Removable edges in a 5-connected graph and a construction method of 5-connected graphs☆

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Abstract

An edge e of a k-connected graph G is said to be a removable edge if G \( \cong e \) is still k-connected. A k-connected graph G is said to be a quasi \( (k + 1) \)-connected if G has no nontrivial k-separator. The existence of removable edges of 3-connected and 4-connected graphs and some properties of quasi k-connected graphs have been investigated [D.A. Holton, B. Jackson, A. Saito, N.C. Wormald, Removable edges in 3-connected graphs, J. Graph Theory 14(4) (1990) 465–473; H. Jiang, J. Su, Minimum degree of minimally quasi \( (k + 1) \)-connected graphs, J. Math. Study 35 (2002) 187–193; T. Politof, A. Satyanarayana, Minors of quasi 4-connected graphs, Discrete Math. 126 (1994) 245–256; T. Politof, A. Satyanarayana, The structure of quasi 4-connected graphs, Discrete Math. 161 (1996) 217–228; J. Su, The number of removable edges in 3-connected graphs, J. Combin. Theory Ser. B 75(1) (1999) 74–87; J. Yin, Removable edges and constructions of 4-connected graphs, J. Systems Sci. Math. Sci. 19(4) (1999) 434–438]. In this paper, we first investigate the relation between quasi connectivity and removable edges. Based on the relation, the existence of removable edges in k-connected graphs \( (k \geq 5) \) is investigated. It is proved that a 5-connected graph has no removable edge if and only if it is isomorphic to \( K_6 \). For a k-connected graph G such that end vertices of any edge of G have at most \( k - 3 \) common adjacent vertices, it is also proved that G has a removable edge. Consequently, a recursive construction method of 5-connected graphs is established, that is, any 5-connected graph can be obtained from \( K_6 \) by a number of \( \theta^+ \)-operations. We conjecture that, if k is even, a k-connected graph G without removable edge is isomorphic to either \( K_{k+1} \) or the graph \( H_{k/2+1} \) obtained from \( K_{k+2} \) by removing \( k/2 + 1 \) disjoint edges, and, if k is odd, G is isomorphic to \( K_{k+1} \).

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1. Introduction

Graph theoretic terminology used here generally follows that of Bondy [4]. We consider only finite and simple graphs.

Connectivity of graphs is a fundamental topic in graph theory research. For properties and constructions of several classes of k-edge-connected graphs and k-connected graphs, many investigations have been made [9,14–17,19,20,24,25].

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A construction method for minimally \( k \)-edge-connected graphs was given by Habib [9]. A construction method for critically \( k \)-edge-connected graphs was given by Zhang et al. [32]. For \( k \)-connected graphs, Tutte [27,28] gave some construction methods for 2-connected graphs and 3-connected graphs, and Dirac [6] gave a construction method for minimally 2-connected graphs. In Ref. [31], Zhang and Guo investigated reducible chains of several classes of 2-connected graphs, and gave a construction method for minimally, critically, and critically and minimally 2-connected graphs. Mader, Maure, Slater, and Su [16,19,20,24] investigated properties of \( k \)-critical \( n \)-connected graphs and \( k \)-minimally \( n \)-edge-connected graphs. Contractible edges in \( k \)-connected graphs and properties of contractible-critical graphs are investigated by Egawa, Enomoto, Kawarabayashi, Ando, Su, and Yuan et al. [1,2,7,8,12,13,18,25,30].

For removable edges of \( k \)-connected graphs, Holton et al. [10] first defined removable edges in a 3-connected graph. Later, Yin [29] defined removable edges in a 4-connected graph. The concept of removable edges in a 3-connected graph and a 4-connected graph can be generalized to \( k \)-connected graphs.

**Definition 1.** Let \( G \) be a \( k \)-connected graph, and let \( e \) be an edge of \( G \). Let \( G \cup e \) denote the graph obtained from \( G \) by the following operation: (1) delete \( e \) from \( G \) to get \( G - e \); (2) for any end vertex of \( e \) with degree \( k - 1 \) in \( G - e \), say \( x \), delete \( x \) and then add edges between any pair of non-adjacent vertices in \( N_{G - e}(x) \). If \( G \cup e \) is \( k \)-connected, then \( e \) is said to be a removable edge of \( G \), otherwise \( e \) is said to be non-removable. The set of all non-removable edges of \( G \) and the set of all removable edges of \( G \) are denoted by \( E_N(G) \) and \( E_R(G) \), respectively.

