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MULTIMONADS AND MULTIMONADIC CATEGORIES

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0. Introduction

The aim of the work is to extend the results of the theory of monads (= triples) and apply them to new situations: local rings, fields, inner spaces, locally compact spaces, locally compact groups, complete ordered sets, etc. We use the notion of multiadjunction developed in [4].

Take, for example, the category **Locc** of commutative local rings and local homomorphisms. The forgetful functor $U:\mathbf{Locc}\to\mathbf{Set}$ has a left multiadjoint. There is on the category **Set** a trace of this multiadjunction: first, the spectrum relative to U which is a functor $S:\mathbf{Set}^{\mathrm{op}}\to\mathbf{Set}$ given by $S(E)=$ Set of prime ideals of the ring $\mathbf{Z}[\langle x\rangle]_{x\in E}$ of polynomials with coefficients in $\mathbf{Z}$, of the variables $\langle x\rangle$ where x runs through E. Second, a functor T which assigns to a pair (E,I) of a set E and a prime ideal $I\in S(E)$, the set $T(E,I)$ of rational fractions $P/Q\in\mathbf{Z}(\langle x\rangle)_{x\in E}$ such that $Q\notin I$, and two natural transformations. This trace is called a multimonad on **Set**. It is remarkable that the category **Locc** can be reconstructed from this multimonad, a local ring appearing then as a triple (E,I,e) of a set E, a member I of $S(E)$ and a structural map $T(E,I)\to E$ fulfilling two axioms.

Any functor $U:\mathbf{A}\to\mathbf{B}$ which has a left multiadjoint generates a multimonad on **B** and we characterize the categories **A** which can be reconstructed from this multimonad and which are called multimonadic. These categories are naturally equipped with a comonad so we get standard resolutions in the homological sense. The multimonadic categories on **Set** can be characterized in the following way: they are regular, with connected limits, with coequalizers of coequalizable pairs, their equivalence relations are effective, their forgetful functors preserve coequalizers of equivalence relations and reflect isomorphisms. But they need not have products, and this is totally different from monadic categories on **Set**. Local rings, fields, inner spaces, locally compact spaces, locally compact groups, complete ordered sets are examples. The descriptions of these well known objects in terms of multimonads, are not ordinary. A locally compact space is described by its relatively compact subsets, a complete ordered set by its nonempty upper bounded subsets, a local ring by the rational fractions defined on it.

1. Categories of S-objects

1.0. Let $\mathbf{B}$ be a category and $S:\mathbf{B}^{\mathrm{op}}\to\mathbf{Set}$ be a functor. An *S-object* is a pair (B, i) of an object B of $\mathbf{B}$ and an element i of SB. A *S-morphism* $(B, i)\to(B', i')$ is a morphism $g:B\to B'$ of $\mathbf{B}$ which satisfies $Sg(i')=i$. They form a category $\mathbf{B}_{/S}$. The functor $U_s:\mathbf{B}_{/s}\to\mathbf{B}$ is defined by $U_s(B, i)=B$ and $U_s(g)=g$. It is a small fibers discrete fibration i.e. a fibration, the fibers of which are small discrete categories i.e. sets. It is said *associated* to the functor S.

The small fibers discrete fibrations can be described as the functors $P:\mathbf{X}\to\mathbf{B}$ which satisfy the two following data

(1) for any object B of $\mathbf{B}$, $\{X\in \mathrm{Obj}(\mathbf{X}):PX=B\}$ is a set,

(2) for any object X of $\mathbf{X}$ and any morphism $g:B\to PX$, there exists a unique morphism f with codomain X such that $Pf=g$. Then, all the morphisms of $\mathbf{X}$ are cartesian for P. A small fibers discrete fibration P has a left multiadjoint, the families $(1_B:B\to PX)_{PX=B}$ of morphisms from B to P being universal. The spectrum relative to P is then the functor $\mathrm{Spec}_P:\mathbf{B}^{\mathrm{op}}\to\mathbf{Set}$ defined by $\mathrm{Spec}_P(B)=$ fiber of P in B and $\mathrm{Spec}_P(g)(X)=g^*(X)$.

It is easy to see that the correspondences described above, between the functors $S:\mathbf{B}^{\mathrm{op}}\to\mathbf{Set}$ and the small fibers discrete fibrations on $\mathbf{B}$, are inverse of each other, up to isomorphisms. From now on, discrete fibration means small fibers discrete fibration.

1.1. Proposition. *A functor has a left multiadjoint if and only if it is the composite of a functor having a left adjoint with a discrete fibration. This factorization is unique up to isomorphisms and the fibration is then associated to the spectrum of the functor.*

Proof. (a) The sufficient condition is a consequence of the fact that the composite of two functors which have left multiadjoints, has a left multiadjoint [4]. Let $U:\mathbf{A}\to\mathbf{B}$ be a functor having a left multiadjoint. For any object B of $\mathbf{B}$, let $(g_i:B\to UA_i)_{i\in\mathrm{Spec}_U(B)}$ be a universal family of morphisms from B to U. Let denote $P:\mathbf{X}\to\mathbf{B}$ the discrete fibration associated to the functor $\mathrm{Spec}_U:\mathbf{B}^{\mathrm{op}}\to \mathit{Set}$. For any object A of $\mathbf{A}$, the morphism $1_{UA}:UA\to UA$ factorizes in a unique way in the form $1_{UA}=(Uf_A)g_{i_A}$ where $i_A\in\mathrm{Spec}_U(UA)$ and $f_A:A_{i_A}\to A$. For any morphism $f:A\to A'$ of $\mathbf{A}$, one has $\mathrm{Spec}_U(Uf)(i_{A'})=i_A$. One defines then a functor $V:\mathbf{A}\to\mathbf{X}$ by $VA=(UA, i_A)$ and $Vf=Uf$. For any morphism $g:B\to B'$ of $\mathbf{B}$ and any $i'\in\mathrm{Spec}_U(B')$, the morphism $g_{i'}g:B\to UA_{i'}$ factorizes in a unique way in the form $g_{i'}g=(Uf)g_i$ where $i=\mathrm{Spec}_U(g)(i')$ and $f:A_i\to A_{i'}$. One defines then a functor $F:\mathbf{X}\to\mathbf{A}$ by $F(B, i)=A_i$ and $Fg=f$. Let us show that F is left adjoint to V. One defines a natural transformation

$$\alpha : \mathrm{Hom}_{\mathbf{A}}(F(\cdot), -) \to \mathrm{Hom}_{\mathbf{X}}(\cdot, V(-))$$

by defining $\alpha_{(B,i),A} : \mathrm{Hom}_{\mathbf{A}}(F(B,i), A) \to \mathrm{Hom}_{\mathbf{X}}((B,i), VA)$ by $\alpha_{(B,i),A}(f) = (Uf)g_i$ for $f : A_i \to A$. For any object A of $\mathbf{A}$ and any object B of $\mathbf{B}$, the map $\coprod_{i \in \mathrm{Spec}_U(B)} \alpha_{(B,i),A}$ is composed of the following bijections:

$$\coprod_{i \in \mathrm{Spec}_U(B)} \mathrm{Hom}_{\mathbf{A}}(A_i, A) \simeq \mathrm{Hom}_{\mathbf{B}}(B, UA) = \mathrm{Hom}_{\mathbf{B}}(B, P(UA, i_A))$$

$$\simeq \coprod_{i \in \mathrm{Spec}_U(B)} \mathrm{Hom}_{\mathbf{X}}((B,i), (UA, i_A)) \simeq \coprod_{i \in \mathrm{Spec}_U(B)} \mathrm{Hom}_{\mathbf{X}}((B,i), VA).$$

It is thus a bijection and so is any map $\alpha_{(B,i),A}$. It follows that α is an isomorphism.

(b) Let $U = QW$ be another factorization of U, where the functor $W : \mathbf{A} \to \mathbf{Y}$ has a left adjoint and where $Q : \mathbf{Y} \to \mathbf{B}$ is a discrete fibration. For any object (B, i) of $\mathbf{X}$, there is a unique morphism $y : Y \to WA_i$ of $\mathbf{Y}$ such that $Qy = g_i : B \to UA_i$. Let us denote $R(B,i)$ the domain of y. If $g : (B,i) \to (B', i')$ is a morphism of $\mathbf{X}$, there is a unique morphism $Rg : R(B,i) \to R(B',i')$ of $\mathbf{Y}$ such that $Q(Rg) = g$. So one defines a functor $R : \mathbf{X} \to \mathbf{Y}$ which satisfies $QR = P$. For any object A of $\mathbf{A}$, there is a canonical morphism $A_{i_A} \to A$ and a morphism $RVA = R(UA, i_A) \to WA_{i_A}$ and then, a morphism $RVA \to WA$. The relation $QRVA = PVA = QWA$ implies the equality $RVA = WA$, and the relation $WRV = PV = QW$ implies the equality $RV = W$. It follows that R is a discrete fibration, the fibers of which must be reduced to one element, because $RV = W$ has a left adjoint. So R is an isomorphism.

