \leq \sqrt{|G|} e^{k|x|\|\text{Re}(w)\|}.
\]  

(2.7)
For complex-valued $k$, there is a detailed investigation of (2.6) by Opdam [25]. Theorem 2.2(i) is a weak version of Opdam’s result. For integral-valued multiplicity function $k$, another proof for Theorem 2.2 can be found in [2] by means of a contraction procedure. When $k \equiv 0$, we have $E_0(z, w) = e^{i(z, w)}$ for $z, w \in \mathbb{C}^N$.

Let $\vartheta_k$ be the weight function on $\mathbb{R}^N$ defined by

$$
\vartheta_k(x) = \prod_{\alpha \in \mathbb{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha}, \quad x \in \mathbb{R}^N.
$$

It is $G$-invariant and homogeneous of degree $2\gamma_k$, with the index

$$
\gamma_k := \sum_{\alpha \in \mathbb{R}^+} k_\alpha.
$$

Let $\mathcal{D}_k$ be the weight function on $\mathbb{R}^N$ defined by

$$
\vartheta_k(x) := \prod_{\alpha \in \mathbb{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha}, \quad x \in \mathbb{R}^N.
$$

Let $dx$ be the Lebesgue measure corresponding to $\langle \cdot, \cdot \rangle$, and set $L^p(\mathbb{R}^N, \vartheta_k(x) dx)$ to be the space of $L^p$-integrable functions on $\mathbb{R}^N$ with respect to $\vartheta_k(x) dx$. Following Dunkl [12], we define the Dunkl transform on the space $L^1(\mathbb{R}^N, \vartheta_k(x) dx)$ by

$$
\mathcal{D}_k f(\xi) = c_k^{-1} \int_{\mathbb{R}^N} f(x) E_k(x, -i\xi) \vartheta_k(x) dx, \quad \xi \in \mathbb{R}^N,
$$

where $c_k$ denotes the Mehta-type constant $c_k := \int_{\mathbb{R}^N} e^{-\|x\|^2/2} \vartheta_k(x) dx$. In view of (2.7), the transform $\mathcal{D}_k$ is well defined. Many properties of the Euclidean Fourier transform carry over to the Dunkl transform. In particular:

**Theorem 2.3.** (Cf. [8,12].) If $k \in \mathcal{K}^+$, then:

(i) The Dunkl transform is a homeomorphism of the Schwartz space $S(\mathbb{R}^N)$. Its inverse is given by $\mathcal{D}_k^{-1} f(\xi) = \mathcal{D}_k f(-\xi)$.

(ii) If $f \in L^1(\mathbb{R}^N, \vartheta_k(x) dx) \cap L^2(\mathbb{R}^N, \vartheta_k(x) dx)$, then $\mathcal{D}_k f \in L^2(\mathbb{R}^N, \vartheta_k(x) dx)$ and $
\|\mathcal{D}_k f\|_2 = \|f\|_2$. Further, $\mathcal{D}_k$ extends uniquely from $L^1(\mathbb{R}^N, \vartheta_k(x) dx) \cap L^2(\mathbb{R}^N, \vartheta_k(x) dx)$ to a unitary operator on $L^2(\mathbb{R}^N, \vartheta_k(x) dx)$.

To conclude this section, we mention that the Dunkl operators are anti-symmetric with respect to the weight function $\vartheta_k$ (cf. [10]): if $f \in S(\mathbb{R}^N)$ and $g$ is smooth such that both $g$ and $T_\xi(k)g$ are at most of polynomial growth, then

$$
\int_{\mathbb{R}^N} \left( T_\xi(k)f \right)(x)g(x) \vartheta_k(x) dx = -\int_{\mathbb{R}^N} f(x) \left( T_\xi(k)g \right)(x) \vartheta_k(x) dx. \quad (2.9)
$$

3. The integrability of an $\mathfrak{sl}(2, \mathbb{R})$-representation

Recall the triples $\{h, e^+, e^-\}$ and $\{H, E, F\}$ from the introduction, and the representation $\omega_k$ of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ on $S(\mathbb{R}^N)$ defined by

$$
\omega_k(h) = J_k, \quad \omega_k(e^+) = i \frac{\|x\|^2}{2}, \quad \omega_k(e^-) = i \frac{\Delta_k}{2}. \quad (3.1)
$$
For \( h \in G \), denote by \( \pi(h) \) the “left regular” action of the Coxeter group \( G \) on \( S(\mathbb{R}^N) \),

\[
\pi(h)f(x) = f(h^{-1}x).
\]

Obviously the actions \( \omega_k \) and \( \pi \) on \( S(\mathbb{R}^N) \) commute (recall (1.2) and (2.5)).

To investigate the structure of the representation \( \omega_k \), we now consider a second TDS triple \( \{k, n^+, n^-\} \subset g_C \) defined by

\[
\{k, n^+, n^-\} = c \{h, e^+, e^-\} c^{-1},
\]

where \( c \) is the unitary matrix

\[
\begin{bmatrix}
1/\sqrt{2} & -i/\sqrt{2} \\
-i/\sqrt{2} & 1/\sqrt{2}
\end{bmatrix}.
\]

That is, \( k = i(e^- - e^+) \), \( n^+ = \frac{1}{2}(-ih + e^+ + e^-) \), and \( n^- = \frac{1}{2}(ih + e^+ + e^-) \).