Barnette and Grünbaum [3] proved that a 3-connected graph of order at least five has a removable edge. Based on the above graph operation and fact, a constructive characterization of minimally 3-connected graphs was given by Dawes [5], which differs from the characterization provided by Tutte [27]. The graph \( C_n^2 \), for an integer \( n \geq 4 \), is defined as follows. Let \( C_n = v_1v_2 \cdots v_nv_1 \) be an \( n \)-cycle. Then \( C_n^2 \) is obtained from \( C_n \) by adding edges \( uv \), satisfying that \( j \equiv i \pm 2 \mod n \), for each \( 1 \leq i \leq n \).

In [29], Yin also proved that the graph without removable edge is either \( C_5^2 \) or \( C_6^2 \). Based on this result, he provided a constructive characterization of 4-connected graphs, which is simpler than Slater’s method [23].

On the other hand, Politof and Satyanarayana [21,22] introduced the concept of quasi-4-connected graphs and investigated their structure and properties. Jiang and Su [11] further investigated some properties of quasi \( k \)-connected graphs.

Let \( S \) be a vertex cut set of a graph \( G \) with \( |S| = k \) \((k \geq 2)\). \( S \) is said to be a nontrivial \( k \)-separator of \( G \), if the set of the components of \( G - S \) can be partitioned into two sets, each of which has to contain at least two vertices. A \( k \)-connected graph is quasi-(\( k + 1 \))-connected if it has no nontrivial \( k \)-separator. Clearly, every \( k \)-connected graph is quasi-\( k \)-connected. A quasi-\( k \)-connected graph \( G \) is minimally quasi-\( k \)-connected if \( G - uv \) is not quasi-\( k \)-connected for all \( uv \in E(G) \).

For the removable edges, non-removable edges, and quasi connectivity of a graph \( G \), the following results are given in [10,11,26,29].

**Theorem 1** (Holton et al. [10]). Let \( G \) be a 3-connected graph of order at least six and \( e = xy \in E(G) \). Then \( e \) is non-removable if and only if there exists \( S \subseteq V(G) \) with \( |S| = 2 \) such that \( G - e - S \) has exactly two components \( A, B \) with \( |A| \geq 2 \) and \( |B| \geq 2 \), moreover \( x \in A, y \in B \).

**Theorem 2** (Yin [29]). Let \( G \) be a 4-connected graph of order at least seven and \( e = xy \in E(G) \). Then \( e \) is non-removable if and only if there exists \( S \subseteq V(G) \) with \( |S| = 3 \) such that \( G - e - S \) has exactly two components \( A, B \) with \( |A| \geq 2 \) and \( |B| \geq 2 \), moreover \( x \in A, y \in B \).

**Theorem 3** (Yin [29]). A 4-connected graph without removable edge is either \( C_5^2 \) or \( C_6^2 \).

**Theorem 4** (Politof and Satyanarayana [22]). If \( G \) is minimally quasi 4-connected, then \( \delta(G) = 3 \).

**Theorem 5** (Jiang and Su [11]). If \( G \) is minimally quasi 5-connected, then \( \delta(G) = 4 \).

Let \( G \) be a minimally quasi-(\( k + 1 \))-connected graph. Let \( xy \) be any edge in \( G \), let \( S \subseteq V(G) \) be a minimum vertex cut of \( G - xy \), and let \( A \) and \( B \) be two connected components of \( G - xy - S \). Then \( A \) and \( B \) are called \((xy, S)\)-fragments of \( G \).
A \((xy, S)\)-fragment of \(G\) is called a \((xy, S)\)-atom of \(G\) if \(A\) has the minimum cardinal number in all \((xy, S)\)-fragments of \(G\).