For any functor $U : \mathbf{A} \to \mathbf{B}$ which has a left multiadjoint, one chooses a functor $\mathrm{Spec}_U : \mathbf{B}^{\mathrm{op}} \to \mathrm{Set}$, and, for any object B of $\mathbf{B}$, a universal family $(g_i : B \to UA_i)_{i \in \mathrm{Spec}_U(B)}$ of morphisms from B to U. Alternatively one chooses a discrete fibration $P : \mathbf{X} \to \mathbf{B}$ and an adjunction (F, V, α) such that $PV = U$.

2. Multimonads

2.0. A *multimonad* on a category $\mathbf{B}$ is a pair $(S, \mathbf{T})$ of a functor $S : \mathbf{B}^{\mathrm{op}} \to \mathbf{Set}$ and a monad $\mathbf{T}$ on the category of S-objects.

If $\mathbf{T} = (T, \eta, \mu)$, $B \in \mathbf{B}$, $B' \in \mathbf{B}$, $i \in SB$, $i' \in SB'$, and $f : (B', i') \to (B, i)$, one denotes: $T(B,i) = (T_i B, m_B(i))$ and $Tf = T_i f$. The axioms of the monad $\mathbf{T}$ can be expressed by the commutativity of the two following diagrams:

$$\begin{array}{ccc} T_iB & \xrightarrow{\eta_{T_iB, m_B(i)}} & T_{m_B(i)}T_iB \\ \Big\downarrow{\scriptstyle T_{m_B(i)}(\eta_{B,i})} & & \Big\downarrow{\scriptstyle \mu_{B,i}} \\ T_{m_B(i)}T_iB & \xrightarrow[\mu_{B,i}]{} & T_iB \end{array} \qquad \begin{array}{ccc} T_{m_{T_iB}(m_B(i))}T_{m_B(i)}T_iB & \xrightarrow{T_{m_B(i)}(\mu_{B,i})} & T_{m_B(i)}T_iB \\ \Big\downarrow{\scriptstyle \mu_{T_iB, m_B(i)}} & & \Big\downarrow{\scriptstyle \mu_{B,i}} \\ T_{m_B(i)}T_iB & \xrightarrow[\mu_{B,i}]{} & T_iB \end{array}$$

2.1. Proposition. *Any functor $U:\mathbf{A}\to\mathbf{B}$ which has a left multiadjoint generates a multimonad on* $\mathbf{B}$.

Proof. By Proposition 1.1., the functor $U:\mathbf{A}\to\mathbf{B}$ is of the form $U=PV$ with P the discrete fibration associated to the spectrum relative to U and where $V:\mathbf{A}\to\mathbf{X}$ has a left adjoint F. The adjunction (F, V) generates a monad $\mathbf{T}$ on $\mathbf{X}$. The pair $(\mathrm{Spec}_U, \mathbf{T})$ is then a multimonad on $\mathbf{B}$, which is said, *generated* by U.

2.2. Algebras. If $(S, \mathbf{T})$ is a multimonad on $\mathbf{B}$, a $S\mathbf{T}$-*algebra* is a triple (B, i, b) such that (B, i) is an S-object and $((B, i), b)$ is a $\mathbf{T}$-algebra, a $S\mathbf{T}$-*homomorphism* of $S\mathbf{T}$-algebras $(B, i, b)\to(B', i', b')$ is a $\mathbf{T}$-homomorphism $((B, i), b)\to((B', i'), b')$. They constitute the category $\mathbf{B}^{\mathbf{T}}_{/s}$ which is, indeed, the category of $\mathbf{T}$-algebras.

With the notations of 2.0, a $S\mathbf{T}$-algebra is thus a triple (B, i, b) of an object B of $\mathbf{B}$, an element i of SB and a morphism $b:T_iB\to B$ such that $Sb(i)=m_B(i)$ and the following two diagrams commute:

$$
\begin{array}{ccc}
B & \xrightarrow{\eta_{B,i}} & T_iB \\
 & {\scriptstyle 1_B}\searrow & \downarrow{\scriptstyle b} \\
 & & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
T_{m_B(i)}T_iB & \xrightarrow{T_ib} & T_iB \\
\downarrow{\scriptstyle \mu_{B,i}} & & \downarrow{\scriptstyle b} \\
T_iB & \xrightarrow{b} & B
\end{array}
$$

and a $S\mathbf{T}$-homomorphism $f:(B, i, b)\to(B', i', b')$ is a morphism $f:B\to B'$ such that $Sf(i')=i$ and the following diagram commutes

$$
\begin{array}{ccc}
T_iB & \xrightarrow{T_if} & T_iB' \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b'} \\
B & \xrightarrow[f]{} & B'
\end{array}
$$

Denote $U^{\mathbf{T}}:\mathbf{B}^{\mathbf{T}}_{/s}\to\mathbf{B}_{/s}$ the forgetful functor of $\mathbf{T}$-structure and $U^{\mathbf{T}}_s:\mathbf{B}^{\mathbf{T}}_{/s}\to\mathbf{B}$ the forgetful functor of $S\mathbf{T}$-structure, composite of $U^{\mathbf{T}}$ with the discrete fibration $U_s:\mathbf{B}_{/s}\to\mathbf{B}$. Observe that $\mathbf{B}_{/s}=\mathbf{B}^{\mathbf{1}}_{/s}$ and $U_s=U^{\mathbf{1}}_s$.

2.3. Proposition. *The functor $U^{\mathbf{T}}_s:\mathbf{B}^{\mathbf{T}}_{/s}\to\mathbf{B}$ has a left multiadjoint. It generates the multimonad $(S, \mathbf{T})$ on $\mathbf{B}$ and, for any functor $U:\mathbf{A}\to\mathbf{B}$ which has a left multiadjoint and generates $(S, \mathbf{T})$, there exists a unique functor $K:\mathbf{A}\to\mathbf{B}^{\mathbf{T}}_{/s}$, called the comparison functor, which satisfies $U^{\mathbf{T}}_sK=U$ and preserves the universal families of morphisms.*

Proof. One knows that the functor $U^{\mathbf{T}}: \mathbf{B}_{/s}^{\mathbf{T}} \to \mathbf{B}_{/s}$ has a left adjoint $F^{\mathbf{T}}$ and that it generates the monad $\mathbf{T}$ on $\mathbf{B}_{/s}$. The relation $U_s^{\mathbf{T}} = U_s U^{\mathbf{T}}$ implies that $U_s^{\mathbf{T}}$ has a left multiadjoint and that the spectrum relative to $U_s^{\mathbf{T}}$ is S. So $(S, \mathbf{T})$ is the multimonad generated by $U_s^{\mathbf{T}}$. Let $U: \mathbf{A} \to \mathbf{B}$ be a functor having a left multiadjoint and generating $(S, \mathbf{T})$. It factorizes in the form $U = U_s V$ with V having a left adjoint F and generating the monad $\mathbf{T}$ on $\mathbf{B}_{/s}$. There exists then, a unique functor $K: \mathbf{A} \to \mathbf{B}_s^{\mathbf{T}}$ satisfying $U^{\mathbf{T}}K = V$ and $KF = F^{\mathbf{T}}$. But this last condition is equivalent to say that, for any object B of $\mathbf{B}$, if $(g_i : B \to UA_i)_{i \in SB}$ is a universal family of morphisms from B to U, then $(g_i : B \to U_s^{\mathbf{T}} KA_i)_{i \in SB}$ is a universal family of morphisms from B to $U_s^{\mathbf{T}}$, that is to say, K preserves universal families of morphisms.

3. Multimonadic categories

3.0. Definition. A functor $U: \mathbf{A} \to \mathbf{B}$ is *multimonadic* if it has a left multiadjoint and if the comparison functor $K: \mathbf{A} \to \mathbf{B}_{/s}^{\mathbf{T}}$ is an equivalence. The category $\mathbf{A}$ is then said to be *multimonadic* on $\mathbf{B}$.

It is obvious that a functor is multimonadic if and only if it is the composite of a monadic functor with a discrete fibration. In particular, a discrete fibration is multimonadic.

