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Partly Alternating Families

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INTRODUCTION

In this note we study families of approximating functions such that some best approximations are characterized by alternation. For $g \in C[a, b]$, define

$$\| g \| = \sup \{ | g(x) | : a \leqslant x \leqslant b \}.$$

Let F be an approximating function with parameter A ranging over a space P, and let $F(A,.) \in C[a, b]$ for all $A \in P$. The Chebyshev problem is: Given $f \in C[a, b]$, find $A^* \in P$ minimizing $\| f - F(A,.) \|$ over P. The corresponding $F(A^*,.)$ is called a best approximation to f.

1. ALTERNATION

DEFINITION. $g \in C[a, b]$ is said to *alternate* n times if there is a $n + 1$ point set $\{x_0, ..., x_n\}$ with $a \leqslant x_0 < \cdots < x_n \leqslant b$ such that

$$| g(x_0) | = \| g \|,$$
$$g(x_i) = (-1)^i g(x_0), \qquad i = 0, ..., n.$$

The set $\{x_0, ..., x_n\}$ is called an *alternant* of g.

DEFINITION. F is said to have property N (property S, property NS) of degree n at A if a necessary (sufficient, necessary and sufficient) condition for $F(A, \cdot)$ to be best to $g \in C[a, b]$ is that $g - F(A, \cdot)$ alternate n times.

Rice [3, pp. 325–327] has characterized pairs (F, P) such that F has property

NS of degree n at A for all $A \in P$. This was later extended to cover (F, P) such that F has property *NS* of variable degree at all $A \in P$ [4, pp. 18–21]. We consider in this note cases such that F has property *NS* of variable degree at some $A \in P$.

There are several reasons for such a study. First, to guarantee existence of best approximations by an alternating family, we may have to add various kinds of limits which do not have alternating properties, and we want a theory for the resulting family. Such families are given in Examples 1 and 7. Second, the alternating characterization property is a special case of a general characterization property called extremum characterizability [2, p. 375]; a study of partly alternating families thus aids in the study of families with partial extremum characterizability. Third, the uniqueness results for partly alternating families are useful in a study of the general uniqueness problem.

DEFINITION. g has n *sign changes* if there exists a set

$$\{x_0, ..., x_n\},\ a \leqslant x_0 < \cdots < x_n \leqslant b$$

such that the g is alternately >0 and <0 on the set.

DEFINITION. F has *weak property Z* of degree n at A if there exists no B such that $F(A, \cdot) - F(B, \cdot)$ has n sign changes.

DEFINITION. F has property $\mathcal{O}\!\!\ell$ of degree n at A, if for any integer $m < n$, any sequence $\{x_1, ..., x_m\}$ with

$$a = x_0 < x_1 < \cdots < x_{m+1} = b$$

any sign σ, and any real ϵ with

$$0 < \epsilon < \min\{x_{j+1} - x_j : 0, ..., m\},$$

there exists a $B \in P$, such that

$$\begin{aligned} \| F(A, \cdot) - F(B, \cdot)\| &< \epsilon, \\ \operatorname{sgn}(F(A, x) - F(B, x)) &= \sigma,\ a \leqslant x \leqslant x_1 - \epsilon \\ &= \sigma(-1)^j,\ x_j + \epsilon \leqslant x \leqslant x_{j+1} - \epsilon \\ &= \sigma(-1)^m,\ x_m + \epsilon \leqslant x \leqslant b. \end{aligned}$$

In case $m = 0$, the above sign condition reduces to

$$\operatorname{sgn}(F(A, \cdot) - F(B, \cdot)) = \sigma.$$

LEMMA 1. *Let F have weak property Z of degree n at A. If $f - F(A, \cdot)$ alternates in sign on $x_0 < \cdots < x_n$ then*

$$\max \{|f(x_i) - F(B, x_i)| : i = 0,..., n\} \geqslant \min \{|f(x_i) - F(A, x_i)| : i = 0,..., n\}.$$

Proof. Suppose not, then $F(A, \cdot) - F(B, \cdot)$ can be shown to have n sign changes.

