Linear Algebra and its Applications 434 (2011) 730–740

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Bidiagonal factorizations with some parameters equal to zero

Rong Huang *,[1], Jianzhou Liu [2]

School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, Hunan, China

A R T I C L E I N F O

Article history:
Received 22 December 2009
Accepted 16 September 2010
Available online 13 October 2010

Submitted by S. Fallat

AMS classification:
15A18
15A42
15A57

Keywords:
Totally nonsingular matrices
Bidiagonal factorization

A B S T R A C T

Motivated by the results of Fiedler and Markham [2], we provide necessary and sufficient conditions for a matrix to have a bidiagonal factorization with some of the parameters of the bidiagonal factors equal to zero.

1. Introduction

Denote by $E_j(x)$ an elementary matrix obtained from I_n by changing the $(j, j-1)$th entry to x. Matrices of the form $E_j(x)$ or $E_j^T(x)$ are called *elementary bidiagonal matrices*. An n-by-n matrix A has an elementary bidiagonal factorization if it can be factorized as

* Corresponding author.
E-mail addresses: rongh98@yahoo.cn (R. Huang), liujz@xtu.edu.cn (J. Liu).

[1] The author was supported by the National Natural Science Foundation for Youths of China (Grant No. 11001233), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20094301120002), China Postdoctoral Science Foundation (Grant No. 20090451103) and the Research Fund of Education Bureau of Hunan Province (Grant No. 09C942).
[2] The author was supported by NSFC (Grant No. 10971176) and the Natural Science Foundation of Hunan Province (Grant No. 10JJ2002).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2010.09.033

$$A = \left(\prod_{k=1}^{n-1}\prod_{j=n}^{k+1} E_j(\alpha_{jk})\right) D \left(\prod_{k=n-1}^{1}\prod_{j=k+1}^{n} E_j^T(\beta_{kj})\right)$$

where D is diagonal, and the parameters α_{jk} and β_{kj} are zero or nonzero. Throughout the paper, let us denote the bidiagonal matrices

$$B_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \alpha_{n-i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & \alpha_{ni} & 1 \end{pmatrix} \quad (1)$$

and

$$C_i = \begin{pmatrix} 1 & 0 & & & & & \\ & \ddots & \ddots & & & & \\ & & 1 & 0 & & & \\ & & & 1 & \beta_{1,n-i+1} & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & \beta_{in} \\ & & & & & & 1 \end{pmatrix} \quad (2)$$

for $i = 1, \ldots, n-1$. Notice that $E_j(x)E_i(y) = E_i(y)E_j(x)$ if $|i-j| \neq 1$. Thus, the elementary bidiagonal factorization above can be easily written as

$$A = B_1 \cdots B_{n-1} D C_{n-1} \cdots C_1, \quad (3)$$

which is called the bidiagonal factorization of A. Most previous contributions in the literature have been devoted to get a bidiagonal factorization of the form (3), see [3–7]. For example, it is well known that the bidiagonal factorization always exists for a nonsingular totally nonnegative matrix [3,6]. In view of applications, the bidiagonal factorization is very useful. Given this factorization, Koev [9,10] presented new algorithms that compute the inverse, *LDU* decomposition, eigenvalues, and *SVD* of totally nonnegative matrices to high relative accuracy, independent of the conventional condition number. The idea of using bidiagonal factorizations can also be applied to solve linear systems, see [8].

Therefore, it is natural to consider necessary and sufficient conditions for a matrix to have a bidiagonal factorization of the form (3). Fiedler and Markham [1,2] first studied the interesting problem, and provided the necessary and sufficient conditions for a matrix A to have a bidiagonal factorization of the form (3) with all the parameters α_{ij} and β_{ij} nonzero. To present their main result, we next need to list some notations. Let $N = \{1, \ldots, n\}$, $N_1 \subset N$ and $N_2 \subset N$. Denote by $A(N_1|N_2)$ the submatrix of A with rows and columns indexed by N_1 and N_2, respectively. In the sequel, we only consider matrices of order n on the complex field $\mathbb{C}$.

Definition 1 [2]. Let $A \in \mathbb{C}^{n\times n}$. If all the submatrices

$$A(i-j+1, \ldots, i|1, 2, \ldots, j) \quad \text{for any } i \geqslant j$$

and

$$A(1, 2, \ldots, i|j-i+1, \ldots, j) \quad \text{for any } i \leqslant j$$

are nonsingular, then A is called totally nonsingular.

The main result by Fiedler and Markham is the following theorem.

Theorem 2 [1,2]. *Let $A \in \mathbb{C}^{n\times n}$ be nonsingular. Then A is a totally nonsingular matrix if and only if it admits a factorization of the form*

$$A = B_1 \cdots B_{n-1} D C_{n-1} \cdots C_1$$

where D is a nonsingular diagonal matrix, all subdiagonal entries α_{ij} are nonzero, and all superdiagonal entries β_{ij} are nonzero.