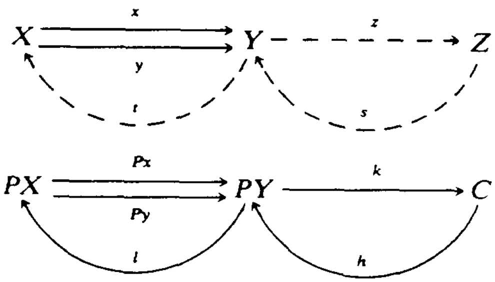
3.1. Theorem. *A functor $U: \mathbf{A} \to \mathbf{B}$ is multimonadic if and only if*

(1) *it has a left multiadjoint,*

(2) *it reflects isomorphisms,*

(3) *the pairs of parallel morphisms of* $\mathbf{A}$, *whose image by U has a split coequalizer, have a coequalizer preserved by U.*

Proof. (a) *Lemma.* A discrete fibration creates isomorphisms and split coequalizers: let $P: \mathbf{X} \to \mathbf{B}$ be a discrete fibration. Let X be an object of $\mathbf{X}$ and $g: B \to PX$ be an isomorphism of $\mathbf{B}$. There is a unique morphism $x: Y \to X$ of $\mathbf{X}$ such that $Px = g$. The morphism x being cartesian, there exists a unique morphism $y: X \to Y$ of $\mathbf{X}$ such that $xy = 1_X$ and $Py = g^{-1}$. The relation $P(yx) = g^{-1}g = 1_B$ implies $yx = 1_Y$. So x is isomorphic. Let $(x, y): X \rightrightarrows Y$ be a pair of parallel morphisms of $\mathbf{X}$ such that the pair $(Px, Py): PX \rightrightarrows PY$ has a split coequalizer in $\mathbf{B}$ [9].



Let $k:PY\to C$, $h:C\to PY$, $l:PY\to PX$ be morphisms of **B** satisfying $k(Px)=k(Py)$, $kh=1_C$, $(Px)l=1_{Py}$, $(Py)l=hk$. The morphism x being cartesian, there is a unique morphism $t:Y\to X$ such that $xt=1_Y$ and $Pt=l$. Let $s:Z\to Y$ be the unique morphism such that $Ps=h$. The morphism s being cartesian, there is a unique morphism $z:Y\to Z$ such that $sz=yt$ and $Pz=k$. The relation $P(zs)=kh=1_C$ implies $zs=1_Z$ and the relation $P(zx)=k(Px)=k(Py)=P(zy)$ implies $zx=zy$. So, one gets a split coequalizer of (x, y).

(b) *Proof of the theorem.* Let $U:\mathbf{A}\to\mathbf{B}$ be a functor having a left multiadjoint and $U=PV$, with $V:\mathbf{A}\to\mathbf{X}$ and $P:\mathbf{X}\to\mathbf{B}$, its canonical factorization. The functor U is multimonadic if and only if the functor V is monadic. By the theorem of J. Beck characterizing monadic categories [9], the functor V is monadic if and only if it reflects isomorphisms and if the pairs of parallel morphisms whose image by V have split coequalizers, have coequalizers preserved by V. Because the functor P creates isomorphisms, V reflects isomorphisms if and only if U reflects them. Because P creates split coequalizers, for any pair $(f,f'):A\rightrightarrows A'$ of parallel morphisms of $\mathbf{A}$, (Vf, Vf') has a split coequalizer if and only if (Uf, Uf') has one. The result follows.

We are going to prove that, in the preceding characterization, in most cases, it is sufficient to consider those pairs of morphisms which are equivalence relations. Note that the categories we are dealing with, need not have products, so we consider kernel pairs of nonempty sets of morphisms with common domain instead of kernel pairs of one morphism.

3.2. Theorem. *If* **B** *is a category with kernel pairs, a functor* $U:\mathbf{A}\to\mathbf{B}$ *is multimonadic if and only if*

(1) *it has a left multiadjoint,*

(2) *it reflects isomorphisms,*

(3) $\mathbf{A}$ *has kernel pairs,*

(4) *the equivalence relations of* $\mathbf{A}$ *whose image by* U *is contractible and has a coequalizer, have a coequalizer preserved by* U.

Proof. Let $P:\mathbf{X}\to\mathbf{B}$ be a discrete fibration.

(a) P creates kernel pairs: let $(x_i:X\to X_i)_{i\in I}$ be a nonempty family of morphisms of $\mathbf{X}$ and let $(m, n):K\rightrightarrows PX$ be a kernel pair of $(Px_i:PX\to PX_i)_{i\in I}$. There is a unique morphism $y:Y\to X$ such that $Py=m$, and a unique morphism $z:Z\to X$ such that $Pz=n$. The relation $P(x_iy)=(Px_i)m=(Px_i)n=P(x_iz)$ for some $i\in I$, implies $Y=Z$. It is then easy to see that (y, z) is a kernel pair of $(x_i)_{i\in I}$.

(b) P preserves and reflects equivalence relations: for any objet X of $\mathbf{X}$ and any object B of $\mathbf{B}$, one has

$$\mathrm{Hom}_{\mathbf{B}}(B, PX)\simeq \coprod_{PZ=B}\mathrm{Hom}_{\mathbf{X}}(Z, X)$$

in a natural way; for two parallel morphisms $(x, y): X \rightrightarrows Y$, the two pairs of maps

$$\mathrm{Hom}_{\mathbf{B}}(B, PX \overset{px}{\underset{py}{\Longrightarrow}} PY) \quad \text{and} \quad \coprod_{PZ=B} \mathrm{Hom}_{\mathbf{X}}(Z, X \overset{x}{\underset{y}{\Longrightarrow}} Y)$$

are also isomorphic. It follows that $\mathrm{Hom}_{\mathbf{B}}(B, PX) \overset{px}{\underset{py}{\Longrightarrow}} PY)$ is an equivalence relation if and only if $\coprod_{PZ=B} \mathrm{Hom}_{\mathbf{X}}(Z, X \overset{x}{\underset{y}{\Longrightarrow}} Y)$ is one, that is to say if $\mathrm{Hom}_{\mathbf{X}}(Z, X \overset{x}{\underset{y}{\Longrightarrow}} Y)$ is an equivalence relation for any Z such that $PZ = B$. Thus, (x, y) is an equivalence relation in $\mathbf{X}$ if and only if (Px, Py) is an equivalence relation in $\mathbf{B}$.

(c) P preserves and reflects contractible equivalence relations: Suppose that $(Px, Py): PX \rightrightarrows PY$ is a contractible equivalence relation i.e. there is a morphism $m: PY \to PX$ such that $(Px)m = 1_{PY}$ and $P(y)mP(y) = (Py)m(Px)$. There is a unique morphism $s: Y \to X$ such that $xs = 1_X$ and $Ps = m$. The relations

$$P(ysy) = (Py)m(Py) = (Py)m(Px) = P(ysx)$$

imply $ysy = ysx$. So (x, y) is a contractible equivalence relation.

(d) P preserves and reflects effective contractible equivalence relations: the coequalizer of a contractible equivalence relation is split. By Lemma 3.1 (a), a contractible equivalence relation (m, n) of $\mathbf{X}$ has a coequalizer z if and only if its image (Pm, Pn) has a coequalizer, and by (a), (m, n) is the kernel pair of z if and only if (Pm, Pn) is the kernel pair of Pz.

(e) *Proof of the theorem.* Let $U = PV$ be the canonical factorization of U. Because $\mathbf{B}$ has kernel pairs, so has $\mathbf{X}$. By a theorem of *J.* Duskin [5, p. 89, Theorem 3.2] the functor V is monadic if and only if: it reflects isomorphisms, $\mathbf{A}$ has kernel pairs (cf. separators) and the equivalence relations in $\mathbf{A}$ whose image by V is contractible and has a coequalizer, have a coequalizer preserved by V. Because of the results above, we can write these conditions for the functor U.

3.3. Proposition. *The functor* $U_S^{\mathbf{T}}: \mathbf{B}_{/S}^{\mathbf{T}} \to \mathbf{B}$ *creates connected limits.*

Proof. Because a monadic functor creates limits, it is sufficient to show that a discrete fibration $P: \mathbf{X} \to \mathbf{B}$ creates connected limits. Let $\mathbf{I}$ be a nonempty connected category and $(X_i)_{i \in \mathbf{I}}$ be a diagram of $\mathbf{X}$ the image of which has a limit $(p_i: B \to PX_i)_{i \in \mathbf{I}}$ in $\mathbf{B}$. For any object i of $\mathbf{I}$, there exists a unique morphism $x_i: Y_i \to X_i$ such that $Px_i = p_i$. For any morphism $\alpha: i \to i'$ of $\mathbf{I}$, one has $P(X_\alpha x_i) = (PX_\alpha)p_i = Px_{i'}$ and thus, $Y_{i'} = Y_i$ and $X_\alpha x_i = x_{i'}$. Because $\mathbf{I}$ is connected, it follows that: $\forall i \in \mathbf{I}$, $\forall i' \in \mathbf{I}$, $Y_{i'} = Y_i$. Denote Y the common value of Y_i. One gets a projective cone $(x_i: Y \to X_i)_{i \in \mathbf{I}}$ based on $(X_i)_{i \in \mathbf{I}}$. Let $(z_i: Z \to X_i)_{i \in \mathbf{I}}$ be a projective cone based on $(X_i)_{i \in \mathbf{I}}$. There is a unique morphism $g: PZ \to B$ such that $p_i g = Pz_i$ for every $i \in \mathbf{I}$. There is then a unique morphism $z: Z \to Y$ such that $Pz = g$. Then $P(x_i z) = p_i g = Pz_i$ implies $x_i z = z_i$, for every $i \in \mathbf{I}$. Such a morphism z is uniquely determined by $x_i z = z_i$ because one has $p_i(Pz) = Pz_i$ and thus $Pz = g$. It follows that $(x_i: Y \to X_i)_{i \in \mathbf{I}}$ is the limit of $(X_i)_{i \in \mathbf{I}}$.