By (3.1) we have

\[
\begin{align*}
\omega_k(k) &= \frac{\|x\|^2 - \Delta_k}{2}, \\
\omega_k(n^+) &= -i \frac{\mathcal{J}_k - \Delta_k/2 - \|x\|^2/2}{2}, \\
\omega_k(n^-) &= i \frac{\mathcal{J}_k + \Delta_k/2 + \|x\|^2/2}{2}.
\end{align*}
\]

The TDS triple \( \{k, n^+, n^-\} \) spans the Lie algebra \( \mathfrak{su}(1, 1) \), which is isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \). Both Lie algebras are real forms of \( \mathfrak{sl}(2, \mathbb{C}) \). It is convenient to use either one or the other suitable form of the two isomorphic Lie algebras according to the problem at hand. Throughout the paper we will denote the operators on the right-hand side of (3.2a), (3.2b), and (3.2c) by \( \tilde{H}, \tilde{E}, \) and \( \tilde{F} \), respectively.

For every polynomial \( p \in \mathcal{P}(\mathbb{R}^N) \), we have

\[
e^{\nu \|x\|^2/2} p(-T(k)) e^{-\nu \|x\|^2/2} = p(\nu x - T(k)) \quad \text{for all } \nu \in \mathbb{R}.
\]

Here \( p(T(k)) \) is the operator derived from \( p(x) \) by replacing \( x_j \) by \( T_j(k) \) (recall Remark 2.1 for the notation). The identity (3.3) follows from the product rule (2.3). In particular, if \( p(x) = \sum_{j=1}^N x_j^2 \), Eq. (3.3) becomes

\[
e^{\nu \|x\|^2/2} \Delta_k e^{-\nu \|x\|^2/2} = v^2 \|x\|^2 + \Delta_k - v \sum_j (x_j T_j(k) + T_j(k) x_j).
\]

Further, using the general recipe \( [A, B, C] = A[B, C] + [A, C]B \), with \( A = B = T_j(k) \) and \( C = \|x\|^2 \), together with formula (2.3), we can verify that

\[
\begin{align*}
[\Delta_k, \|x\|^2] &= 2 \sum_j (x_j T_j(k) + T_j(k) x_j).
\end{align*}
\]

Thus, in view of (1.2), Eq. (3.4) becomes

\[
e^{\nu \|x\|^2/2} \Delta_k e^{-\nu \|x\|^2/2} = v^2 \|x\|^2 + \Delta_k - 2v \mathcal{J}_k, \quad \text{for all } v \in \mathbb{R}.
\]
This formula allows us to rewrite the triple \( \{ \widetilde{R}, \widetilde{F}, \widetilde{H} \} \) as

\[
\widetilde{R} = \frac{i}{4} e^{\|x\|^2/2} \Delta_k e^{-\|x\|^2/2},
\]

\[\widetilde{F} = -\frac{i}{4} e^{\|x\|^2/2} \Delta_k e^{\|x\|^2/2},\]

\[\widetilde{H} = e^{-\|x\|^2/2} \left( \mathcal{J}_k - \frac{\Delta_k}{2} \right) e^{\|x\|^2/2}.\]

According to (3.6b), the kernel of \( \widetilde{R} \) consists of functions of the form \( e^{-\|x\|^2/2} P(x) \), where \( P \) is \( h \)-harmonic, i.e. \( \Delta_k P = 0 \). Now by (3.6c), we get \( \widetilde{H}(e^{-\|x\|^2/2} P(x)) = e^{-\|x\|^2/2} \mathcal{J}_k P(x) \). Thus, \( e^{-\|x\|^2/2} P(x) \) is an eigenvector for \( \widetilde{H} \) with eigenvalue \( (|m| + \frac{N}{2} + \gamma_k) \) if and only if \( P \) is a homogeneous polynomial of degree \( |m| \). Here \( m = (m_1, \ldots, m_N) \in \mathbb{Z}_+^N \) and \( |m| = m_1 + \cdots + m_N \).

In conclusion, if \( P \) is an \( h \)-harmonic polynomial of degree \( |m| \), then

\[\widetilde{R}(e^{-\|x\|^2/2} P(x)) = 0\]

and

\[\widetilde{H}(e^{-\|x\|^2/2} P(x)) = \left( |m| + \frac{N}{2} + \gamma_k \right) e^{-\|x\|^2/2} P(x).\]

Henceforth, for \( m \in \mathbb{Z}_+^N \), we set \( \mathcal{H}_{|m|,k} \) to be the space of \( h \)-harmonic homogeneous polynomials on \( \mathbb{R}^N \) of degree \( |m| \). Further, we will write

\[
\lambda_k := \gamma_k + \frac{N}{2}.
\]

From (3.8), we see that for \( s \in \mathbb{N} \) and \( P \in \mathcal{H}_{|m|,k} \), the orthogonal vectors \( v_s := \widetilde{R}^s(e^{-\|x\|^2/2} P(x)) \) are eigenvectors for \( \widetilde{H} \) with eigenvalues \( \lambda_k + |m| + 2s \).