**Theorem 6** (Jiang and Su [11]). Let \(G\) be a minimally quasi-\((k + 1)\)-connected graph with \(\delta(G) = k + 1\), and let \(A\) be a \((xy, S)\)-atom of \(G\). Then \(|A| = 2\). Let \(A = \{x, y\}\), then \(xy \in E(G)\), \(d_G(x) = d_G(y) = k + 1\), and \(|N_G(x) \cap N_G(y)| = k - 1\).

**Theorem 7** (Jiang and Su [11]). Let \(G\) be a minimally quasi-\((k + 1)\)-connected graph with girth \(g(G) \geq 4\). Then \(\delta(G) = k\).

In fact, from Theorem 6, we have the following stronger conclusion.

**Theorem 8.** Let \(G\) be a minimally quasi-\((k + 1)\)-connected graph for which two end vertices of any edge of \(G\) have at most \(k - 2\) common adjacent vertices. Then \(\delta(G) = k\).

An \(s\)-hyperoctahedral graph \(H_s\), for an integer \(s \geq 2\), is the graph obtained from \(K_{2s}\) by removing \(s\) disjoint edges.

In this paper, we first investigate the relation between quasi connectivity and removable edge. Based on the relation, the existence of removable edge in \(k\)-connected graphs \((k \geq 3)\) is investigated. It is proved that a \(5\)-connected graph has no removable edge if and only if it is isomorphic to \(K_6\). For a \(k\)-connected graph \(G\) such that end vertices of any edge of \(G\) have at most \(k - 3\) common adjacent vertices, it is also proved that \(G\) has a removable edge. Consequently, a recursive construction method of \(5\)-connected graphs is established, that is, any \(5\)-connected graph can be obtained from \(K_6\) by a number of \(\theta^+\)-operations. We conjecture that, if \(k\) is even, a \(k\)-connected graph \(G\) without removable edge is isomorphic to either \(K_{k+1}\) or \(H_{k+1}\), and, if \(k\) is odd, \(G\) is isomorphic to \(K_{k+1}\). It is pointed out that the conclusion of Theorem 3 can be more easily proved by using the method in this paper.

## 2. Removable edges in a \(5\)-connected graph

For \(k\)-connected graphs \((k \geq 3)\), we have the following conclusion which is a generalization of Theorems 1 and 2.

**Theorem 9.** Let \(G\) be a \(k\)-connected graph of order at least \(k + 3\) \((k \geq 3)\), and \(e = xy \in E(G)\). Then \(e\) is non-removable if and only if there exists \(S \subseteq V(G)\) with \(|S| = k - 1\) such that \(G - e - S\) has exactly two components \(A, B\) with \(|A| \geq 2\) and \(|B| \geq 2\), moreover \(x \in A, y \in B\).

**Proof.** Suppose \(e = xy\) is non-removable, then \(G \ominus e\) is not \(k\)-connected. Since \(|G| \geq k + 3\), \(|G \ominus e| \geq k + 1\), there exists \(S \subseteq V(G \ominus e)\) with \(|S| = k - 1\) such that \((G \ominus e) - S\) is disconnected. Then \(G - e - S\) is disconnected. Since \(G\) is \(k\)-connected, \(G - e - S\) has exactly two components \(A, B\) with \(x \in A, y \in B\). Assume \(|A| = 1\). Then \(A = \{x\}\). \(d_{G-e}(x) = k - 1\). Hence \(x\) is deleted in \(G \ominus e\). This implies that \((G \ominus e) - S\) is connected, a contradiction. Thus we have \(|A| \geq 2\). Similarly, we have \(|B| \geq 2\).

Conversely, suppose there exists \(S \subseteq V(G)\) with \(|S| = k - 1\) such that \((G \ominus e) - S\) is connected, but \((G \ominus e) - S\) has exactly two components \(A, B\) with \(|A| \geq 2\) and \(|B| \geq 2\). Since \((V(G) = V(G \ominus xy) \cup \{x, y\}\), \(G \ominus e - S\) has two components \(A' \subseteq A\) and \(B' \subseteq B\) with \(A' \cup \{x\} = A\) and \(B' \cup \{y\} = B\). Since \(|A| \geq 2\) and \(|B| \geq 2\), then \(|A'| \geq 1\) and \(|B'| \geq 1\). Hence \(S\) is a \((k - 1)\)-separator of \(G \ominus e\), \(G \ominus e\) is \((k - 1)\)-connected, so \(e\) is non-removable.