COROLLARY. *Let F have weak property Z of degree n at A, then F has property S of degree n at A.*

Proof. Let $f - F(A, \cdot)$ alternate n times with alternant $\{x_0, ..., x_n\}$. Apply the lemma.

LEMMA 2. *Let F have property $\mathcal{A}$ of degree n at A, then F has property N of degree n at A.*

This is proven by Rice [3, pp. 18–19].

LEMMA 3. *Let F have property NS of degree n at A, then F has weak property Z of degree n at A.*

Proof. Suppose $F(A, \cdot) - F(B, \cdot)$ has n sign changes. We can construct continuous f such that

$$\operatorname{sgn}(f - F(A, \cdot)) = \operatorname{sgn}(F(B, \cdot) - F(A, \cdot)),$$

$f - F(A, \cdot)$ alternates n times and $\|f - F(A, \cdot)\| > \|f - F(B, \cdot)\|$.

LEMMA 4. *If F has property NS of degree n at A then F has property $\mathcal{A}$ of degree n at A.*

This is proven by Rice [3, p. 21].

DEFINITION. F is said to have *weak degree n* at A if F has weak property Z and property $\mathcal{A}$ of degree n at A.

From the four lemmas we obtain immediately

THEOREM 1. *A necessary and sufficient condition that F have property NS of degree n at A is that F have weak degree n at A.*

The following definitions are useful in a study of uniqueness.

DEFINITION. A double zero of $g \in C[a, b]$ is a point x in (a, b) at which g vanishes without a sign change.

DEFINITION. F has *strong property* Z of degree n at A if $F(A, \cdot) - F(B, \cdot)$ having n zeros, counting double zeros twice, implies $F(A, \cdot) \equiv F(B, \cdot)$.

DEFINITION. F has *strong degree* n at A if f has strong property Z and property $\mathcal{O}\!\mathcal{l}$ of degree n at A.

2. EXAMPLES

EXAMPLE 1. This is taken from [2, p. 383]. Let $[a, b] = [0, 1]$, $P = [0, \infty)$,

$$F(\alpha, x) = \left(1 + \frac{1}{\alpha}\right)\Big/(1 + \alpha x) \qquad \alpha > 0$$

$$= 0 \qquad \alpha = 0$$

F has strong degree 1 at all $\alpha > 0$. To guarantee existence the parameter 0 is required.

EXAMPLE 2. Let $F(\alpha, \cdot) = \alpha$ and let P be a subset of the real line containing 0 as well as sequences $\{a_k\} \to 0$, $\{a_k'\} \to 0$, $a_k > 0$, $a_k' < 0$. Then F has strong degree 1 at 0.

EXAMPLE 3. Let $[a, b] = [-1, 1]$ in Example 2 and add the function $|x|$ as an approximant; then F has weak degree 1 at 0.

EXAMPLE 4. Let $[a, b] = [0, 1]$ and $F(A, x) = a_1 + a_2 x + a_3 x^2$. Let P_1 be the set of all (a_1, a_2, a_3) for which $\| F(A, x) \| < 1$. Let P_2 be any other set of triples (a_1, a_2, a_3), and let $P = P_1 \cup P_2$. Then F has strong degree 3 at all $A \in P_1$.

EXAMPLE 5. Let us choose P_2 in Example 4 to be the set of triples $(a_1, a_2, 0)$ for which $-1 < a_1 + a_2 < 1$ and the line $y = a_1 + a_2 x$ has slope 3. Then $P_1 \cap P_2$ is empty. For any $B \in P_2$ there is an $A \in P_1$ such that $F(A, \cdot) - F(B, \cdot)$ has a sign change. Hence F has no degree at any $B \in P_2$. It can be shown that F is not extremum characterizable at any $B \in P_2$.