However, the condition that all the parameters α_{ij} and β_{ij} are nonzero can be weakened as shown by the following example.

Example 1. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}.$$

Then A can be factorized as

$$A = B_1 B_2 D C_2 C_1$$

where $D = \operatorname{diag}(1, -1, 1)$, but the parameters

$$\alpha_{31} = 0, \quad \alpha_{21} = 2, \quad \alpha_{32} = -3, \quad \beta_{13} = 0, \quad \beta_{12} = 1, \quad \beta_{23} = 0.$$

Therefore, our aim of this paper is to provide necessary and sufficient conditions for a matrix to have a bidiagonal factorization of the form (3) with some of the parameters of the bidiagonal factors equal to zero. For our main result, we first introduce a new class of matrices as follows.

Given a matrix $A \in \mathbb{C}^{n\times n}$. Consider the (i,j)-place of A with $i \geqslant j$. Let t be the maximal integer, $1 \leqslant t \leqslant j$, such that $A(i-j+t|1,2,\ldots,t-1)$ is a zero or void matrix. Then

$$A_{(ij)} = A(i-j+t,\ldots,i|t,t+1,\ldots,j)$$

is called *a relevant submatrix in the (i,j)-place* by referring to [1]. Similarly, for the (i,j)-place with $i \leqslant j$, let t be the maximal integer, $1 \leqslant t \leqslant i$, such that $A(1,2,\ldots,t-1|j-i+t)$ is a zero or void matrix. Then

$$A_{(ij)} = A(t,t+1,\ldots,i|j-i+t,\ldots,j)$$

is called *a relevant submatrix in the (i,j)-place*. In general, we call all these matrices *relevant submatrices*. For example, let

$$A = \begin{pmatrix} * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

where $*$ means the corresponding entry is nonzero. Then the relevant submatrix in the $(4,3)$-place is $A_{(43)} = A(4|3)$, and the relevant submatrix in the $(4,4)$-place is $A_{(44)} = A(3,4|3,4)$.

Definition 3. Let $A \in \mathbb{C}^{n\times n}$. If the relevant submatrices in all places of A satisfy that

$$\det A_{(ij)} \neq 0 \text{ and } i \geqslant j \Rightarrow \det A_{(sr)} \neq 0, \quad \text{whenever} \quad i \geqslant s \geqslant r \geqslant j, \tag{4}$$

and

$$\det A_{(ij)} \neq 0 \text{ and } i \leqslant j \Rightarrow \det A_{(sr)} \neq 0, \quad \text{whenever} \quad j \geqslant r \geqslant s \geqslant i, \tag{5}$$

then A is called almost totally nonsingular.

Example 2. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 \\ 3 & 8 & 17 & 4 \\ 0 & 0 & 2 & 7 \end{pmatrix}.$$

Thus, $A_{(31)} = (3)$ is nonsingular, and $A_{(21)}, A_{(11)}, A_{(32)}, A_{(22)}$ and $A_{(33)}$ are nonsingular; $A_{(43)}$ and $A_{(44)}$ are nonsingular; $A_{(12)}, A_{(23)}$ and $A_{(34)}$ are nonsingular. Therefore, it is verified that A is almost totally nonsingular.

Our main result is the following theorem.

Theorem 4. *Let $A \in \mathbb{C}^{n\times n}$ be nonsingular. Then A is almost totally nonsingular if and only if it admits a factorization of the form*

$$A = B_1 \cdots B_{n-1} D C_{n-1} \cdots C_1$$

where D is a nonsingular diagonal matrix, all subdiagonal entries α_{ij} $(i > j)$ satisfy that

$$\alpha_{ij} \neq 0 \Rightarrow \alpha_{sr} \neq 0, \quad \text{whenever} \quad i \geqslant s > r \geqslant j; \tag{6}$$

and all superdiagonal entries β_{ij} $(i < j)$ satisfy that

$$\beta_{ij} \neq 0 \Rightarrow \beta_{sr} \neq 0, \quad \text{whenever} \quad j \geqslant r > s \geqslant i. \tag{7}$$

The proof will be given in the final part.

Remark 1. Suppose that $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is almost totally nonsingular. If $A_{(n1)}$ is nonsingular, i.e, $a_{n1} \neq 0$, then we have by (4) that $a_{11} \neq 0, \ldots, a_{n-1,1} \neq 0$, and

$$A_{(ij)} = A(i-j+1, \ldots, i|1, 2, \ldots, j) \text{ is nonsingular for any } i \geqslant j;$$

if $A_{(1n)}$ is nonsingular, i.e, $a_{1n} \neq 0$, then we have by (5) that $a_{11} \neq 0, \ldots, a_{1,n-1} \neq 0$, and

$$A_{(ij)} = A(1, 2, \ldots, i|j-i+1, \ldots, j) \text{ is nonsingular for any } i \leqslant j.$$

So A is totally nonsingular. Thus, it shows that the class of almost totally nonsingular matrices is a proper extension of the class of totally nonsingular matrices. This also means that Theorem 4 is a proper extension of Theorem 2 by Fiedler and Markham.