A calculation shows that for all \( \psi \) in the Schwartz space \( \mathcal{S}(\mathbb{R}^+) \) and \( P \in \mathcal{H}_{|m|,k} \), we have

\[\mathcal{J}_k \left( P(x) \psi \left( \|x\|^2 \right) \right) = \left\{ \left( |m| + \lambda_k \right) \psi \left( \|x\|^2 \right) + 2\|x\|^2 \psi' \left( \|x\|^2 \right) \right\} P(x),\]

and

\[\Delta_k \left( P(x) \psi \left( \|x\|^2 \right) \right) = 4 \left\{ \|x\|^2 \psi'' \left( \|x\|^2 \right) + \left( |m| + \lambda_k \right) \psi' \left( \|x\|^2 \right) \right\} P(x).\]

To prove (3.10) one needs to use the identities (1.2) and (3.5), that is \( \sum_{j=1}^N (x_j T_j(k) + T_j(k)x_j) = 2\mathcal{J}_k \). Combining (3.9) and (3.10) we deduce that for every \( s \in \mathbb{N} \) and \( P \in \mathcal{H}_{|m|,k} \), the operator \( \widetilde{R}^s \) leaves the set \( \mathcal{I} P := \{ \psi(\|x\|^2) P \mid \psi \in \mathcal{S}(\mathbb{R}^+) \} \) invariant. It follows that the vectors \( v_s = \widetilde{R}^s(e^{-\|x\|^2/2} P(x)) \) belong to the space \( e^{-\|x\|^2/2} P(\mathbb{R}^N) \), which is dense in \( \mathcal{S}(\mathbb{R}^N) \). We have now all ingredients to give the following theorem.

**Theorem 3.1.** (Cf. [4].) Assume that \( k \in \mathcal{K}^+ \) and \( N \geq 1 \). Let \( \mathfrak{k} = \mathfrak{so}(2) \) be the Lie algebra of the compact group \( \text{SO}(2, \mathbb{R}) \).
(i) The space \( \sum_{m \in \mathbb{Z}_+^N} \mathcal{H}_{|m|,k} \cdot \mathcal{I}(\mathbb{R}^N) \), where \( \mathcal{I}(\mathbb{R}^N) \) denotes the space of \( O(N) \)-invariant Schwartz functions on \( \mathbb{R}^N \), is dense in \( S(\mathbb{R}^N) \).

(ii) As a \( G \times \mathfrak{sl}(2, \mathbb{R}) \)-module, the \( G \times k \)-finite vectors in the Schwartz space admit the following decomposition

\[ S(\mathbb{R}^N)_{G \times \mathfrak{t}} = \bigoplus_{m \in \mathbb{Z}_+^N} \tilde{\mathcal{H}}_{|m|,k} \otimes W_{|m|+\lambda_k}, \]

where \( W_{|m|+\lambda_k} \) is the \( \mathfrak{sl}(2, \mathbb{R}) \)-representation of lowest weight \( |m|+\lambda_k \), and \( \tilde{\mathcal{H}}_{|m|,k} \) is the \( G \)-module \( e^{-\|x\|^2/2} \mathcal{H}_{|m|,k} \). The summands are mutually orthogonal with respect to the inner product on \( L^2(\mathbb{R}^N, \vartheta_k(x) \mathrm{d}x) \).

Remark 3.2. The space \( \tilde{\mathcal{H}}_{|m|,k} \) is a unitary representation of \( G \), in general not irreducible. It would be interesting to decompose it further.

Next we will investigate the integrability of \( \omega_k \). Our approach uses a result of E. Nelson [24]. For ease of reference, we recall it here. Let \( g \) be a Lie algebra over \( \mathbb{R} \), and \( \mathcal{G} \) be the simply connected Lie group with Lie algebra \( g \). A skew-symmetric \( g \)-module \( \omega \) is said to be integrable, if there exists a continuous unitary representation \( \Omega \) of \( G \) in a Hilbert space \( H \) such that \( \omega = d\Omega \).

Theorem 3.3. Let \( X_1, \ldots, X_l \) be a basis of \( g \) and \( \omega \) a densely defined \( g \)-module in \( H \). Then \( \omega = d\Omega \) for some continuous unitary representation \( \Omega \) of \( \mathcal{G} \) if and only if (i) for all \( X \in g \), \( \omega(X) \) is a skew-symmetric operator on \( H \), and (ii) the operator \( \omega(X_1^2 + \cdots + X_l^2) \) is essentially self-adjoint.

We shall prove that \( \omega_k \) obeys the conditions of Nelson’s theorem. Theorem 3.1(i) points us to consider the map

\[ \alpha^N_{m,k} : \mathcal{H}_{|m|,k} \otimes S(\mathbb{R}^+) \rightarrow S(\mathbb{R}^N) \]

defined by

\[ \alpha^N_{m,k}(h \otimes \psi)(x) := h(x) \psi(\|x\|^2), \quad (3.11) \]

with \( h \in \mathcal{H}_{|m|,k} \) and \( \psi \in S(\mathbb{R}^+) \). Moreover, using the representation \( \omega_k \), we construct a representation \( \pi^N_{m,k} \) of \( \mathfrak{sl}(2, \mathbb{R}) \) on \( S(\mathbb{R}^+) \) by

\[ \alpha^N_{m,k}(h \otimes \pi^N_{m,k}(X)\psi) = \omega_k(X)(\alpha^N_{m,k}(h \otimes \psi)), \quad X \in \mathfrak{sl}(2, \mathbb{R}). \quad (3.12) \]

Using the definitions (3.1) of the operators \( \{h, e^+, e^-\} \) in the formula (3.12), we can compute that the action on \( S(\mathbb{R}^+) \) is given by

\[ \pi^N_{m,k}(h) = 2i \frac{d}{dr} + (|m| + \lambda_k), \quad (3.13a) \]
\[ \pi^N_{m,k}(e^+) = i \frac{t}{2}, \quad (3.13b) \]
\[ \pi^N_{m,k}(e^-) = 2it \frac{d^2}{dt^2} + 2i(|m| + \lambda_k) \frac{dt}{dt}, \quad (3.13c) \]

where \( t \) denotes the positive variable of \( \mathbb{R}^+ \). The following is then immediate.