4. Multimonadic categories on Set

4.0. Proposition. *If* $(S, \mathbf{T})$ *is a multimonad on* **Set**, *the category* $\mathbf{Set}^{\mathbf{T}}_{/s}$ *is regular, with effective equivalence relations, has connected limits and coequalizers of coequalizable pairs of morphisms, is well-powered, regularly cowell-powered and the functor* $U^{\mathbf{T}}_s : \mathbf{Set}^{\mathbf{T}}_{/s} \to \mathbf{Set}$ *preserves and reflects connected limits and regular epimorphisms.*

Proof. In **Set**, equivalence relations are contractible. By Theorem 3.2, equivalence relations in $\mathbf{Set}^{\mathbf{T}}_{/s}$ have coequalizer preserved by $U^{\mathbf{T}}_s$. Let $(m,n):X \rightrightarrows Y$ be an equivalence relation in $\mathrm{Set}^{T}_{/s}$ and $p:Y \to Z$ its coequalizer. $(U^T_s m, U^T_s n)$ is an equivalence relation in **Set** and Up is its coequalizer. Then $(U^T_s m, U^T_s n)$ is the kernel pair of $U^{\mathbf{T}}_s p$. Because $U^{\mathbf{T}}_s$ reflects kernel pairs (Proposition 3.3), (m, n) is the kernel pair of p. So the equivalence relations are effective in $\mathrm{Set}^{\mathbf{T}}_{/s}$ and $U^{\mathbf{T}}_s$ preserves regular epimorphisms, for any regular epimorphism is coequalizer of its kernel pair. Let $p:Y \to T$ be a morphism of $\mathbf{Set}^{\mathbf{T}}_{/s}$ such that $U^{\mathbf{T}}_s p$ is a regular epimorphism. Let $(m,n):X \rightrightarrows Y$ be the kernel pair of $p, q:Y \to Z$ the coequalizer of (m,n) and $k:Z \to T$ the unique morphism satisfying $kq=p$. Then $(U^{\mathbf{T}}_s m, U^{\mathbf{T}}_s n)$ is the kernel pair of $U^{\mathbf{T}}_s p$ and also of $U^{\mathbf{T}}_s q$ for $U^{\mathbf{T}}_s q$ is coequalizer of $(U^{\mathbf{T}}_s m, U^{\mathbf{T}}_s n)$. It follows that $U^{\mathbf{T}}_s k : U^{\mathbf{T}}_s Z \to U^{\mathbf{T}}_s T$ is isomorphic. Because $U^{\mathbf{T}}_s$ reflects isomorphisms, k is isomorphic and $p \simeq q$ is a regular epimorphism in $\mathbf{Set}^{\mathbf{T}}_{/s}$. Thus $U^{\mathbf{T}}_s$ reflects regular epimorphisms. For the functor $U^{\mathbf{T}}_s$ preserves and reflects fiber products (Proposition 3.3), regular epimorphisms are universal in $\mathbf{Set}^{\mathbf{T}}_{/s}$. So the category $\mathbf{Set}^{\mathbf{T}}_{/s}$ is regular in the sense of M. Barr [2]. The category $\mathbf{Set}_{/s}$ being well-powered, so is the category $\mathbf{Set}^{\mathbf{T}}_{/s}$ [10]. Let $f:X \to Y$ and $g:X \to Z$ two regular epimorphisms in $\mathbf{Set}^{\mathbf{T}}_{/s}$ such that $U^{\mathbf{T}}_s f$ and $U^{\mathbf{T}}_s g$ are isomorphic i.e. there is an isomorphism $h: U^{\mathbf{T}}_s Y \to U^{\mathbf{T}}_s Z$ satisfying $h(U^{\mathbf{T}}_s f) = U^{\mathbf{T}}_s g$. If (m, n) is the kernel pair of f, one has

$$
\begin{aligned}
U^{\mathbf{T}}_s &= (U^{\mathbf{T}}_s g)(U^{\mathbf{T}}_s m) = h(U^{\mathbf{T}}_s f)(U^{\mathbf{T}}_s m) = h(U^{\mathbf{T}}_s fm) \\
&= h(U^{\mathbf{T}}_s fn) = h(U^{\mathbf{T}}_s f)(U^{\mathbf{T}}_s n) = (U^{\mathbf{T}}_s g)(U^{\mathbf{T}}_s n) = U^{\mathbf{T}}_s (gn).
\end{aligned}
$$

Because $U^{\mathbf{T}}_s$ is faithful, one has $gm = gn$. Then there is a unique morphism $k: Y \to Z$ satisfying $kf = g$. The relation $U^{\mathbf{T}}_s g = U^{\mathbf{T}}_s (kf) = (U^{\mathbf{T}}_s k)(U^{\mathbf{T}}_s f)$ implies $U^{\mathbf{T}}_s k = h$. It follows that k is isomorphic. The category **Set** being cowell-powered, one deduces that $\mathbf{Set}^{\mathbf{T}}_{/s}$ is regularly cowell-powered. There remains to show that the pairs of morphisms $(f,g):X \rightrightarrows Y$ of $\mathbf{Set}^{\mathbf{T}}_{/s}$ which are coequalizable, have a coequalizer. Let $(q_i: Y \to Z_i)_{i \in I}$ be a representative set of regular quotients of Y. Let $I' = \{i \in I : q_i f = q_i g\}$. The set I' is not empty because there is a morphism $h: Y \to Z$ which coequalizes (f, g) and which factorizes, then, in the form $h = rq_i$ with $i \in I$ and r monomorphic, so $i \in I'$. Because $\mathbf{Set}^{\mathbf{T}}_{/s}$ has connected limits, the kernel pair $(m,n): T \rightrightarrows Y$ of the family $(q_i)_{i \in I'}$ exists. It is an equivalence relation. Let $q: Y \to Z$ be the coequalizer of (m, n). There is a unique morphism $k: X \to T$ such that $mk = f$ and $nk = g$. The relation $q_i n = q_i m$ implies the existence of a morphism $r_i: Z \to Z_i$ satisfying $r_i q = q_i$. Let us show that $q: Y \to Z$ is coequalizer of (f, g). One has

$qf = qmk = qnk = qg$. If $h: Y \to R$ coequalizes (f, g), it is of the form $h = rq_i$ with r monomorphic and $i \in I'$, then q_i coequalizes (f, g) and h factorizes through q in a unique way.

4.1. Theorem. *A functor $U : \mathbf{A} \to \mathbf{Set}$ is multimonadic if and only if*
(1) *it has a left multiadjoint,*
(2) *it reflects isomorphisms,*
(3) *$\mathbf{A}$ has kernel pairs,*
(4) *equivalence relations in $\mathbf{A}$ have coequalizers preserved by U.*

Proof. It is the Theorem 3.2 for $\mathbf{B} = \mathbf{Set}$.

4.2. Theorem. *A functor $U : \mathbf{A} \to \mathbf{Set}$ is multimonadic if and only if*
(1) *it has a left multiadjoint,*
(2) *it reflects isomorphisms,*
(3) *$\mathbf{A}$ has kernel pairs,*
(4) *equivalence relations in $\mathbf{A}$ are effective,*
(5) *U preserves regular epimorphisms.*

Proof. The conditions are necessary by Proposition 4.0. They imply that U preserves coequalizers of equivalence relations for if (m, n) is an equivalence relation with coequalizer p, then (m, n) is the kernel pair of p, thus (Um, Un) is the kernel pair of Up and, because Up is a regular epimorphism, Up is coequalizer of (Um, Un).

4.3. Remarks. (a) It is easy to see that the preceding results remain valid, if instead of **Set**, one takes a category **B** with kernel pairs, effective equivalence relations and whose regular epimorphisms are split. For example, $\mathbf{B} = \mathbf{Set}^I$ with I a set.

(b) The functor $P^I : \mathbf{Set}^I \to \mathbf{Set}$ defined by $P^I((E_i)_{i \in I}) = \coprod_{i \in I} E_i$ is a discrete fibration. It is thus multimonadic. Moreover any category which is multimonadic on $\mathbf{Set}^I$ is multimonadic on **Set**.