2. Almost totally nonsingular matrices

In this section, we will provide some results on almost totally nonsingular matrices. Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with no zero row and no zero column. Denote

$$\begin{aligned} & j_0 = 1, \text{ for } t = 1, 2, \ldots: \\ & \quad i_t = \max\left\{i | a_{i,j_{t-1}} \neq 0\right\} \\ & \quad j_t = \min\left\{j | a_{i_t+1,j} \neq 0\right\}. \end{aligned} \tag{8}$$

Analogously, denote

$$\begin{aligned} & \tilde{i}_0 = 1, \text{ for } t = 1, 2, \ldots: \\ & \quad \tilde{j}_t = \max\left\{j | a_{\tilde{i}_{t-1},j} \neq 0\right\} \\ & \quad \tilde{i}_t = \min\left\{i | a_{i,\tilde{j}_t+1} \neq 0\right\}. \end{aligned} \tag{9}$$

In the sequel, we write the index sets

$$I = \{i_1, \ldots, i_l\}, \quad J = \{j_1, \ldots, j_{l-1}\}, \quad \widetilde{I} = \{\tilde{i}_1, \ldots, \tilde{i}_{r-1}\}, \quad \widetilde{J} = \{\tilde{j}_1, \ldots, \tilde{j}_r\} \tag{10}$$

where $i_l = n$ and $\tilde{j}_r = n$. If $i_t < i_{t+1}$ for all t, then we say that the index set I is strictly increasing.

Definition 5. Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with strictly increasing index sets $I, J, \widetilde{I}$ and $\widetilde{J}$ by (10). If $a_{ij} = 0$ for all $i_k < i \leqslant i_{k+1}, 1 \leqslant j < j_k$ $(k = 1, \ldots, l-1)$; and $a_{ij} = 0$ for all $1 \leqslant i < \tilde{i}_k, \tilde{j}_k < j \leqslant \tilde{j}_{k+1}$ $(k = 1, \ldots, r-1)$, then we say A has a $(I, J; \widetilde{I}, \widetilde{J})$ zero pattern.

Theorem 6. *Suppose that a nonsingular matrix $A \in \mathbb{C}^{n\times n}$ is almost totally nonsingular. Let the index sets $I, J, \widetilde{I}$ and $\widetilde{J}$ be obtained by (8) and (9), respectively. Then A has a $(I, J; \widetilde{I}, \widetilde{J})$ zero pattern, i.e., A is of the following form*

$$\begin{bmatrix} * & & 0 & \cdots & 0 \\ * & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * & \end{bmatrix} \tag{11}$$

where $*$ means the corresponding entry is nonzero.

Proof. Since $A = (a_{ij})$ is nonsingular, by using the procedures (8) and (9), it is true to assume that the index sets $I, J, \widetilde{I}$ and $\widetilde{J}$ are of the from (10). Furthermore, both the index sets I and $\widetilde{J}$ are strictly increasing. Now assume that $a_{i'j} \neq 0$ for some $i_k < i' \leqslant i_{k+1}$ and $1 \leqslant j < j_k$ with i' minimal $(1 \leqslant k \leqslant l)$. Obviously $i' \neq i_k + 1$; otherwise $j \geqslant j_k$ by (8), a contradiction. So $i_k < i' - 1 \leqslant i_{k+1}$, and thus

$$a_{i'-1,s} = 0 \quad \text{for all} \quad 1 \leqslant s < j_k \tag{12}$$

by the minimality of i'. Now let us choose $a_{i'j'} \neq 0$ with $1 \leqslant j' < j_k$ minimal. According to the minimality of j', we have $A(i'|1, 2, \ldots, j'-1) = 0$. Thus it follows that the relevant submatrix

$$A_{(i'j')} = A(i'|j')$$

is nonsingular. Since A is almost totally nonsingular, it follows by (4) that the relevant submatrix $A_{(i'-1,j')}$ is nonsingular, which is impossible because $A_{(i'-1,j')} = 0$ by (12). Therefore, we get that $a_{ij} = 0$ for any $i_k < i \leqslant i_{k+1}$ and $1 \leqslant j < j_k$ $(k = 1, \ldots, l-1)$. Applying the same argument to A^T, we have that $a_{ij} = 0$ for any $1 \leqslant i < \tilde{i}_k$ and $\tilde{j}_k < j \leqslant \tilde{j}_{k+1}$ $(k = 1, \ldots, r-1)$.