**Lemma 3.4.** The action of \( k^+, n^+, \) and \( n^- \) on \( \mathcal{S}(\mathbb{R}^+) \) is given by

\[ \pi^N_{m,k}(k) = -2t \frac{d^2}{dt^2} - 2(|m| + \lambda_k) \frac{dt}{dt} + \frac{1}{2} \left( |m| + \lambda_k \right), \]
\[ \pi^N_{m,k}(n^+) = i \left( t \frac{d^2}{dt^2} + (|m| + \lambda_k) - t \right) \frac{dt}{dt} + \frac{t}{4} - \frac{1}{2} \left( |m| + \lambda_k \right), \]
\[ \pi^N_{m,k}(n^-) = i \left( t \frac{d^2}{dt^2} + (|m| + \lambda_k) + t \right) \frac{dt}{dt} + \frac{t}{4} + \frac{1}{2} \left( |m| + \lambda_k \right). \]

**Remark 3.5.**

(i) Observe that \( \pi^N_{m,k} \) depends only on \( |m| + \lambda_k \).

(ii) The infinitesimal representation \( \pi^N_{m,k} \) appears also in [6] and in [22] (denoted by \( \lambda_\alpha \) and \( D_r \), respectively) from a different point of view and for a different reason.

Let \( \mathcal{U}(g) \) be the universal enveloping algebra of \( g_{\mathbb{C}} \), and let \( \mathcal{C} \) be the quadratic Casimir element corresponding to the Killing form of \( g_{\mathbb{C}} \). It is well known that

\[ \mathcal{C} = h^2 + 2e^+ e^- + 2e^- e^+. \quad (3.14) \]

A straightforward calculation based on formulas (3.13a)–(3.13c), shows that:

**Proposition 3.6.** The differential operator \( \pi^N_{m,k}(\mathcal{C}) \) is the scalar operator given by

\[ \pi^N_{m,k}(\mathcal{C}) = \left( |m| + \lambda_k \right) \left( |m| + \lambda_k - 2 \right). \quad (3.15) \]

Equation (3.12) and Proposition 3.6 now combine to give the following:

**Corollary 3.7.** For all \( h \in \mathcal{H}_{|m|,k} \) and \( \psi \in \mathcal{S}(\mathbb{R}^+) \), we have

\[ \omega_k(\mathcal{C}) \omega^N_{m,k}(h \otimes \psi) = \left( |m| + \lambda_k \right) \left( |m| + \lambda_k - 2 \right) \omega^N_{m,k}(h \otimes \psi). \]

Recall that \( L^2(\mathbb{R}^N, \vartheta_k(x) dx) \) denotes the space of square integrable functions on \( \mathbb{R}^N \) with respect to the measure \( \vartheta_k(x) dx \). Let \( d\omega \) be the normalized rotation-invariant measure on the unit sphere \( S_+^{N-1} \subset \mathbb{R}^N \). It is well known that \( L^2(S_+^{N-1}, \vartheta_k(\theta) d\omega(\theta)) = \sum_{m \in Z^N_+} \mathcal{H}_{|m|,k} \). Let
\{h^{(m)}_j\}_{j \in J_m}\) be an orthonormal basis of \(\mathcal{H}|_{m,k}\). Further, for \(m \in \mathbb{Z}^N_+, \ j \in J_m\), and a non-negative integer \(\ell\), define

\[
c_{\ell,m} := \left(\frac{\Gamma(N/2)\ell!}{\pi^{N/2}\Gamma(\lambda_k + |m| + \ell)}\right)^{1/2},
\]

and

\[
\phi_{\ell,m,j}(x) := c_{\ell,m}h^{(m)}_j(x)L_{\ell}^{[m]+\lambda_k-1}(\|x\|^2)e^{-\|x\|^2/2}.
\]

Here \(L_\ell^\alpha\) denotes the classical Laguerre polynomial given by

\[
L_\ell^\alpha(x) = \frac{(\alpha + 1)\ell}{\ell!} \sum_{j=0}^\ell (-\ell)_j x^j (\alpha + 1)_j j!
\]

By [12, Proposition 2.4, Theorem 2.5], the functions

\[
\phi_{\ell,m,j}, \ \ell \in \mathbb{N}, \ m \in \mathbb{Z}^N_+, \ j \in J_m,
\]

form an orthonormal basis of \(L^2(\mathbb{R}^N, \vartheta_k(x) dx)\).

**Proposition 3.8.** The dense subspace, in \(L^2(\mathbb{R}^N, \vartheta_k(x) dx)\), spanned by the functions \(\{\phi_{\ell,m,j} \mid \ell \in \mathbb{N}, \ m \in \mathbb{Z}^N_+, \ j \in J_m\}\), is stable under the action of \(\omega_k(\mathfrak{sl}(2, \mathbb{C}))\). More precisely

\[
\omega_k(\mathfrak{k})\phi_{\ell,m,j}(x) = (|m| + \lambda_k + 2\ell)\phi_{\ell,m,j}(x),
\]

\[
\omega_k(\mathfrak{n}^+)\phi_{\ell,m,j}(x) = -i(\ell + 1)^{1/2}(\lambda_k + |m| + \ell)^{1/2}\phi_{\ell+1,m,j}(x),
\]

\[
\omega_k(\mathfrak{n}^-)\phi_{\ell,m,j}(x) = -i\ell^{1/2}(|m| + \lambda_k + \ell - 1)^{1/2}\phi_{\ell-1,m,j}(x),
\]

with \(\phi_{-1,m,j} \equiv 0\). We may think of \(\omega_k(\mathfrak{n}^+)\) and \(\omega_k(\mathfrak{n}^-)\) as a creation and an annihilation operator, respectively.