EXAMPLE 6. Let $[a, b] = [0, 1]$. Let $F(A, x) = a_1 + a_2 x$. Let P' consist of all pairs $(a_1, 0)$ with $a_1 \leqslant 0$, and let P'' consist of all (a_1, a_2) for which $a_1 + a_2 x > 0$ throughout $[a, b]$. Let $P = P' \cup P'' \cup (0, 1)$. Then F has strong degree 1 at all $A \in P' \sim (0, 0)$, and strong degree 2 at all $A \in P''$. It has weak degree 1 at (0, 0) and no degree at (0, 1).

3. Uniqueness

Lemma 5. *Let F have weak degree n at A. Let A and B be best to f, then $F(A, \cdot) - F(B, \cdot)$ has n zeros, counting double zeros twice.*

Proof. $f - F(A, \cdot)$ must alternate n times. Let $\{x_0, ..., x_n\}$ be an alternant of $f - F(A, \cdot)$. Assume without loss of generality that $f(x_0) - F(A, x_0) > 0$; then

$$(-1)^i (F(B, x_i) - F(A, x_i)) \geqslant 0, \qquad i = 0, ..., n.$$

By drawing a diagram it can be seen that the num ber of zeros is at least n, counting double zeros twice.

Lemma 6. *Let F have weak property Z of degree n at A and $F(A, \cdot) - F(B, \cdot)$ have n zeros counting double zeros twice. There exists $f \in C[a, b]$ with A, B best.*

Proof. Define $e = \| F(A, \cdot) - F(B, \cdot)\|/2$. Suppose first that $F(A, \cdot) - F(B, \cdot)$ has n distinct zeros $z_1, ..., z_n$. Let x' be a point such that $F(A, x') \neq F(B, x')$, say $F(B, x') > F(A, x')$. Let $\{x_0, ..., x_n\} = \{z_1, ..., z_n\} \cup \{x'\}$, with $x_0 < \cdots < x_n$. Let j be the subscript for which $x_j = x'$

Define

$$f(x_i) = F(A, x_i) + (-1)^{i-j} e. \tag{1}$$

By construction, $|f(x_i) - F(B, x_i)| \leqslant e$, $i = 0, ..., n$. There is a continuous extension of f to $[a, b]$ such that $\|f - F(A, \cdot)\| = e$, $\|f - F(B, \cdot)\| = e$. By (1), $f - F(A, \cdot)$ alternates n times, and so A is best, hence B is also best. The other case to consider is where there are less than n zeros but at least n zeros when double zeros are counted twice. At a double zero x of $F(A, \cdot) - F(B, \cdot)$ let $f(x) = F(A, x) - s \cdot e$, where s is the sign of $F(B, \cdot) - F(A, \cdot)$ near x. If an endpoint x is a zero let $f(x) = F(A, x) - s \cdot e$, s the sign of $F(B, \cdot) - F(A, \cdot)$ near x. Between any two successive zeros of $F(A, \cdot) - F(B, \cdot)$, select a point x and let $f(x) = F(A, x) + s \cdot e$, where s is the sign of $F(B, \cdot) - F(A, \cdot)$ at x. If an endpoint x is not a zero, define f the same. It can be seen that $f - F(A, \cdot)$ alternates in sign on the points of definition with amplitude e. By construction, $|f(x) - F(B, x)| \leqslant e$ for such x. The number of points of definition of f is the number of endpoint zeros plus the number of double zeros plus (1 plus the number of interior zeros, double or not), and is thus $\geqslant n + 1$. There is a continuous extension of f to $[a, b]$ such that $\| f - F(A, \cdot)\| = e$, $\|f - F(B, \cdot)\| = e$. As $f - F(A, \cdot)$ alternates n times, $F(A, \cdot)$ is best and so $F(B, \cdot)$ is best. From the two previous lemmas we obtain

Theorem 2. *Let F have property NS of degree n at A. A necessary and sufficient condition for $F(A, \cdot)$ to be unique when it is best is that F has strong property Z of degree n at A.*