5. Examples

5.0. Locally compact spaces

Denote **Locomp** the category of locally compact (Hausdorff) spaces and proper continuous maps i.e continuous maps such that the inverse image of any compact subset is compact. The forgetful functor $U : \mathbf{Locomp} \to \mathbf{Set}$ has a left multiadjoint because it is the composite of the functor $\mathbf{Locomp} \to \mathbf{Complreg}$ which has a left multiadjoint [4, Section 10] with the forgetful functor $\mathbf{Complreg} \to \mathbf{Set}$ which has a left adjoint. We claim that the functor $U : \mathbf{Locomp} \to \mathbf{Set}$ is multimonadic, for it satisfies the conditions of Theorem 4.1. U reflects isomorphisms for if E, F are locally compact spaces and $f:E\to F$ is a proper continuous map which is bijective, then f is a closed map and thus is an homeomorphism. **Locomp** has connected limits [4, Section 10], in particular kernel pairs. Let $(m,n):R\rightrightarrows E$ be an equivalence relation in **Locomp**. The pair $(Um, Un): UR\rightrightarrows UE$ is an equivalence relation in **Set**. If X is a compact subset of E, the saturated subset $\tilde{X}$ of X for the equivalence relation, is compact for $\tilde{X}=(m(n^{-1}(X))$ with n, m proper continuous maps. It follows that the quotient E/R is locally compact and that the canonical map $p:E\to E/R$ is proper continuous [3, Chapter I, Section 10, Prop. 17]. It follows immediately that p is the coequalizer of (m,n) in **Locomp**. So the equivalence relations in **Locomp** have coequalizer preserved by U.

The spectrum of a set E relatively to U is, by [4, Section 10], the set $S(E)$ of sets $\mathcal{K}$ of subsets of E such that

(1) $\emptyset\in\mathcal{K}$,
(2) $\forall X\in\mathcal{K}, \forall Y\subset X, Y\in\mathcal{K}$,
(3) $\forall X\in\mathcal{K}, \forall Y\in\mathcal{K}, X\cup Y\in\mathcal{K}$,
(4) $\forall x\in E, \{x\}\in\mathcal{K}$.

The spectrum relative to U is the functor $S:\mathbf{Set}^{op}\to\mathbf{Set}$ which, to any E assigns $S(E)$ and, to any $f:E\to E'$ assigns the map $S(f):S(E')\to S(E)$ defined by $S(f)(\mathcal{K})=\{X\subset E: f(X)\in\mathcal{K}\}$.

The category $\mathbf{Set}_{/s}$ has S-sets for objects i.e. pairs $(E,\mathcal{K})$ of a set E and a member $\mathcal{K}$ of $S(E)$, and has S-morphisms for morphisms $(E,\mathcal{K})\to(E',\mathcal{K}')$ i.e. maps $f:E\to E'$ such that: $\forall X\subset E, (X\in\mathcal{K}\Leftrightarrow f(X)\in\mathcal{K}')$.

The functor $V:\mathbf{Locomp}\to\mathbf{Set}_{/s}$ assigns to a locally compact space E, the pair $(E,\mathcal{K}_E)$ where $\mathcal{K}_E$ is the set of relatively compact subsets of E, and, to a proper continuous map the underlying map. The functor $F:\mathbf{Set}_{/s}\to\mathbf{Locomp}$ left adjoint to V, assigns to the S-set $(E,\mathcal{K})$ and locally compact space $E_{\mathcal{K}}=\{\mathcal{U}\in\beta(E):\mathcal{U}\cap\mathcal{K}\neq\emptyset\}$ = set of ultrafilters which meet $\mathcal{K}$, equipped with the topology induced by the topology of $\beta(E)$ [4], and to a S-map $f:(E,\mathcal{K})\to(E',\mathcal{K}')$ assigns the proper continuous map $Ff:E_{\mathcal{K}}\to E_{\mathcal{K}'}$ given by $Ff(\mathcal{U})=\{Y\subset E': f^{-1}(Y)\in\mathcal{U}\}$.

Let $\mathbf{T}=(T,\eta,\mu)$ be the monad on $\mathbf{Set}_{/s}$ generated by the adjunction (F,V). The functor $T:\mathbf{Set}_{/s}\to\mathbf{Set}_{/s}$ assigns to $(E,\mathcal{K})$ the pair $T(E,\mathcal{K})=(E_{\mathcal{K}},\mathcal{L}_{\mathcal{K}})$ where $\mathcal{L}_{\mathcal{K}}$ is the set of relatively compact subsets of $E_{\mathcal{K}}$ i.e. subsets $\mathcal{X}$ of $E_{\mathcal{K}}$ such that: $\forall\mathcal{U}\in\beta(E)$, $(\cap\mathcal{X}\subset\mathcal{U}\Rightarrow\mathcal{U}\in E_{\kappa})$. The natural transformation $\eta_{E,\mathcal{K}}:E\to E_{\mathcal{K}}$ assigns to x, the principal ultrafilter (x) generated by x. We identify E with a subset of $E_{\mathcal{K}}$ par $\eta_{E,\mathcal{K}}$. The natural transformation $\mu_{E,\mathcal{K}}:(E_{\mathcal{K}})_{\mathcal{L}_{\mathcal{K}}}\to E_{\mathcal{K}}$ is given by

$$\mu_{E,\mathcal{K}}(\mathcal{V})=\{X\subset E:\{\mathcal{U}\in\beta(E):X\in\mathcal{U}\}\in\mathcal{V}\}.$$

The multimonad generated by U is then $(S,\mathbf{T})$. A $S\mathbf{T}$-algebra is a triple $(E,\mathcal{K},\lambda)$ of a set E, a S-structure $\mathcal{K}$ on E and a map $\lambda:E_{\mathcal{K}}\to E$ such that:

(1) $\forall x\in E, \lambda((x))=x$,
(2) $\forall\mathcal{X}\subset E_{\mathcal{K}}(\lambda(\mathcal{X})\in\mathcal{K}\Leftrightarrow\forall\mathcal{U}\in\beta(E)(\cap\mathcal{X}\subset\mathcal{U}\Rightarrow\mathcal{U}\in E_{\mathcal{K}}))$,
(3) $\forall\mathcal{V}\in(E_{\mathcal{K}})_{\mathcal{L}_{\mathcal{K}}}, \lambda\{\lambda(\mathcal{X}):\mathcal{X}\in\mathcal{V}\}=\lambda\{X\subset E:\{\mathcal{U}\in\beta(E):X\in\mathcal{U}\}\in\mathcal{V}\}$.

A $S\mathbf{T}$-homomorphism $(E, \mathcal{K}, \lambda) \to (E', \mathcal{K}', \lambda')$ is a map $f: E \to E'$ such that

(1) $\forall X \subset E (X \in \mathcal{K} \Leftrightarrow f(X) \in \mathcal{K}')$,

(2) $\forall \mathcal{U} \in E_{\mathcal{K}}, f(\lambda(\mathcal{U})) = \lambda'\{Y \subset E' : f^{-1}(Y) \in \mathcal{U}\}$.

They constitute the category $\mathbf{Set}^{\mathbf{T}}_{/s}$.

The comparison functor $W : \mathbf{Locomp} \to \mathbf{Set}^{\mathbf{T}}_{/s}$ assigns to any locally compact space E, the triple $(E, \mathcal{K}_E, \lambda_E)$ where $\mathcal{K}_E$ is the set of relatively compact subsets of E and $\lambda_E : E_{\mathcal{K}_E} \to E$ is the map given by $\lambda_E(\mathcal{U}) = \lim_E \mathcal{U}$. This functor is an equivalence of categories. The quasi-inverse functor assigns to a $S\mathbf{T}$-algebra $(E, \mathcal{K}, \lambda)$ the locally compact topological space E, the topology of which is given by the closure $\bar{X} = \{\lambda_E(\mathcal{U}) : \mathcal{U} \in E_{\mathcal{K}} \text{ and } X \in \mathcal{U}\}$, for any $X \subset E$.

5.1. Complete ordered sets

An ordered set is *complete* if any nonempty upper bounded subset has a supremum. If E, F are complete ordered sets, a map $f: E \to F$ is *sup-continuous* if it preserves the suprema of nonempty subsets. It is *proper sup-continuous* if, moreover the inverse image of any upper bounded subset is upper bounded. Complete ordered sets and proper sup-continuous maps form the category **Ordcompl**. The forgetful functor $U : \mathbf{Ordcompl} \to \mathbf{Set}$ has a left multiadjoint for it is the composite of the functor $\mathbf{Ordcompl} \to \mathbf{Ord}$ which has a left multiadjoint [4, Section 11] with the forgetful functor $\mathbf{Ord} \to \mathbf{Set}$ which has a left adjoint.