**Proof.** Using the following well-known recursion relations (cf. [28, Section 6.14])

\[
t^2 d^2/dt^2 L_\ell^\alpha(t) + (\alpha + 1 - t) d/dt L_\ell^\alpha(t) = -\ell L_\ell^\alpha(t),
\]

\[
t d/dt L_\ell^\alpha(t) = \ell L_\ell^\alpha(t) - (\ell + \alpha)L_{\ell-1}^\alpha(t),
\]

\[
t d/dt L_{\ell+1}^\alpha(t) = (\ell + 1)L_{\ell+1}^\alpha(t) - (\ell + \alpha + 1 - t)L_{\ell}^\alpha(t),
\]

we obtain

\[
\pi_{m,k}^N(\mathfrak{k})\{e^{-t/2}L_{\ell}^{[m]+\lambda_k-1}(t)\} = (|m| + \lambda_k + 2\ell)\{e^{-t/2}L_{\ell}^{[m]+\lambda_k-1}(t)\}, \quad (3.16a)
\]

\[
\pi_{m,k}^N(\mathfrak{n}^+)\{e^{-t/2}L_{\ell}^{[m]+\lambda_k-1}(t)\} = -i(\ell + 1)e^{-t/2}L_{\ell+1}^{[m]+\lambda_k-1}(t), \quad (3.16b)
\]

\[
\pi_{m,k}^N(\mathfrak{n}^-)\{e^{-t/2}L_{\ell}^{[m]+\lambda_k-1}(t)\} = -i(|m| + \lambda_k + \ell - 1)e^{-t/2}L_{\ell-1}^{[m]+\lambda_k-1}(t). \quad (3.16c)
\]

Now the statement holds by means of (3.12). \(\square\)
If \( f \) and \( g \) belong to \( L^2(\mathbb{R}^N, \vartheta_k(x) \, dx) \), we will write
\[
\langle\langle f, g \rangle\rangle_k := \int_{\mathbb{R}^N} f(x) g(x) \vartheta_k(x) \, dx.
\]

Writing \( J_k \) as in (1.2), it follows from the symmetry properties of \( \Delta_k \) and \( \|x\|^2 \) that \( J_k \) is skew-symmetric. Thus, by (3.1), \( \omega_k(X) \), for any \( X \in \mathfrak{g} \), regarded as an operator on \( S(\mathbb{R}^N) \), is skew-symmetric. Furthermore, if one denotes by \( \mathcal{F}_k \) the dense subspace, in \( L^2(\mathbb{R}^N, \vartheta_k(x) \, dx) \), spanned by the functions \( \{ \phi_{\ell,m,j} \} \), then the following holds:

**Proposition 3.9.** Let \( X \mapsto X^* \) be the conjugate linear map on \( \mathfrak{g}_\mathbb{C} \) defined so that \( X^* = -X \) for \( X \in \mathfrak{g} \). Then for every \( f, g \in \mathcal{F}_k \)
\[
\langle\langle \omega_k(X) f, g \rangle\rangle_k = \langle\langle f, \omega_k(X^*) g \rangle\rangle_k, \tag{3.17}
\]
for all \( X \in \mathfrak{g}_\mathbb{C} \).

**Proof.** Since \( X \mapsto \langle\langle f, \omega_k(X^*) g \rangle\rangle_k \) is complex linear, it suffices to show (3.17) for the basis \( \{ \mathfrak{k}, \mathfrak{n}^+, \mathfrak{n}^- \} \) of \( \mathfrak{g}_\mathbb{C} \). In the light of Proposition 3.8, the statement follows from the fact that \( \mathfrak{k}^* = \mathfrak{k} \) and \( (\mathfrak{n}^+)^* = -\mathfrak{n}^- \). \( \square \)

An operator \( \mathcal{O} \) is called essentially self-adjoint, if it is symmetric and its closure is self-adjoint. Let \( \mathcal{O} \) be a symmetric operator on a Hilbert space \( \mathcal{H} \) with domain \( \mathbb{D}(\mathcal{O}) \), and let \( \{ f_n \}_{n} \) be a complete orthonormal set in \( \mathcal{H} \). If each \( f_n \in \mathbb{D}(\mathcal{O}) \) and there exists \( \lambda_n \in \mathbb{R} \) such that \( \mathcal{O} f_n = \lambda_n f_n \), for every \( n \), then \( \mathcal{O} \) is essentially self-adjoint. We refer to [7, Chapter 1] for more details on this matter.

Henceforth, \( \mathcal{G} \) denotes the simply connected covering Lie group with Lie algebra \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \). We have now all ingredients to apply Nelson’s Theorem 3.3 to \( \omega_k \).

**Theorem 3.10.** The representation \( \omega_k \) exponentiates to define a unique unitary representation \( \Omega_k \) of \( \mathcal{G} \) on \( L^2(\mathbb{R}^N, \vartheta_k(x) \, dx) \).