Strong property Z is a difficult property to verify directly and we consider when it can be replaced by weaker properties, in particular weak property Z on part of P. The following lemma is a generalization of Lemma 1 of [1], for which no complete proof was given,

LEMMA 7. *Let F have weak property Z of degree n at A and F have property $\mathcal{A}$ of degree n at B. If $F(A, \cdot) - F(B, \cdot)$ has n zeros, counting double zeros twice, but does not vanish identically, there exists $C \in P$ such that $F(A, \cdot) - F(C, \cdot)$ has n sign changes.*

Proof. The first possibility is that $F(A, \cdot) - F(B), \cdot)$ vanishes on a nondegenerate interval I. Without loss of generality we can suppose that there exists y such that $F(A, y) - F(B, y) > \epsilon$. By property $\mathcal{A}$ of degree n at B there exists $C \in P$ such that $F(B, \cdot) - F(C, \cdot)$ changes sign $n-1$ times in the interior of I,

$$\| F(B, \cdot) - F(C, \cdot)\| < \epsilon \qquad \text{and} \quad F(B, x) - F(C, x) < 0$$

for x between y and I. Then $F(A, \cdot) - F(C, \cdot)$ has a sign change between I and y, and $n - 1$ sign changes in I. We have n sign changes, contradicting property Z of degree n at A. The first possibility cannot occur and between any two zeros of $F(A, \cdot) - F(B, \cdot)$ there is a point at which $F(A, \cdot) - F(B, \cdot)$ does not vanish. Next suppose that $F(A, \cdot) - F(B, \cdot)$ does not change sign, say $F(A, \cdot) - F(B, \cdot) > 0$. Select a finite number of zeros $\{z_k : k = 1,..., m\}$. Between z_k and z_{k+1}, select x_k such that $F(A, x_k) - F(B, x_k) > 0$. Define

$$\epsilon = \min\{F(A, x_k) - F(B, x_k) : k = 1,..., m - 1\}.$$

By property $\mathcal{A}$ of degree n at B we can select C such that

$$F(C, \cdot) - F(B, \cdot) > 0, \| F(C, \cdot) - F(B, \cdot)\| < \epsilon/2.$$

It is not difficult to see that for every zero of $F(A, \cdot) - F(B, \cdot)$, counting double zeros twice, there is a sign change of $F(A, \cdot) - F(C, \cdot)$. Finally suppose that $F(A, \cdot) - F(B, \cdot)$ has exactly k sign changes which occur at $z_1,..., z_k$ (there can be at most $n - 1$ sign changes). Select a finite set Z of zeros of $F(A, \cdot) - F(B, \cdot)$ which includes $z_1,..., z_k$. Select a finite point set X such that between any two elements of Z there is an element of X and

$$F(A, x) - F(B, x) \neq 0 \qquad \text{for} \quad x \in X.$$

Define

$$\epsilon_1 = \min\{| F(A, x) - F(B, x)| : x \in X\}.$$

Let $\epsilon_2 = \inf\{|x - z| : x \in X, z \in Z\}$ and set $\epsilon = 1/4 \min\{\epsilon_1, \epsilon_2\}$. By property $\mathcal{Cl}$ of degree n at B there exists C such that

$$\|F(C, \cdot) - F(B, \cdot)\| < \epsilon, F(C, \cdot) - F(B, \cdot)$$

changes sign in an ϵ-neighborhood of z_i, $i = 1,..., k$, and outside the ϵ-neighbourhood, $\operatorname{sgn}(F(C, y) - F(B, y))$ is the sign of $F(A, \cdot) - F(B, \cdot)$ at or near y. It is not difficult to see that $F(A, \cdot) - F(C, \cdot)$ has a sign change for every zero of $F(A, \cdot) - F(B, \cdot)$, counting double zeros twice.
In case (F, P) is an alternating family, all elements have a degree (as defined in ref. 1, p. 225) and we have