Next we show that the index set J is strictly increasing. In fact, if we assume that there exists a k such that $j_{k-1} > j_k$ $(2 \leqslant k \leqslant l)$, according to the conclusion above, then

$$a_{i_k,1} = 0, \quad a_{i_k,2} = 0, \ldots, \quad a_{i_k j_k} = 0, \ldots, \quad a_{i_k j_{k-1}-1} = 0.$$

So the relevant submatrix $A_{(i_k j_k)} = 0$. However, since

$$A_{(i_k+1, j_k)} = A(i_k + 1|j_k)$$

is nonsingular, $A_{(i_k j_k)}$ is nonsingular because A is almost totally nonsingular, a contradiction. So the index set J is strictly increasing. Similarly, we have that the index set $\widetilde{I}$ is strictly increasing. Thus A has a $(I, J; \widetilde{I}, \widetilde{J})$ zero pattern of the form (11). $\square$

According to Theorem 6, we immediately have the following result.

Corollary 7. *Suppose that a nonsingular matrix $A \in \mathbb{C}^{n\times n}$ is almost totally nonsingular. If $A(i|1, \ldots, j) = 0$ for some $i \geqslant j$, then $A(i, \ldots, n|1, \ldots, j) = 0$; if $A(1, \ldots, i|j) = 0$ for some $i \leqslant j$, then $A(1, \ldots, i|j, \ldots, n) = 0$.*

Theorem 8. *Suppose that a nonsingular matrix $A \in \mathbb{C}^{n\times n}$ is almost totally nonsingular. Then A has a factorization $A = BDC$, where D is a nonsingular diagonal matrix, B (C) is unit lower (upper) triangular.*

Proof. Set $N_k = \{1, 2, \ldots, k\}$ for $k = 1, 2, \ldots, n$. We start the proof by proving that $A(N_k|N_k)$ is nonsingular for all k. Now assume that $A(N_t|N_t)$ is singular for some $1 \leqslant t \leqslant n$ with t minimal. Then we must have that $a_{t1} = a_{t2} = \cdots = a_{tt} = 0$; otherwise it is easy to verify by (4) that the relevant submatrix $A_{(tt)}$ is nonsingular because A is almost totally nonsingular. From that it follows that

- if $A_{(tt)} = A(N_t|N_t)$, then $A(N_t|N_t)$ is nonsingular, a contradiction;
- if $A_{(tt)} = A(r, \ldots, t|r, \ldots, t)$ $(r \leqslant t)$, then $A(r|1, \ldots, r-1) = 0$, and thus using Corollary 7 we have

  $$\det A(N_t|N_t) = \det A(N_{r-1}|N_{r-1}) \cdot \det A(r, \ldots, t|r, \ldots, t) \neq 0$$

  by considering that t is minimal, a contradiction.

So $A(t|1, \ldots, t) = 0$. However, we have from Corollary 7 that $A(t, \ldots, n|1, \ldots, t) = 0$, which implies that A is singular, a contradiction. Hence, we must have that $A(N_k|N_k)$ is nonsingular for all k, and the result follows. □

3. Bidiagonal factorizations

Theorem 9. *Let $A \in \mathbb{C}^{n\times n}$ be nonsingular. Then A is almost totally nonsingular if and only if it can be factorized as $A = BDC$, where D is a nonsingular diagonal matrix, and B (C) is a unit lower (upper) triangular and almost totally nonsingular matrix.*

Proof. First assume that A is almost totally nonsingular. According to Theorem 8, it is sufficient to show that both B and C are almost totally nonsingular. Now consider the relevant submatrix $B_{(ij)} = B(i-j+t, \ldots, i|t, t+1, \ldots, j)$ for any $i \geqslant j$. So $B(i-j+t|1, 2, \ldots, t-1) = 0$ with t maximal by the definition. Since $A = BDC$, it is not difficult to show that

$$A(i-j+t|1, 2, \ldots, t-1) = 0 \text{ with } t \text{ maximal;}$$

otherwise if there exists $t_1 > t$ such that $A(i-j+t_1|1, 2, \ldots, t_1-1) = 0$, then it is easy to show that $B(i-j+t_1|1, 2, \ldots, t_1-1) = 0$, a contradiction. Thus the relevant submatrix $A_{(ij)} = A(i-j+t, \ldots, i|t, t+1, \ldots, j)$, and

$$\begin{aligned} A(i-j+t|1, 2, \ldots, t-1) = 0 &\Rightarrow A(i-j+t, \ldots, n|1, 2, \ldots, t-1) = 0 \text{ (by Corollary 7)} \\ &\Rightarrow B(i-j+t, \ldots, n|1, 2, \ldots, t-1) = 0. \end{aligned}$$