The functor $U : \mathbf{Ordcompl} \to \mathbf{Set}$ is multimonadic. We shall use Theorem 4.1 for the proof. U reflects isomorphisms because if $f: E \to F$ is a bijective proper sup continuous map, then for any nonempty upper bounded subset Y of F, $f^{-1}(Y)$ is also nonempty upper bounded and $f(\sup_E f^{-1}(Y)) = \sup_F(ff^{-1}(Y)) = \sup_F Y$, and thus, $\sup_E(f^{-1}(Y)) = f^{-1}(\sup_F Y)$, so f^{-1} is sup-continuous and also proper sup-continuous. The category **Ordcompl** has connected limits [4, Section 11], and thus has kernel pairs. Let $(m, n) : R \rightrightarrows E$ be an equivalence relation in **Ordcompl**. One may suppose that the set R is an equivalence relation on E in the usual sense and that $m(x, y) = x$ and $n(x, y) = y$. For $(x, x') \in R$ and $(y, y') \in R$, one has: $\sup_R\{(x, x'), (y, y')\}$ exists $\Leftrightarrow \sup_E(x, y)$ exists $\Leftrightarrow \sup_E(x', y')$ exists. For any $x \in E$, denote $\bar{x}$ the equivalence class of x modulo R. Let us prove that $\bar{x}$ has a maximum in E. Because m is proper sup-continuous, $\sup_R m^{-1}(x) = (x_0, x_1)$ exists. Then

$$x_0 = m(x_0, x_1) = m(\sup_R m^{-1}(x)) = \sup_E\{x\} = x$$

and

$$x_1 = n(x_0, x_1) = n(\sup_R m^{-1}(x)) = \sup_R(nm^{-1}(x)) = \sup_E \bar{x}.$$

The relation $(x_0, x_1) \in R$ implies $x_1 \in \bar{x}$ and thus $x_1 = \max_E \bar{x}$. Denote F the set of maxima of equivalence classes modulo R. It is a representative set of equivalence classes; we consider it as the quotient set of E by R by denoting $p : E \to F$ the map given by $p(x) = \max_E \bar{x}$. Let us equip F with the order induced by the order of E. Let us show that the map p is order-preserving. If $x \leq y$ in E, then $\sup_R\{(x, p(x)), (y, p(y))\}$ exists and is $(y, \sup_E\{p(x), p(y)\})$. Thus $\sup_E\{p(x), p(y)\} \in$

$\bar{y}$ and $\sup_E\{p(x), p(y)\} \leqslant p(y)$ so $\sup_E\{p(x), p(y)\} = p(y)$ i.e. $p(x) \leqslant p(y)$. The order preserving map $p : E \to F$ is left adjoint to the inclusion map $F \to E$, because, for any $x \in E$ and $y_0 \in F$,

$$x \leqslant y_0 \Leftrightarrow \max_E(\bar{x}) \leqslant \max_E(\bar{y}_0) \Leftrightarrow \max_E(\bar{x}) \leqslant y_0 \Leftrightarrow p(x) \leqslant y_0.$$

It follows that p is sup-continuous. It immediately follows that p is proper, that F is complete and that p is the coequalizer of (m, n) in **Ordcompl**. So the equivalence relations in **Ordcompl** have coequalizer preserved by U. Thus, U is multimonadic.

The spectrum of a set E relatively to U is, by [4, Section 11], the set $S(E)$ of sets $\mathcal{M}$ of subsets of E, such that:

(1) $\emptyset \notin \mathcal{M}$,

(2) $\forall X \in \mathcal{M}, \forall Y \subset E\ (\emptyset \neq Y \subset X \Rightarrow Y \in \mathcal{M})$,

(3) $\forall x \in E, \{x\} \in \mathcal{M}$.

The spectrum relative to U is the functor $S : \mathbf{Set}^{op} \to \mathbf{Set}$ which to E assigns $S(E)$ and to $f : E \to E'$ assigns the map $S(f) : S(E') \to S(E)$ defined by

$$S(f)(\mathcal{M}') = \{X \subset E : f(X) \in \mathcal{M}'\}.$$

The category $\mathbf{Set}_{/s}$ has S-sets for objects i.e. pairs $(E, \mathcal{M})$ of a set E and a member $\mathcal{M}$ of $S(E)$, and has for morphisms $(E, \mathcal{M}) \to (E', \mathcal{M}')$ the maps $f : E \to E'$ such that: $\forall X \subset E (X \in \mathcal{M} \Leftrightarrow f(X) \in \mathcal{M}')$.

The functor $V : \mathbf{Ordcompl} \to \mathbf{Set}_{/s}$ assigns to a complete ordered set E, the set E equipped with the set of its nonempty upper bounded subsets and to a proper sup-continuous map, the underlying map. The functor $F : \mathbf{Set}_{/s} \to \mathbf{Ordcompl}$, left adjoint to V, assigns to any S-set $(E, \mathcal{M})$, the set $\mathcal{M}$ ordered by inclusion and to any S-map $f : (E, \mathcal{M}) \to (E', \mathcal{M}')$ the proper sup-continuous map $Ff : \mathcal{M} \to \mathcal{M}'$ given by $Ff(X) = f(X)$.

If $\mathbf{T} = (T, \eta, \mu)$ is the monad generated by the adjunction (F, V), the functor $T : \mathbf{Set}_{/s} \to \mathbf{Set}_{/s}$ is defined by $T(E, \mathcal{M}) = (\mathcal{M}, \mathcal{M}_{\mathcal{M}})$ where $\mathcal{M}_{\mathcal{M}}$ is the set of nonempty upper bounded subsets of $\mathcal{M}$, the map $\eta_{E,\mathcal{M}} : E \to \mathcal{M}$ is defined by $\eta_{E,\mathcal{M}}(x) = \{x\}$ and the map $\mu_{E,\mathcal{M}} : \mathcal{M}_{\mathcal{M}} \to \mathcal{M}$ by $\mu_{E,\mathcal{M}}(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} X$. The multimonad generated by U is then $(S, \mathbf{T})$.

A $S\mathbf{T}$-algebra is a triple $(E, \mathcal{M}, \sigma)$ of a set E, a S-structure $\mathcal{M}$ on E and a map $\sigma : E_{\mathcal{M}} \to E$ such that

(1) $\forall x, \sigma(\{x\}) = x$,

(2) $\forall \mathcal{X} \subset \mathcal{M}, \bigcup_{X \in \mathcal{X}} X \in \mathcal{M} \Leftrightarrow \{\sigma(X) : X \in \mathcal{X}\} \in \mathcal{M}$,

(3) and then, $\sigma(\bigcup_{X \in \mathcal{X}} X) = \sigma\{\{\sigma(X) : X \in \mathcal{X}\}\}$.

A $S\mathbf{T}$-homomorphism $(E, \mathcal{M}, \sigma) \to (E', \mathcal{M}', \sigma')$ is a map $f : E \to E'$ such that

(1) $\forall X \subset E (X \in \mathcal{M} \Leftrightarrow f(X) \in \mathcal{M}')$,

(2) $\forall X \in \mathcal{M}, f(\sigma(X)) = \sigma'(f(X))$.

They constitute the category $\mathbf{Set}^{\mathbf{T}}_{/s}$.

The comparison functor $W : \mathbf{Ordcompl} \to \mathbf{Set}^{\mathbf{T}}_{/s}$ assigns to a complete ordered set E, the triple $(E, \mathcal{M}_E, \sigma_E)$ where $\mathcal{M}_E$ is the set of nonempty upper bounded subsets of E and $\sigma_E : \mathcal{M}_E \to E$ is the map given by $\sigma_E(X) = \sup_E X$. This functor is an equivalence of categories. The quasi-inverse functor assigns to a $S\mathbf{T}$-algebra $(E, \mathcal{M}, \sigma)$ the set E equipped with the order given by: $x \leq y \Leftrightarrow (\{x, y\} \in \mathcal{M}$ and $\sigma\{x, y\} = y)$.

5.2. Local rings

Locc is the category of (commutative unitary) local rings and local homomorphisms. The forgetful functor $U:\mathbf{Locc} \to \mathbf{Set}$ has a left multiadjoint as the composite of the functor $\mathbf{Locc} \to \mathbf{CRng}$ which has a left multiadjoint [4, 6.5.4] with the forgetful functor $\mathbf{CRng} \to \mathbf{Set}$ which has a left adjoint. Let us show that U is multimonadic. It reflects isomorphism for a local homomorphism which is bijective is an isomorphism. The category **Locc** having connected limits [4, Section 5.4] has kernel pairs. Let $(m, n): R \rightrightarrows A$ be an equivalence relation in **Locc**. One may consider that the set UR is an equivalence relation on UA in the usual sense and that $m(x, y) = x$ and $n(x, y) = y$. One knows that the quotient set UA/UR has a ring structure denoted B, and that the canonical map $p: A \to B$ is a ring homomorphism. We claim that p is local. The equivalence relation R satisfies the property:

$$\forall (x, y) \in R, \quad (x, y) \text{ invertible} \Leftrightarrow x \text{ invertible} \Leftrightarrow y \text{ invertible}.$$

Let $x \in A$ such that $p(x)$ is invertible. There is an element y of A such that $p(y)p(x) = 1$ and then $p(yx) = 1$. The element $(yx, 1)$ of R is then invertible. Thus yx is invertible and so is x. The ring B is then local because (x invertible) or ($1-x$ invertible) implies ($p(x)$ invertible) or ($1-p(x)$ invertible). It is then easy to see that p is coequalizer of (m, n) in **Locc**. So equivalence relations in **Locc** have coequalizers preserved by U.