**Theorem 10.** Let \(G\) be a \(k\)-connected graph of order at least \(k + 3\) \((k \geq 3)\), and \(e = xy \in E(G)\). Then \(e\) is non-removable if and only if \(G - e\) is not quasi-\(k\)-connected.

**Proof.** Suppose \(e\) is non-removable, by Theorem 9, there exists \(S \subseteq V(G - e)\) with \(|S| = k - 1\) such that \((G - e - S)\) has exactly two components \(A, B\) with \(|A| \geq 2\) and \(|B| \geq 2\). Hence \(S\) is a nontrivial \((k - 1)\)-separator of \(G - e\), \(G - e\) is not quasi-\(k\)-connected.

Suppose \(G - e\) is not quasi-\(k\)-connected, then there exists a nontrivial \((k - 1)\)-separator \(S\) of \(G - e\) such that the components of \(G - e - S\) can be partitioned into two sets, each of which has at least two vertices. Since \(G\) is \(k\)-connected, \(G - e - S\) has exactly two components. By Theorem 9, \(e\) is non-removable.
Theorem 11. Let $G$ be a $k$-connected graph of order at least $k + 3$ ($k \geq 3$). Then $G$ has no removable edge if and only if $G$ is minimally quasi-$k$-connected.

Proof. Suppose $G$ has no removable edge, then by Theorem 10, for any edge $e$ of $G$, $G - e$ is not quasi-$k$-connected. On the other hand, since $G$ is $k$-connected, $G$ is quasi-$k$-connected, hence $G$ is minimally quasi-$k$-connected.

Suppose that $G$ is minimally quasi-$k$-connected, for any edge $e$ of $G$, $G - e$ is not quasi-$k$-connected, by Theorem 10, $e$ is non-removable, then $G$ has no removable edge. □

Theorem 12. Let $G$ be a 5-connected graph. Then $G$ has no removable edge if and only if $G \cong K_6$.

Proof. Suppose $G$ has no removable edge. If $|G| \geq 8$, $G$ is minimally quasi 5-connected by Theorem 11. By Theorem 5, $\delta(G) = 4$. This contradicts that $G$ is 5-connected. Hence $|G| \leq 7$.

If $|G| = 6$, it is only possible that $G \cong K_6$. $K_6$ has no removable edge obviously.

If $|G| = 7$, then the only possibility is that $G \cong K_7$ or $G$ is obtained by removing $s$ disjoint edges from $K_7$ ($s = 1, 2, 3$).

If $G = K_7$, obviously, $G$ has removable edges, a contradiction.

If $G$ is obtained by removing one edge or two disjoint edges from $K_7$, then there exists an edge of $G$ whose end vertices have degree 6. Hence $G \oplus e = G - e$, and $\delta(G \oplus e) = 5$. Then $G \oplus e$ is 5-connected, $e$ is a removable edge of $G$, a contradiction.

If $G$ is obtained by removing three disjoint edges from $K_7$, it can be checked directly that an edge of $G$ with an end vertex of degree 6 is a removable edge, again a contradiction. □

Since the minimum degree of a minimally quasi 4-connected graph is three, by a similar reasoning as in the proof of Theorem 12, the conclusion of Theorem 3 can be proved more easily.

Theorem 13. Let $G$ be a $k$-connected graph ($k \geq 6$) such that end vertices of any edge of $G$ have at most $k - 3$ common adjacent vertices. Then $G$ has a removable edge.

Proof. Suppose that $G$ has no removable edge.

If $|G| \geq k + 3$, it follows from Theorems 8 and 11 that $\delta(G) = k - 1$, contradicting that $G$ is $k$-connected.

If $|G| = k + 1$, then $G \cong K_{k+1}$. Obviously end vertices of any edge of $K_{k+1}$ have $k - 1$ common adjacent vertices, which contradicts the condition of the theorem.

If $|G| = k + 2$, then $G$ can only be the graphs obtained from $K_{k+2}$ by removing $s$ disjoint edges ($s = 1, 2, \ldots, \lfloor k/2 + 1 \rfloor$). End vertices of any edge of $G$ have at least $k - 2$ common adjacent vertices since they have $k$ common adjacent vertices in supergraph $K_{k+2}$ of $G$. This contradicts the condition of the theorem.