Hence, we have

$$A_{(ij)} = B_{(ij)} D_{(j)} C_{(j)} \tag{13}$$

where $D_{(j)} = D(t, \ldots, j|t, \ldots, j)$ and $C_{(j)} = C(t, \ldots, j|t, \ldots, j)$. Therefore, since A is almost totally nonsingular, it follows from (13) that

$$\begin{aligned} \det B_{(ij)} \neq 0 \text{ and } i \geqslant j &\Rightarrow \det A_{(ij)} \neq 0 \\ &\Rightarrow \det A_{(sr)} \neq 0 \quad \text{whenever} \quad i \geqslant s \geqslant r \geqslant j \\ &\Rightarrow \det B_{(sr)} \neq 0 \quad \text{whenever} \quad i \geqslant s \geqslant r \geqslant j. \end{aligned}$$

Thus the lower triangular matrix B is almost totally nonsingular. Similarly, we have that the upper triangular matrix C is almost totally nonsingular.

Conversely, let us consider the relevant submatrix $A_{(ij)} = A(i-j+t, \ldots, i|t, t+1, \ldots, j)$ for any $i \geqslant j$. So $A(i-j+t|1, 2, \ldots, t-1) = 0$ with t maximal by the definition. Since $A = BDC$, it is not difficult to show that

$$B(i-j+t|1,2,\ldots,t-1)=0 \text{ with } t \text{ maximal};$$

otherwise if there exists $t_1 > t$ such that $B(i-j+t_1|1,2,\ldots,t_1-1)=0$, then it is easy to show that $A(i-j+t_1|1,2,\ldots,t_1-1)=0$, a contradiction. Thus the relevant submatrix $B_{(ij)} = B(i-j+t,\ldots,i|t,t+1,\ldots,j)$, and $B(i-j+t,\ldots,n|1,2,\ldots,t-1)=0$ by Corollary 7. Hence, we get

$$A_{(ij)} = B_{(ij)}D_{(j)}C_{(j)} \tag{14}$$

where $D_{(j)} = D(t,\ldots,j|t,\ldots,j)$ and $C_{(j)} = C(t,\ldots,j|t,\ldots,j)$. Therefore, since B is almost totally nonsingular, it follows from (14) that

$$\begin{aligned}\det A_{(ij)} \neq 0 \text{ and } i \geqslant j \Rightarrow \det B_{(ij)} \neq 0 & \\ \Rightarrow \det B_{(sr)} \neq 0 & \quad \text{whenever} \quad i \geqslant s \geqslant r \geqslant j \\ \Rightarrow \det A_{(sr)} \neq 0 & \quad \text{whenever} \quad i \geqslant s \geqslant r \geqslant j.\end{aligned}$$

Similarly, we have

$$\det A_{(ij)} \neq 0 \text{ and } i \leqslant j \Rightarrow \det A_{(sr)} \neq 0 \quad \text{whenever} \quad j \geqslant r \geqslant s \geqslant i.$$

So A is almost totally nonsingular. □

Example 3. Let

$$A = \begin{pmatrix} 1 & -5 & 0 & 0 \\ 7 & -33 & 4 & 0 \\ 0 & -6 & -8 & 3 \\ 0 & -12 & 0 & 27 \end{pmatrix}.$$

Then A is almost totally nonsingular. It is easy to show that

$$A = LDU = \begin{pmatrix} 1 & & & \\ 7 & 1 & & \\ 0 & -3 & 1 & \\ 0 & -6 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 4 & \\ & & & 9 \end{pmatrix} \begin{pmatrix} 1 & -5 & 0 & 0 \\ & 1 & 2 & 0 \\ & & 1 & \frac{3}{4} \\ & & & 1 \end{pmatrix},$$

where L, D and U are almost totally nonsingular, respectively.

Lemma 10. *Let $B = B_1B_2\cdots B_{n-1} = (b_{ij})$ where all subdiagonal entries α_{ij} satisfy (6). If $\alpha_{rs} = 0$, then $b_{ij} = 0$ for all $i \geqslant r$ and $j \leqslant s$.*