**Proof.** Let \( \mathbf{u}_1 = \mathbf{e}^+ - \mathbf{e}^- \), \( \mathbf{u}_2 = \mathbf{e}^+ + \mathbf{e}^- \), and \( \mathbf{u}_3 = \mathbf{h} \), so that \( \{ \mathbf{u}_i \} \) is a basis of \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \). In particular
\[
-\mathbf{u}_1^2 + \mathbf{u}_2^2 + \mathbf{u}_3^2 = \mathbf{h}^2 + 2\mathbf{e}^+ \mathbf{e}^- + 2\mathbf{e}^- \mathbf{e}^+ = \mathcal{C}' \quad \text{and} \quad \mathbf{u}_1^2 = -\mathbf{k}^2.
\]

By Corollary 3.7 and Proposition 3.8, the elements of the orthonormal basis \( \{ \phi_{\ell,m,j} : \ell \in \mathbb{N}, m \in \mathbb{Z}_+^N, j \in J_{|m|} \} \) of \( L^2(\mathbb{R}^N, \vartheta_k(x) \, dx) \) are eigenvectors of
\[
\omega_k(\mathbf{u}_1^2 + \mathbf{u}_2^2 + \mathbf{u}_3^2) = \omega_k(\mathcal{C}' - 2\mathbf{k}^2),
\]
and the eigenvalues are real. Thus, the operator \( \omega_k(\sum \mathbf{u}_i^2) \) is essentially self-adjoint. Moreover, since \( \omega_k \) is skew-symmetric, it follows then from Theorem 3.3 that \( \omega_k \) exponentiates to define on \( L^2(\mathbb{R}^N, \vartheta_k(x) \, dx) \) a unique unitary representation \( \Omega_k \) of the simply connected Lie group \( \mathcal{G} \). \( \square \)

**Remark 3.11.** The above theorem proves the conjectured integrability of \( \omega_k \) stated in [3].
Remark 3.12.

(i) Set $\Delta = u_1^2 + u_2^2 + u_3^2$. Since the elements $\phi_{\ell, m, j}$ are eigenfunctions normalized to one of $\omega_k(\Delta)$, it follows that these eigenfunctions are analytic vectors for $\omega_k(\Delta)$. Thus, by virtue of [24, Theorem 3], the set $\{\phi_{\ell, m, j}\}$ provides a dense set of analytic vectors for the representation $\Omega_k$ of $\mathfrak{G}$.

(ii) After this paper was finished, it came to our attention another beautiful theory of integrability of Lie algebras representations elaborated by Flato, Simon, Snellman and Sternheimer [14]. In contrast to Nelson’s theory it gives integrability criteria in terms of the properties of the generators of the Lie algebra. For the three-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, the level of difficulty in applying either one or the other theory is quite similar. However, generally Flato et al.’s criteria is more effective in practical applications, especially for higher-dimensional Lie algebras.

(iii) As suggested by the referee, one may also use the explicit action of $\{k, n^+, n^−\}$ on $\mathcal{S}(\mathbb{R}^+)$, and the Euclidean Fourier transform to prove the integrability of $\omega_k$. Since $G$ is simply connected, the following map $\mathbb{R} \to K, t \mapsto \exp(t(e^- - e^+))$, is a diffeomorphism. Moreover, if $Z$ denotes the center of $\mathfrak{G}$, then $Z \subset K$, and

$$Z = \{\exp(r\pi(e^- - e^+)) \mid r \in \mathbb{Z}\} \simeq \mathbb{Z}.$$ 

This is a consequence of the fact that $Z$ is the kernel of the adjoint representation and of the fact that $(e^- - e^+) = -i k$. On the other hand, by Proposition 3.8, for every element $\exp(t(e^- - e^+)) = \exp(-itk) \in K$ we have

$$\Omega_k(\exp(t(e^- - e^+)))\phi_{\ell, m, j} = \Omega_k(\exp(-itk))\phi_{\ell, m, j} = e^{-it(|m| + \lambda_k + 2\ell)}\phi_{\ell, m, j}. \quad (3.18)$$

In particular the following two facts hold:

(i) For $\lambda_k \in \mathbb{N}$, the element $\exp(r\pi(e^- - e^+)) \in \text{Ker} \Omega_k$ if and only if $r\pi(e^- - e^+) \in \mathbb{Z}^2$.

(ii) For $\lambda_k \in \frac{\mathbb{N}}{2}$, the element $\exp(r\pi(e^- - e^+)) \in \text{Ker} \Omega_k$ if and only if $r\pi(e^- - e^+) \in \mathbb{Z}^4$.

For $d = 1, 2, \ldots$, the quotient $\mathfrak{G}/\mathbb{Z}^d$ is the $d$th fold covering of the adjoint group $PSL(2, \mathbb{R})$. We may identify

$$SL(2, \mathbb{R}) \equiv \mathfrak{G}/\mathbb{Z}^2 \quad \text{and} \quad Mp(2, \mathbb{R}) \equiv \mathfrak{G}/\mathbb{Z}^4,$$

where $Mp(2, \mathbb{R})$ is the metaplectic group, i.e. the double covering of $SL(2, \mathbb{R})$. In the light of all the above discussions, the following holds:

**Proposition 3.13.** For all $k \in \mathcal{K}^+$ we have:

(i) The unitary representation $\Omega_k$ descends to $SL(2, \mathbb{R})$ if and only if $\lambda_k \in \mathbb{N}$.

(ii) The unitary representation $\Omega_k$ descends to $Mp(2, \mathbb{R})$ if and only if $\lambda_k \in \frac{\mathbb{N}}{2}$.