The proof is thus completed. □

From the proof of Theorem 13, it can be seen that, if the minimum degree of a minimally quasi-$k$-connected graph $G$ is equal to $k - 1$, then the conclusion in the following conjecture would hold.

Conjecture 14. Let $G$ be a $k$-connected ($k \geq 3$). $G$ has no removable edge if and only if either $G \cong K_{k+1}$ for $k$ being odd, or $G$ is isomorphic to either $K_{k+1}$ or $H_{k/2+1}$ for $k$ being even.

3. A recursive construction method of 5-connected graphs

By the definition of a removable edge of $k$-connected graphs, we can define the following operations.

Definition 2. Let $G$ be a $k$-connected graph, let $e$ be a removable edge of $G$, and let $H = G \oplus e$. Then $H$ is said to be obtained from $G$ by a $\theta^+$-operation, denoted by $H = \theta^+(G)$, and $G$ is said to be obtained from $H$ by a $\theta^-$-operation, denoted by $G = \theta^-(H)$. A $\theta^+$-operation is said to be the inverse operation of $\theta^-$-operation, and vice versa.

Let $G$ be a 5-connected graph, and let $e = xy$ be a removable edge of $G$. Let $E_x = \{x_i x_j | x_i, x_j \in N_{G-e}(x), x_i x_j \notin E(G)\}$, and let $E_y = \{y_i y_j | y_i, y_j \in N_{G-e}(y), y_i y_j \notin E(G)\}$.
A $\theta^-$-operation $H = \theta^-(G) = G \ominus e$ is one of the following three operations:

1. if $d_G(x) \geq 6$ and $d_G(y) \geq 6$, $H = G \ominus e = \theta^-(G) = G - e$;
2. if $d_G(x) = 5$ and $d_G(y) \geq 6$, $H = G \ominus e = \theta^-(G) = G - x + E_x$;
3. if $d_G(x) = d_G(y) = 5$, $H = G \ominus e = \theta^-(G) = G - x - y + E_x + E_y$.

In order to give an exact definition of a $\theta^+$-operation, we need the following theorem.

For a $k$-connected graph $G$ and a minimum vertex cut $T$ of $G$, the vertex set of a connected component of $G - T$ is called a $T$-fragment of $G$. A subset $S$ of $V(G)$ is called a fragment of $G$ if there is a minimum vertex cut $T$ of $G$ such that $S$ is a $T$-fragment. A fragment of $G$ is called an end fragment of $G$ if its any proper subset is not a fragment of $G$.

**Theorem 15.** Let $H$ be a 5-connected graph, let $X = \{x_1, x_2, x_3, x_4\} \subset V(H)$ and $Y = \{y_1, y_2, y_3, y_4\} \subset V(H)$ such that $H[X] \cong K_4$ and $H[Y] \cong K_4$, and let $E_X \subseteq E(H[X])$ and $E_Y \subseteq E(H[Y])$. Then

(i) $G_X = (H - E_X) + x + \{xx_i|i = 1, 2, 3, 4\} + xy$ is 5-connected if and only if $\kappa(H - E_X) = \kappa(G_X - x) \geq 4$, where $x \notin V(H)$, $y \in V(H) \setminus X$;

(ii) $G_{XY} = (H - E_X - E_Y) + x + y + xy + \{xx_i|i = 1, 2, 3, 4\} + \{yy_i|i = 1, 2, 3, 4\}$ is 5-connected if and only if $|X \cap Y| \leq 3$, $\kappa(H - E_X - E_Y) = \kappa(G_{XY} - x - y) \geq 3$, and, if $\kappa(H - E_X - E_Y) = \kappa(G_{XY} - x - y) = 3$, any end fragment of $H - E_X - E_Y$ contains a vertex in $X$ and a vertex in $Y$, where $x, y \notin V(H)$.