The spectrum of a set E relatively to U is, by [4, Section 5.4], the set $S(E)$ of prime ideals of the ring $\mathbf{Z}[\langle x\rangle]_{x \in E}$ of polynomials of variables $\langle x\rangle$ with x running through E and with coefficients in $\mathbf{Z}$. The spectrum relative to U is the functor $S:\mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ which assigns to E the set $S(E)$ and to $f: E \to E'$ the map $S(f): S(E') \to S(E)$ given by

$$S(f)(I') = \{P(\langle x_1\rangle, \ldots, \langle x_n\rangle): P(\langle f(x_1)\rangle, \ldots, \langle f(x_n)\rangle) \in I'\}.$$

The category $\mathbf{Set}_{/s}$ has S-sets for objects i.e. pairs (E, I) of a set E and a prime ideal of $\mathbf{Z}[\langle x\rangle]_{x\in E}$, and has for morphisms $(E, I) \to (E', I')$ the maps $f: E \to E'$ such that:

$$P(\langle x_1\rangle, \ldots, \langle x_n\rangle) \in I \Leftrightarrow P(\langle f(x_1)\rangle, \ldots, \langle f(x_n)\rangle) \in I'.$$

In the sequel, let us denote $\mathbf{Z}[E]$ the ring $\mathbf{Z}[\langle x\rangle]_{x\in E}$, $\mathbf{Z}(E)$ the field of fractions of $Z[E]$ and, for any prime ideal I of $\mathbf{Z}[E]$, $\mathbf{Z}[E]_I$ the localized ring of $\mathbf{Z}[E]$ at I i.e. the ring of rational fractions

$$R(\langle x_1\rangle, \ldots, \langle x_n\rangle) = P(\langle x_1\rangle, \ldots, \langle x_n\rangle)/Q(\langle x_1\rangle, \ldots, \langle x_n\rangle) \quad \text{with } Q \notin I.$$

The functor $V:\mathbf{Locc} \to \mathbf{Set}_{/s}$ assigns, to a local ring A, the pair (A, I) where $I = \{P(\langle x_1\rangle, \ldots, \langle x_n\rangle) \in \mathbf{Z}[A]: P(x_1, \ldots, x_n)$ is non invertible in $A\}$ and, to a local homomorphism, the underlying map. The functor $F:\mathbf{Set}_{/s} \to \mathbf{Locc}$ left adjoint to V, assigns to the S-set (E, I) the local ring $F(E, I) = \mathbf{Z}[E]_I$, and to the S-map $f:(E, I) \to (E', I')$ the map $Ff:(E, I) \to F(E', I')$ given by $Ff(R\langle x_1\rangle, \ldots, \langle x_n\rangle) = R(\langle f(x_1)\rangle, \ldots, \langle f(x_n)\rangle)$.

If $\mathbf{T} = (T, \eta, \mu)$ is the monad generated by the adjunction (F, V), the functor $T: \mathbf{Set}_{/s} \to \mathbf{Set}_{/s}$ is given by $T(E, I) = (\mathbf{Z}[E]_I, I_{E,I})$ where $I_{E,I}$ is the set of polynomials belonging to $\mathbf{Z}[\mathbf{Z}[E]_I]$ the value in $\mathbf{Z}[E]_I$ of which has a numerator not belonging to I, the map $\eta_{E,I}: E \to F(E, I)$ is given by $\eta_{E,I}(x) = \langle x\rangle$ and the map $\mu_{E,I}: F(F(E, I), I_{E,I}) \to F(E, I)$ by

$$\mu_{E,I}(R(\langle S_1(\langle x_1\rangle, \ldots, \langle x_n\rangle)\rangle, \ldots, \langle S_p(\langle x_1\rangle, \ldots, \langle x_n\rangle)\rangle))$$
$$= R(S_1(\langle x_1\rangle, \ldots, \langle x_n\rangle), \ldots, S_p(\langle x_1\rangle, \ldots, \langle x_n\rangle)).$$

The multimonad generated by U is then $(S, \mathbf{T})$.

A $S\mathbf{T}$-algebra is a triple (E, I, ν) of a set E, a prime ideal I of $\mathbf{Z}[E]$ and a map $\nu: \mathbf{Z}[E]_I \to E$ such that

(1) $\forall x \in E,\ \nu(\langle x\rangle) = x$.

(2) $\forall R(\langle S_1(\langle x_1\rangle, \ldots, \langle x_n\rangle)\rangle, \ldots, \langle S_p(\langle x_1\rangle, \ldots, \langle x_n\rangle)\rangle) \in \mathbf{Z}(\mathbf{Z}[E]_I)$,

$$R(S_1(\langle x_1\rangle, \ldots, \langle x_n\rangle), \ldots, S_p(\langle x_1\rangle, \ldots, \langle x_n\rangle)) \in \mathbf{Z}[E]_I$$
$$\Leftrightarrow R(\nu(S_1(\langle x_1\rangle, \ldots, \langle x_n\rangle)), \ldots, \nu(S_p(\langle x_1\rangle, \ldots, \langle x_n\rangle))) \in \mathbf{Z}[E]_I$$

(3) In the conditions of (2)

$$\nu(R(\nu(S_1(\langle x_1\rangle, \ldots, \langle x_n\rangle)), \ldots, \nu(S_p(\langle x_1\rangle, \ldots, \langle x_n\rangle))))$$
$$= \nu(R(S_1(\langle x_1\rangle, \ldots, \langle x_n\rangle), \ldots, S_p(\langle x_1\rangle), \ldots, \langle x_n\rangle)))$$

A $S\mathbf{T}$-homomorphism $(E, I, \nu) \to (E', I', \nu')$ is a map $f: E \to E'$ which satisfies

(1) $\forall R(\langle x_1\rangle, \ldots, \langle x_n\rangle) \in \mathbf{Z}(E)$,

$$R(\langle x_1\rangle, \ldots, \langle x_n\rangle) \in \mathbf{Z}[E]_I \quad \Leftrightarrow \quad R(\langle f(x_1)\rangle, \ldots, \langle f(x_n)\rangle) \in \mathbf{Z}[E']_{I'}$$

(2) and then

$$f(\nu(R(\langle x_1\rangle, \ldots, \langle x_n\rangle))) = \nu'(R(\langle f(x_1)\rangle, \ldots, \langle f(x_n)\rangle)).$$

The comparison functor $W: \mathbf{Locc} \to \mathbf{Set}^{\mathbf{T}}_{/s}$ assigns to a local ring A, the triple (A, I_A, ν_A) where I_A is the ring of polynomials $P(\langle x_1\rangle, \ldots, \langle x_n\rangle) \in \mathbf{Z}[A]$ such that $P(x_1, \ldots, x_n)$ is not inversible in A and $\nu_A: \mathbf{Z}[A]_{I_A} \to A$ is the map given by

$$\nu_A(R(\langle x_1\rangle, \ldots, \langle x_n\rangle)) = R(x_1, \ldots, x_n).$$

This functor is an equivalence of categories.

5.3. Other multimonadic categories on Set

Locomp(σ), Locompara, Locompdis: the objects are (Hausdorff) locally compact spaces which are, respectively, σ-compact, paracompact, totally disconnected, and the morphisms are the proper continuous maps.

LocompGr: locally compact groups and proper continuous homomorphisms. Also locally compact rings, fields, modules, algebras, etc.

Ordab: ordered abelian groups and proper order-preserving homomorphisms i.e. $(x \geq 0 \Leftrightarrow f(x) \geq 0)$. Also ordered rings, fields, etc.

Loclat: local distributive lattices i.e. $(x \vee y = 1 \Rightarrow (x = 1 \text{ or } y = 1))$ and local homomorphisms $(f(x) = 1 \Rightarrow x = 1)$.

Loc: local rings not necessary commutative and local homomorphisms.

6. Idempotent multimonads

A multimonad $(S, \mathbf{T})$ is *idempotent* if the monad $\mathbf{T}$ is idempotent. They are generated by the following functors. A functor $U : \mathbf{A} \to \mathbf{B}$ is *relatively full and faithful* if, for any pair of morphisms $h : X \to Z$, $g : Y \to Z$ of $\mathbf{A}$ with the same codomain and for any morphism $m : UX \to UY$ of $\mathbf{B}$ satisfying $(Ug)m = Uh$, there exists a unique morphism $f : X \to Y$ of $\mathbf{A}$ such that $gf = h$ and $Uf = m$. A subcategory is *relatively full* if the inclusion functor is relatively full and faithful. One can easily prove the following:

(1) A discrete fibration is relatively full and faithful

(2) the composite of two relatively full and faithful functors is relatively full and faithful

(3) A functor is relatively full and faithful if its composite with a relatively full and faithful functor is relatively full and faithful.