(iii) The unitary representation $\Omega_k$ descends to the universal covering $\tilde{SL}(2, \mathbb{R})$ if and only if $\lambda_k \in \mathbb{R}$. 
Recall that the Dunkl transform on the space $L^1(\mathbb{R}^N, \partial_k(x) \, dx)$ is given by

$$D_k f(\xi) = c_k^{-1} \int_{\mathbb{R}^N} f(x) E_k(x, -i\xi) \partial_k(x) \, dx, \quad \xi \in \mathbb{R}^N.$$ 

We close the paper by giving an application of Theorem 3.10.

By (3.18), we have

$$\Omega_k \left( \exp \left( -i \frac{\pi}{2} k \right) \right) \phi_{\ell, m, j}(x) = (-i)^{2\ell + |m|} e^{-i \frac{\pi}{4} \lambda_k} \phi_{\ell, m, j}(x).$$

On the other hand, by [12, Theorem 2.6], the Dunkl transform of $\phi_{\ell, m, j}$ is given by

$$D_k(\phi_{\ell, m, j})(x) = (-i)^{2\ell + |m|} \phi_{\ell, m, j}(x).$$

Now, since $e^{i \frac{\pi}{2} \lambda_k} \Omega_k(\exp(-i \frac{\pi}{2} k))$ and $D_k$ are unitary operators on $L^2(\mathbb{R}^N, \partial(x) \, dx)$, it follows that the Dunkl transform can be written as

$$D_k = e^{i \frac{\pi}{2} \lambda_k} e^{-i \frac{\pi}{4} (\|x\|^2 - \Delta_k)}.$$

That is, up to a scalar factor, $D_k$ is an element of the integrated form of the representation $\omega_k$ formulated above. This statement was also proved in [3, Corollary 4.14] using a generalized Segal–Bargmann transform associated with the finite reflection group $G$.

**Remark 3.14.** From a representation theory point of view, representing a Fourier-type transform by a group element, up to a scalar factor, is not a surprising phenomenon. See for instance [9,17].

In the light of (3.19), we may define a transform $\mathcal{F}^N_{m,k}$ on $S(\mathbb{R}^+)$ by

$$\alpha^N_{m,k}(h \otimes \mathcal{F}^N_{m,k}(\psi)) := \alpha^N_{m,k} \left( h \otimes \Pi^N_{m,k} \left( e^{i \frac{\pi}{2} \lambda_k} \exp \left( -i \frac{\pi}{2} k \right) \right) \psi \right)$$

$$= D_k \left( \alpha^N_{m,k}(h \otimes \psi) \right) \quad \text{(by (3.12) and (3.19))},$$

where $h \in \mathcal{H}_{|m|,k}$, and $\Pi^N_{m,k}$ is the unique unitary representation of the Lie group $G$ such that $d\Pi^N_{m,k} = \pi^N_{m,k}$ (see Theorem 3.17 below for more details). Now the following holds.

**Theorem 3.15 (Bochner-type formula).** Let $k \in \mathcal{K}^+$.  

(i) If $f(x) = h(x) \psi(\|x\|^2)$, with $h \in \mathcal{H}_{|m|,k}$ and $\psi \in S(\mathbb{R}^+)$, then

$$D_k(f)(\xi) = h(\xi) \mathcal{F}^N_{m,k}(\psi)(\|\xi\|^2),$$

where $\mathcal{F}^N_{m,k}$ depends only on $|m| + \lambda_k$ up to a constant, i.e. if

$$|m| + \lambda_k = |m'| + \lambda'_{k'},$$

(3.20)
with \( \gamma' = \gamma_k + N'/2 \), then
\[
e^{-i \frac{\pi}{2} \lambda_k} F^N_{m,k} = e^{-i \frac{\pi}{2} \lambda'_k} F^{N'}_{m',k'}.
\]

(ii) The transform \( F^N_{m,k} \) coincides with the classical Hankel transform. More precisely, for \( \psi \in \mathcal{S}(\mathbb{R}^+) \),
\[
F^N_{m,k}(\psi)(r^2) = e^{-i \frac{\pi}{2} |m|} \mathcal{H}_{|m|+\lambda_k-1}(\psi \circ \Upsilon)(r),
\]
where \( \Upsilon(t) := t^2 \) for \( t \in \mathbb{R} \), and
\[
\mathcal{H}_v f(r) := \int_0^{\infty} f(s) \frac{J_v(rs)}{(rs)^v} s^{2v+1} ds
\]
denotes the Hankel transform, with \( J_v \) is the Bessel function of the first kind. In these circumstances, part (i) reads
\[
D_k(h \psi(\| \cdot \|))(\xi) = e^{-i \frac{\pi}{2} |m|} h(\xi) \mathcal{H}_{|m|+\lambda_k-1}(\psi \circ \Upsilon)(\| \xi \|),
\]
for \( h \in \mathcal{H}_{|m|,k} \) and \( \psi \in \mathcal{S}(\mathbb{R}^+) \).