**Proof.** The necessity is obvious. We need only to prove the sufficiency.

(i) If $\kappa(H - E_X) = \kappa(G_X - x) \geq 5$, then $G_X$ is 5-connected clearly. Now suppose $\kappa(H - E_X) = \kappa(G_X - x) = 4$. Let $T$ be any minimum vertex cut of $H - X$. Since $H$ is 5-connected, any fragment of $H - E_X$ contains a vertex in $X$, and so $T$ will not be a vertex cut in $G_X$. Hence $G_X$ is 5-connected.

(ii) If $\kappa(H - E_X - E_Y) = \kappa(G_{XY} - x - y) \geq 4$, by a similar reasoning as in the proof of (i), $G_{XY}$ is 5-connected.

Suppose $\kappa(H - E_X - E_Y) = \kappa(G_{XY} - x - y) = 3$. For any minimum vertex cut $T$ of $H - E_X - E_Y$, since any end fragment of $H - E_X - E_Y$ contains both a vertex in $X$ and a vertex in $Y$, any connected component of $H - E_X - E_Y - T$ contains both a vertex in $X$ and a vertex in $Y$, and so any one of $T, T \cup \{x\}$, and $T \cup \{y\}$ will not be a vertex cut of $G_{XY}$. For a vertex cut $S$ of $H - E_X - E_Y$ with $|S| = 4$, any connected component of $H - E_X - E_Y - S$ contains either a vertex in $X$ or a vertex in $Y$, since $H$ is 5-connected. Therefore, $S$ is also not a vertex cut of $G_{XY}$. Now it follows that $G_{XY}$ is 5-connected. □

**Definition 3.** Let $H$ be a 5-connected graph, and let $X = \{x_1, x_2, x_3, x_4\} \subset V(H)$ and $Y = \{y_1, y_2, y_3, y_4\} \subset V(H)$ such that $H[X] \cong K_4$ and $H[Y] \cong K_4$. Let $G$ be a 5-connected graph obtained from $H$ by a $\theta^+$-operation. Then the $\theta^+$-operation is one of the following three operations:

1. $G = \theta^+(H) = H + xy$, where $x, y \in V(H)$, and $xy \notin E(H)$;
2. $G = \theta^+(H) = H - E_X + x + \{xx_i|i = 1, 2, 3, 4\} + xy$, where $x \notin V(H)$, $y \in V(H) \setminus X$, and $E_X \subseteq E(H[X])$ such that $\kappa(H - E_X) = \kappa(G - x) \geq 4$;
3. $G = \theta^+(H) = H - E_X - E_Y + x + y + xy + \{xx_i|i = 1, 2, 3, 4\} + \{yy_i|i = 1, 2, 3, 4\}$, where $x, y \notin V(H)$, $|X \cap Y| \leq 3$, and $E_X \subseteq E(H[X])$ and $E_Y \subseteq E(H[Y])$ such that $\kappa(H - E_X - E_Y) = \kappa(G - x - y) \geq 3$, and, if $\kappa(H - E_X - E_Y) = \kappa(G - x - y) = 3$, any end fragment of $H - E_X - E_Y$ contains both a vertex in $X$ and a vertex in $Y$.

By Theorem 12, we can give a recursive construction method of 5-connected graphs.

**Theorem 16.** Let $G$ be a 5-connected graph. Then (i) $G$ can be transformed into $K_6$ by a number of $\theta^-$-operations; (ii) $G$ can be obtained from $K_6$ by a number of $\theta^+$-operations.

**Proof.** (i) Suppose $G$ is 5-connected graph, and $G$ is not isomorphic to $K_6$. Then, by Theorem 12, $G$ has a removable edge, say $e_1$, and $G_1 = \theta^-(G) = G \ominus e_1$ is a 5-connected graph with less edges or less vertices than $G$. Repeating the above discussion, by the finiteness of $G$, we can obtain a series of 5-connected graphs $G_1, G_2, \ldots, G_t$ so that $G_{i+1} = \theta^-(G_i)$, $i = 1, 2, \ldots, t - 1$, and $G_t \cong K_6$.

(ii) By the inverse operations of $\theta^-$-operations in (i), $G$ can be obtained from $K_6$ by a number of $\theta^+$-operations. □
Remark. We want to point out that the above recursive construction method of 5-connected graphs would be able to be generalized into \( k \)-connected graphs if Conjecture 14 could be proved.

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