6.0. Proposition. *A relatively full and faithful functor which has a left multiadjoint generates an idempotent multimonad and is multimonadic.*

Proof. Such a functor $U : \mathbf{A} \to \mathbf{B}$ has a canonical factorization $U = PV$, where $V : \mathbf{A} \to \mathbf{X}$ has a left adjoint F. Let us show that V is full and faithful. V is relatively full and faithful by (3) above. Let $\eta : 1_X \to VF$ and $\varepsilon : FV \to 1_{\mathbf{A}}$ be the natural transformations of the adjunction. For any object A of $\mathbf{A}$, one has $1_{VA} = (V\varepsilon_A)(\eta VA)$. Then there exists a unique morphism $\alpha_A : A \to FVA$ such that $V\alpha_A = \eta VA$ and $1_A = \varepsilon_A \alpha_A$. The relations

$$V(\alpha_A \varepsilon_A)(\eta VA) = (V\alpha_A)(V\varepsilon_A)(\eta VA) = V\alpha_A = \eta VA$$

imply $\alpha_A \varepsilon_A = 1_{FVA}$. Thus ε_A is isomorphic and so is ε. Hence V is full and faithful. The monad $\mathbf{T}$ generated by V is then idempotent and the functor V is monadic. The multimonad $(S, \mathbf{T})$ is thus idempotent and the functor U multimonadic.

6.1. Examples. The following categories are multimonadic on **Set**. Their forgetful functors are relatively full and faithful and generate idempotent multimonads.

K:	fields and ring homomorphisms,
Kc:	commutative fields and ring homomorphisms,
Kc(p):	fields of characteristic p and ring homomorphisms,
Dom:	integral domains and injective ring homomorphisms,
Red:	commutative reduced rings and injective ring homomorphisms,
Prim:	commutative primary rings and injective ring homomorphisms,
etc.	...
Kcdiff:	differential fields and differential homomorphisms,
etc.	...
Ordt:	totally ordered sets and strictly order-preserving maps,
OrdRng:	commutative totally ordered rings and strictly order-preserving maps,
OrdKc:	totally ordered commutative fields,
etc.	...
Norm(**R**):	normed real vector spaces and norm-preserving linear maps,
Ban(**R**):	Banach spaces and norm-preserving linear maps,
Norm Alg(**R**):	normed real algebras and norm-preserving homomorphisms,
Ban Alg(**R**):	Banach algebras and norm-preserving homomorphism,
etc.	...
Inn(**R**):	real inner spaces and scalar-preserving linear maps,
Hilb(**R**):	Hilbert spaces and scalar-preserving linear maps,
etc.	...
Met:	metric spaces and isometries,
Metcompl:	complete metric spaces and isometries,
etc.	...

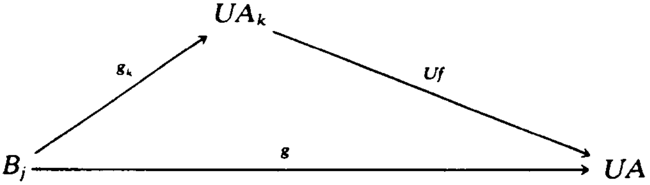
7. Multimonadic triangles.

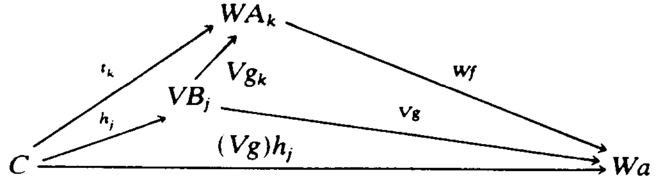
7.0. Theorem. *Let* $U:\mathbf{A}\to\mathbf{B}$, $V:\mathbf{B}\to\mathbf{C}$, $W:\mathbf{A}\to\mathbf{C}$ *be three functors such that* $W=VU$. *If* V *and* W *are multimonadic and if coequalizable pairs of morphisms in* $\mathbf{A}$ *have coequalizers, then* U *has a left multiadjoint and is multimonadic.*

Proof. For monads, one knows that any algebra is coequalizer of two morphisms between free algebras. It gives here, for the multimonadic functor V, that any object Z of $\mathbf{B}$ is coequalizer of $(m,n):X\rightrightarrows Y$ where X, Y are codomains of morphisms belonging to universal families relatively to V. So we start studying such objects.

(a) Let C be an object of $\mathbf{C}$, $(h_j:C\to VB_j)_{j\in J}$ be a universal family of morphisms from C to V and j an element of J. Let us construct a universal family of morphisms from B_j to U.

Let $(t_k:C\to WA_k)_{k\in K}$ be a universal family of morphisms from C to W. Denote I the set of $k\in K$ such that t_k factorizes (in a unique way) in the form $t_k=(Vg_k)h_j$. We



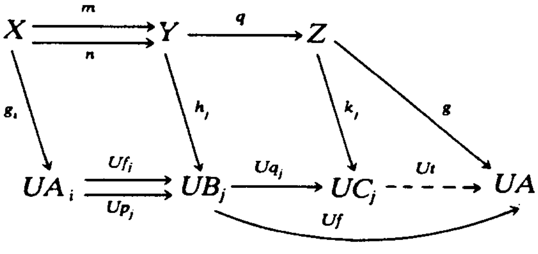


claim that $(g_k : B_j \to UA_k)_{k \in I}$ is a universal family of morphisms from B_j to U. Let $g : B_j \to UA$. The morphism $(Vg)h_j : C \to WA$ factorizes in a unique way in the form $(Vg)h_j = (Wf)t_k$ with $k \in K$. The morphism t_k is of the form $(V_{g'})h_{j'}$ with $j' \in J$. But the relation $V((Uf)g')h_{j'} = (Wf)t_k = (Vg)h_j$ implies $j' = j$ and $(Uf)g' = g$. It follows that $k \in I$ and $g' = g_k$. So g factorizes in the form $g = (Uf)g_k$. Let $k' \in I$ and $f' : A_{k'} \to A$ satisfying $g = (Uf)g_k = (Uf')g_{k'}$. Then

$$(Wf)t_k = V(Uf)(Vg_k)h_j = (Uf')(Vg_{k'})h_j = (Wf')t_{k'}$$

and thus $k = k'$ and $f = f'$.

(b) Let Z be an object of **B**. By (a) there is an object X in **B** and a universal family $(g_i : X \to UA_i)_{i \in I}$, an object Y in **B** and a universal family $(h_j : Y \to UB_j)_{j \in J}$, two morphisms $(m, n) : X \rightrightarrows Y$ and a coequalizer $q : Y \to Z$ of (m, n). Denote K the set of $j \in J$ such that there is $i \in I$ such that the morphisms $h_j m$, $h_j n$ factorize, in a necessary unique way, in the form $h_j m = (Uf_i)g_i$ and $h_j n = (Up_i)g_i$ and such that the pair $(f_i, p_i) : A_i \rightrightarrows B_j$ has a coequalizer denoted $q_j : B_j \to C_j$



The morphism $(Uq_j)h_j$ satisfying

$$(Uq_j)h_j m = (Uq_j)(Uf_i)g_i = (Uq_j)(Up_i)g_i = (Uq_j)h_j n$$

factorizes in a unique way in the form $(Uq_j)h_j = k_j q$. We claim that $(k_j : Z \to UC_j)_{j \in K}$ is a universal family of morphisms from Z to U. Let $g : Z \to UA$ be a morphism. The morphism $gq : Y \to UA$ factorizes in $gq = (Uf)h_j$ with $j \in J$. The two morphisms $h_j m$,

$h_j n : X \to UB_j$ satisfying

$$(Uf)h_j m = gqm = gqn = (Uf)h_j n$$

factorizes by a same morphism $g_i : X \to UA_i$ in the forms $h_j m = (Uf_i)g_i$ and $h_j n = (Up_i)g_i$. Then

$$U(ff_i)g_i = (Uf)h_j m = gqm = gqn = (Uf)h_j n = U(fp_i)g_i$$

implies $ff_i = fp_i$. So $j \in K$. Then $(Uf)h_j = gq$ implies the existence of a morphism $t : C_j \to A$ such that $(Ut)k_j = g$. If $j' \in K$ and $t' : C_{j'} \to A$ satisfy $g = (Ut')k_{j'}$, then

$$U(tq_j)h_j = (Ut)k_j q = (Ut')k_{j'} q = U(t'q_{j'})h_{j'}$$

implies $j = j'$ and $tq_j = t'q_{j'}$, thus $t = t'$.

(c) We have proved that U has a left multiadjoint. U reflects isomorphisms because W reflects them. If (m, n) is a pair of morphisms of **A** such that its image by U has a split coequalizer in **B**, then its image by W has a split coequalizer in **C**, thus the pair (m, n) has a coequalizer preserved by W, but also preserved by U because V reflects isomorphisms. By Theorem 3.1, U is multimonadic.

7.1. Examples. By Proposition 4.0, multimonadic categories on **Set** have coequalizer for any coequalizable pair and thus, a forgetful functor between two multimonadic categories on **Set**, is also multimonadic. With preceding notations, it is the case for the forgetful functors: **OrdRng** → **OrdAb**, **OrdAb** → **Ab**, **LocompGr** → **Gr**, **Kdiff** → **K**, **OrdKc** → **Ordt**, **Norm(R)** → **Met**, **Hilb(R)** → **Ban(R)**, etc.

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