**Proof.** The first statement holds from the definition of \( F^N_{m,k} \), and from Lemma 3.4. To prove (ii), let us start with \( m \in \mathbb{Z}_+^N \) such that \( |m| = 0 \), i.e. \( m = 0 \). In this case \( f(x) = \psi(\|x\|^2) \), i.e. \( f \) is a radial function. This case was basically done in [30]. However, for completeness we shall briefly include the argument for radial functions. Assume that \( F(x) = F_o(\|x\|) \). Using the polar coordinates and the homogeneity of \( \vartheta_k \), we have
\[
D_k(F)(\xi) = c_k^{-1} \int_0^{\infty} F_o(r) r^{2\lambda_k-1} \left\{ \int_{\mathbb{S}^{N-1}} E_k(-ir \xi, \theta) \vartheta_k(\theta) d\omega(\theta) \right\} dr.
\]
By [30, Corollary 2.2], we have
\[
\int_{\mathbb{S}^{N-1}} E_k(-ir \xi, \theta) \vartheta_k(\theta) d\omega(\theta) = c_k \frac{J_{2\lambda_k-1}(r \xi \|)}{(r \xi \|)^{\lambda_k-1}}.
\]
Thus \( D_k(F)(\xi) = \mathcal{H}_{\lambda_k-1}(F_o)(\| \xi \|) \). This implies that
\[
D_k(f)(\xi) = \mathcal{H}_{\lambda_k-1}(\psi \circ \Upsilon)(\| \xi \|)
\]
whenever \( f(x) = \psi(\|x\|^2) \), and therefore \( F^N_{0,k}(\psi)(\| \xi \|^2) = \mathcal{H}_{\lambda_k-1}(\psi \circ \Upsilon)(\| \xi \|) \). Now let \( m \in \mathbb{Z}_+^N \) such that \( |m| \neq 0 \). Equation (3.20) gives
\[
F^N_{m,k}(\psi)(r^2) = e^{-i \frac{\pi}{2} |m|} F^{2|m|+N}_{0,k}(\psi)(r^2) = e^{-i \frac{\pi}{2} |m|} \mathcal{H}_{|m|+\lambda_k-1}(\psi \circ \Upsilon)(r). \quad \square
\]
Example 3.16 (Hecke-type formula). If $\psi(s) = e^{-\frac{s^2}{2}}$, then

$$D_k\left(e^{-\frac{\|x\|^2}{2}} h(\xi) \mathcal{H}[m]_{\lambda_k-1}\right)\left(\|\xi\|\right) = e^{-\frac{i}{2}m} e^{-\frac{\|\xi\|^2}{2}} h(\xi).$$

Thus, we recover the Hecke formula for the Dunkl transform which was initially proved by Dunkl in [12], and later in [3] by Ørsted and the present author using an $\mathfrak{sl}(2, \mathbb{R})$-argument similar to the one illustrated above.

We conclude this paper by putting together a few facts regarding the representation $\pi_{m,k}^N$, since they fit in naturally with the development here. We believe that the theorem below must essentially be known.

For $s \in \mathbb{R}$, write $e^s = \exp G(s e^+)$, $h^s = \exp G(s h)$, and $\kappa = \exp G(\pi \frac{i}{2} (e^- - e^+)) = \exp G(-i \frac{\pi}{2} k)$. Here $\exp G$ denotes the exponential map of $\mathfrak{sl}(2, \mathbb{R})$ into $G$.

Recall that the Laguerre polynomials satisfy the orthogonality relation

$$\int_0^\infty e^{-t} L_\alpha^\mu(t)L_\beta^\nu(t)t^{\alpha-1} dt = \delta_{\alpha,\beta} \frac{\Gamma(\alpha + 1)}{\Gamma(\mu + 1)}.$$

Theorem 3.17. Let $k \in \mathcal{K}^+$ and $m \in \mathbb{Z}_+^N$.

(i) The dense subspace, in $L^2(\mathbb{R}_+, t^{\|m\| + \lambda_k - 1} dt)$, spanned by the Laguerre functions $\{e^{-t/2} L_\ell^{\|m\| + \lambda_k - 1}(t)\}_{\ell \in \mathbb{N}}$, is stable under the action of $\pi_{m,k}^N(\mathfrak{sl}(2, \mathbb{C}))$, and the spectrum of $\pi_{m,k}^N(k)$ is positive.

(ii) There exists a unique unitary representation $\Pi_{m,k}^N$ of the simply connected Lie group $\mathfrak{G}$ on $L^2(\mathbb{R}_+, t^{\|m\| + \lambda_k - 1} dt)$, such that $\pi_{m,k}^N(X)f = (d/ds)|_{s=0}\Pi_{m,k}^N(\exp(sX))f$ for $f \in S(\mathbb{R}_+)$. Further, the unitary representation $\Pi_{m,k}^N$ may be described by the formulas:

1. $\Pi_{m,k}^N(e_s)f : t \mapsto e^{its/2} f(t)$;
2. $\Pi_{m,k}^N(h_s)f : t \mapsto e^{(1/2)(\|m\| + \lambda_k)s} f(e^{2s}t)$;
3. $\Pi_{m,k}^N(\kappa)f : t \mapsto \frac{1}{2} e^{-i \frac{\pi}{2} (\|m\| + \lambda_k)} \mathcal{H}[m]_{\lambda_k-1}(f)(t)$, where $\mathcal{H}_v$ is the Hankel-type transform given by

$$\mathcal{H}_v f(t) = \int_0^\infty f(u) J_v((ut)^{1/2}) \frac{1}{(ut)^{v/2}} u^{v} du,$$

where $J_v$ is the Bessel function of the first kind.

Proof. (i) This statement is just (3.16a)–(3.16c).

(ii) The existence and the uniqueness of $\Pi_{m,k}^N$ follow from Theorem 3.10 and (3.12). However, in the light of part (i), one may also prove the integrability of $\pi_{m,k}^N$ in a direct fashion using Nelson’s theorem, as we did previously with $\omega_k$. Both formulas (1) and (2) are clear. Formula (3) follows from the definition of $\mathcal{H}_v$ and (3.21). $\square$
Acknowledgment

We thank the referee for the insightful comments that helped us to improve the paper.

References