Proof. To prove the result, we apply induction on the order n of B. The case $n = 2$ is trivial. Now assume that the result is true for all orders less than n. Observe that

$$B_i = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & \alpha_{n-i+1,1} & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & 0 & 1 & & & \\ & & & & \alpha_{n-i+2,2} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \alpha_{ni} & 1 \end{pmatrix}$$

$$= E_{n-i+1}(\alpha_{n-i+1,1}) \begin{pmatrix} 1 & \\ & B_i' \end{pmatrix}.$$

Hence, it is not difficult to show that

$$\begin{aligned} B &= E_n(\alpha_{n1})\begin{pmatrix}1 & \\ & B_1'\end{pmatrix}E_{n-1}(\alpha_{n-1,1})\begin{pmatrix}1 & \\ & B_2'\end{pmatrix}\cdots E_2(\alpha_{21})\begin{pmatrix}1 & \\ & B_{n-1}'\end{pmatrix} \\ &= E_n(\alpha_{n1})\cdots E_2(\alpha_{21})\begin{pmatrix}1 & \\ & B_1'\end{pmatrix}\cdots\begin{pmatrix}1 & \\ & B_{n-1}'\end{pmatrix} \\ &= E_n(\alpha_{n1})\cdots E_2(\alpha_{21})\begin{pmatrix}1 & \\ & B_1'\cdots B_{n-1}'\end{pmatrix}. \end{aligned} \tag{15}$$

Set $\alpha_{rs} = 0$. Then $\alpha_{ij} = 0$ for all $i \geqslant r$ and $j \leqslant s$ because of the condition (6). Thus, if $s = 1$, obviously $b_{i1} = 0$ for all $i \geqslant r$ by (15); if $s > 1$, then $\alpha_{i1} = 0$ for all $i \geqslant r$, and thus by applying our inductive assumption, we easily conclude that the result is true by (15). □

Lemma 11. *Suppose $H = B_{n-k}\cdots B_{n-1} = (h_{ij})$ for any $1 \leqslant k \leqslant n-1$. Then H is a lower banded matrix with at most k nonzero subdiagonals, where $h_{ij} = 0$ for any $i - j > k$ and $h_{ij} = \alpha_{j+1,j}\cdots\alpha_{ij}$ for any $i - j = k$.*

Proof. To prove the result, we apply induction on k. The case $k = 1$ is trivial. Now assume that the result is true for any r with $r < k$. Then it is easy to show that the result is true if we consider the form

$$H = B_{n-k}(B_{n-(k-1)}\cdots B_{n-1})$$

by using the inductive assumption. □

Next we point out some important facts that will be used. Suppose $B = B_1\cdots B_{n-1}$ where all subdiagonal entries α_{ij} satisfy the condition (6). Now consider the relevant submatrix $B_{(rs)} = B(r - s + t, \ldots, r|t, t+1, \ldots, s)$ for any $r \geqslant s$. Then

$$B(r - s + t|1, \ldots, t-1) = 0 \text{ with } t \text{ maximal.} \tag{16}$$

Thus the following statements are true:

- Since $B(r - s + t|1, \ldots, t-1) = 0$, we must have
  $$\alpha_{r-s+t,1} = 0, \alpha_{r-s+t,2} = 0, \ldots, \alpha_{r-s+t,t-1} = 0. \tag{17}$$
  In fact, if we assume that $\alpha_{r-s+t,1} = \cdots = \alpha_{r-s+t,k-1} = 0$ and $\alpha_{r-s+t,k} \neq 0$ ($k \leqslant t-1$), then using Lemma 11 it is not difficult to show that
  $$B(r-s+t|k) = (B_{n-(r-s+t)+k}\cdots B_{n-1})(r-s+t|k) = \alpha_{k+1,k}\cdots\alpha_{r-s+t,k} \neq 0$$
  by considering (6), a contradiction.
- If $\alpha_{rs} \neq 0$, then
  $$\alpha_{r-s+t,t} \neq 0, \alpha_{r-s+t+1,t+1} \neq 0, \ldots, \alpha_{rs} \neq 0. \tag{18}$$
  In fact, if $\alpha_{r-s+i,i} = 0$ ($t \leqslant i \leqslant s$), by Lemma 10, then $B(r-s+i, \ldots, n|1, \ldots, i) = 0$. So $B(r-s+i+1|1, \ldots, i) = 0$, which contradicts (16).
- It follows that
  $$\det B_{(rs)} \neq 0 \Rightarrow \alpha_{rs} \neq 0; \tag{19}$$
  otherwise if $\alpha_{rs} = 0$, by Lemma 10, then $B(r, \ldots, n|1, \ldots, s) = 0$. So the relevant submatrix $B_{(rs)} = B(r|s) = 0$, a contradiction.

Lemma 12. *Suppose $B = B_1\cdots B_{n-1}$ where all subdiagonal entries α_{ij} satisfy the condition (6). Then B is almost totally nonsingular.*

Proof. Consider the relevant submatrix $B_{(rs)} = B(r-s+t, \ldots, r|t, t+1, \ldots, s)$ for any $r \geqslant s$. Now we show that if $\alpha_{rs} \neq 0$, then $\det B_{(rs)} \neq 0$.

Partition the matrices

$$B_k = \begin{pmatrix} B_{11}^{(k)} & 0 & 0 \\ B_{21}^{(k)} & B_{22}^{(k)} & 0 \\ B_{31}^{(k)} & B_{32}^{(k)} & B_{33}^{(k)} \end{pmatrix}, \quad k = 1, \ldots, n-r+s-1$$

where all $B_{11}^{(k)}$ are $(r-s+t-1) \times (r-s+t-1)$, and all $B_{22}^{(k)}$ are $(s-t+1) \times (s-t+1)$. Since $B(r-s+t|1, 2, \ldots, t-1) = 0$, using (17) we have

$$\begin{cases} \alpha_{r-s+t,1} = 0 \Rightarrow B_{21}^{(n-r+s-t+1)} = 0, \quad B_{31}^{(n-r+s-t+1)} = 0 \\ \alpha_{r-s+t,2} = 0 \Rightarrow B_{21}^{(n-r+s-t+2)} = 0, \quad B_{31}^{(n-r+s-t+2)} = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \alpha_{r-s+t,t-1} = 0 \Rightarrow B_{21}^{(n-r+s-1)} = 0, \quad B_{31}^{(n-r+s-1)} = 0. \end{cases}$$

Thus

$$(B_1 \cdots B_{n-r+s-t})(B_{n-r+s-t+1} \cdots B_{n-r+s-1}) = \begin{pmatrix} * & 0 & 0 \\ 0 & L & 0 \\ 0 & * & * \end{pmatrix}$$

where $L = B_{22}^{(1)} \cdots B_{22}^{(n-r+s-1)}$ is unit lower triangular. Set $H = B_{n-r+s} \cdots B_{n-1}$. Then it is easy to show that

$$B_{(rs)} = LH(r-s+t, \ldots, r|t, \ldots, s) = LU.$$

It follows by Lemma 11 that $U = (u_{ij})$ is upper triangular because H has at most $r-s$ nonzero subdiagonals, where

$$\begin{aligned} u_{ii} &= H(r-s+t+i-1|t+i-1) \\ &= \alpha_{t+i,t+i-1} \cdots \alpha_{r-s+t+i-1,t+i-1}, \quad (i = 1, \ldots, s-t+1). \end{aligned}$$

Thus, if $\alpha_{rs} \neq 0$, then using (18) and (6) we have

$$\begin{cases} \alpha_{r-s+t,t} \neq 0 \Rightarrow \alpha_{t+1,t} \neq 0, \ldots, \alpha_{r-s+t-1,t} \neq 0, \\ \alpha_{r-s+t+1,t+1} \neq 0 \Rightarrow \alpha_{t+2,t+1} \neq 0, \ldots, \alpha_{r-s+t,t+1} \neq 0, \\ \cdots\cdots\cdots\cdots \\ \alpha_{rs} \neq 0 \Rightarrow \alpha_{s+1,s} \neq 0, \ldots, \alpha_{r-1,s} \neq 0, \end{cases}$$

which means that $u_{ii} \neq 0$ for all $i = 1, \ldots, s-t+1$. Hence, we obtain that if $\alpha_{rs} \neq 0$, then

$$\det B_{(rs)} = \det L \cdot \det U = \prod_{i=1}^{s-t+1} u_{ii} \neq 0.$$

Thus, it follows by using (19) that

$$\begin{aligned} \det B_{(ij)} \neq 0 \text{ and } i \geqslant j \Rightarrow \alpha_{ij} \neq 0 \\ &\Rightarrow \alpha_{rs} \neq 0 \quad \text{whenever} \quad i \geqslant r \geqslant s \geqslant j \\ &\Rightarrow \det B_{(rs)} \neq 0 \quad \text{whenever} \quad i \geqslant r \geqslant s \geqslant j \end{aligned}$$

which means the lower triangular matrix B is almost totally nonsingular. □

Lemma 13. *Let $B \in \mathbb{C}^{n \times n}$ be a unit lower triangular matrix. If B is almost totally nonsingular, then B can be factorized as $B = B_1 \cdots B_{n-1}$ where all subdiagonal entries α_{ij} satisfy* (6).

Proof. To prove the result, we apply induction on the order n of B. The case $n = 1, 2$ is trivial. Now assume that the result is true for all orders less that n. Set $B = (b_{ij})$. Since B is almost totally nonsingular, we can assume that $b_{11} \neq 0, \ldots, b_{r1} \neq 0$, and $b_{r+1,1} = \cdots = b_{n1} = 0$. Let $\alpha_{21} = \frac{b_{21}}{b_{11}}, \ldots, \alpha_{r1} = \frac{b_{r1}}{b_{r-1,1}}$. So $\alpha_{21} \neq 0, \ldots, \alpha_{r1} \neq 0$. Thus it is not difficult to show that B can be factorized as

$$B = E_r(\alpha_{r1}) \cdots E_2(\alpha_{21}) \begin{pmatrix} 1 & \\ & B' \end{pmatrix}$$

where B' is $(n-1) \times (n-1)$. Furthermore, the $(n-1) \times (n-1)$ unit lower triangular matrix $B' = (b_{ij}^{(1)})$ satisfies the following statements:

- Since B is almost totally nonsingular, by considering that $b_{11} \neq 0, \ldots, b_{r1} \neq 0$, we obtain that $b_{11}^{(1)} = 1$,

$$\begin{aligned} b_{21}^{(1)} &= \frac{\det B(2,3|1,2)}{b_{21}} = \frac{\det B_{(32)}}{\det B_{(21)}} \neq 0, \ldots, b_{r-1,1}^{(1)} \\ &= \frac{\det B(r-1,r|1,2)}{b_{r-1,1}} = \frac{\det B_{(r2)}}{\det B_{(r-1,1)}} \neq 0. \end{aligned}$$

  Set $j \leqslant i \leqslant r-1$. Thus the relevant submatrix $B'_{(ij)} = B'(i-j+1, \ldots, i|1, \ldots, j)$, and

$$\begin{aligned} \det B'_{(ij)} &= \frac{1}{b_{i-j+1,1}} \det B(i-j+1, i-j+2, \ldots, i+1|1, 2, \ldots, j+1) \\ &= \frac{1}{b_{i-j+1,1}} \det B_{(i+1,j+1)}. \end{aligned}$$

- Set $j \leqslant i$ and $i > r-1$. Since $b_{r+1,1} = \cdots = b_{n1} = 0$, it is not difficult to show that the relevant submatrix $B'_{(ij)} = B_{(i+1,j+1)}$.

Therefore, it follows that B' is also almost totally nonsingular. Applying the inductive assumption to B', we have the factorization

$$B = E_r(\alpha_{r1}) \cdots E_2(\alpha_{21}) \begin{pmatrix} 1 & \\ & B'_1 \cdots B'_{n-1} \end{pmatrix}$$

where each $(n-1) \times (n-1)$ matrix B'_i is of the form (1) with all subdiagonal entries α_{ij} $(2 \leqslant i, j \leqslant n)$ satisfy (6). In particular, according to the argument above, we have

$$\alpha_{r2} = \frac{b_{r-1,1}^{(1)}}{b_{r-2,1}^{(1)}} \neq 0.$$

Notice that $\alpha_{r1} \neq 0$ and $\alpha_{l1} = 0$ for $l > r$. Thus we easily conclude that the result is true by using (15). □

We are now ready to prove Theorem 4.

Proof of Theorem 4. Let $A \in \mathbb{C}^{n \times n}$ be nonsingular. First assume that A is almost totally nonsingular. Theorem 9 implies that it can be factorized as $A = BDC$, where D is a nonsingular diagonal matrix, and B (C) is a unit lower (upper) triangular and almost totally nonsingular matrix. Thus the result is true by applying Lemma 13 to B and C^T. Conversely, using Theorem 9 and Lemma 12 we have that A is almost totally nonsingular. □

Acknowledgment

The authors are very grateful to the referees and Prof. S.M. Fallat for their helpful suggestions.

References

[1] M. Fiedler, T. Markham, Consecutive-column and -row properties of matrices and the Loewner–Neville factorization,Linear Algebra Appl. 266 (1997) 243–259.
[2] M. Fiedler, T. Markham, A factorization of totally nonsingular matrices over a ring with identity, Linear Algebra Appl. 304 (2000) 161–171.
[3] S.M. Fallat, Bidiagonal factorizations of totally nonnegative matrices, Amer. Math. Monthly 108 (2001) 697–712.
[4] M. Gasca, C.A. Micchelli, J.M. Peña, Almost strictly totally positive matrices, Numer. Algorithms 2 (1992) 225–236.
[5] M. Gasca, J.M. Peña, On the characterization of almost strictly totally positive matrices, Adv. Comput. Math. 3 (1995) 239–250.
[6] M. Gasca, J.M. Peña, On factorizations of totally positive matrices, Total Positivity Appl. (1996) 109–130.
[7] M. Gasca, J.M. Peña, Characterizations and decompositions of almost strictly positive matrices, SIAM J. Matrix Anal. Appl. 28 (2006) 1–8.
[8] N.J. Higham, Stability analysis of algorithms for solving confluent Vandermonde-like systems, SIAM J. Matrix Anal. Appl. 11 (1990) 23–41.
[9] P. Koev, Accurate eigenvalues and SVDs of totally nonnegative matrices, SIAM J. Matrix Anal. Appl. 27 (2005) 1–23.
[10] P. Koev, Accurate computations with totally nonnegative matrices, SIAM J. Matrix Anal. Appl. 29 (2007) 731–751.