COROLLARY. *Let F have degree n at A and some degree at B. If $F(A, \cdot) - F(B, \cdot)$ has n zeros, counting double zeros twice, then*

$$F(A, \cdot) \equiv F(B, \cdot).$$

Proof. We go through the same arguments as in the proof of the lemma. $F(A, \cdot) - F(B, \cdot)$ cannot vanish on an interval without vanishing everywhere. In case $F(A, \cdot) - F(B, \cdot)$ is not identically zero and does not change sign, there exists $C \in P$ such that $F(A, \cdot) - F(C, \cdot)$ has n sign changes, contrary to hypothesis. In case $F(A, \cdot) - F(B, \cdot)$ has k sign changes, there must be at least one other zero of $F(A, \cdot) - F(B, \cdot)$, hence the degree of F at B must be at least $k + 2$. As F has property $\mathcal{Cl}$ of degree $k + 2$ at B, there exists $C \in P$ such that $F(A, \cdot) - F(C, \cdot)$ has n sign changes, contrary to hypothesis.

THEOREM 3. *Let Q be the set of elements of P at which F has a weak degree. Let F have weak degree n at A. If $F(A, \cdot)$ is a best approximation to f there is no other best approximation in Q. A necessary and sufficient condition for $F(A, \cdot)$ to be unique whenever it is best is that for all $B \in P \sim Q$, $F(A, \cdot) - F(B, \cdot)$ has less than n zeros, counting double zeros twice.*

Proof. Suppose A, B are best and in Q. Let F have weak degree m at B. Assume without loss of generality that $m \geqslant n$. F has property $\mathcal{Cl}$ of degree m at B, hence property $\mathcal{Cl}$ of degree n at B. By Lemma 5, $F(A, \cdot) - F(B, \cdot)$ has n zeros, counting double zeros twice and by Lemma 7, $F(A, \cdot) \equiv F(B, \cdot)$. It follows that if $F(A, \cdot)$ is best and $F(B, \cdot)$ is a different best approximation, $B \in P \sim Q$. By lemma 5, $F(A, \cdot) - F(B, \cdot)$ has n zeros counting double zeros twice, establishing sufficiency. If $F(A, \cdot) - F(B, \cdot)$ has n zeros, counting double zeros twice, Lemma 6 guarantees nonuniqueness, proving necessity.

Let us consider Example 4 of Section 2. F has strong property Z of degree 3 at all elements. By the above theorem nonuniqueness at best approximations is possible only if two best parameters exist in P_2. If we select P_2 so that best approximations from parameter space P_2 are unique, we have uniqueness for all continuous functions. In particular, we have uniqueness for Example 5, even though F has neither degree nor extremum characterizability at any parameter in P_2.

4. Noncontinuous Approximations

To get existence of best approximations, we may have to add noncontinuous limits to an alternating family.

Example 7. Let

$$F(\alpha, x) = 1/(1 + \alpha x) + (1/\alpha), \qquad x < 0$$
$$= 1 + (1/\alpha), \qquad x \geqslant 0$$

for $\alpha > 0$. Let $P = (1, \infty)$. If $\alpha_1 < \alpha_2$ then $F(\alpha_1, x) < F(\alpha_2, x)$, so F has strong property Z of degree 1 at all $\alpha \in P$. Using Dini's theorem we can show that F has property $\mathcal{A}$ of degree 1 at all $\alpha \in P$. However, best approximations do not exist to all continuous functions. To assure such existence we must add $F(1, \cdot)$, which is continuous, and $F(\infty, \cdot)$, which is not continous, as approximations.

The theory obtained previously is valid for noncontinuous approximants providing F having a weak degree at A and $F(B, \cdot)$ being noncontinuous imply that $F(A, \cdot) - F(B, \cdot)$ has no sign changes or zeros.

5. Computation of Best Approximations

An algorithm (a variant of the Remez algorithm) for computing best approximations by alternating families is described in [1, p. 228 ff.]. This algorithm can be used to compute best approximations by partly alternating families. Suppose there is a unique parameter A^* that is best and suppose F has strong degree n (the maximum degree) at all A in a neighborhood of A^*. If the hypotheses of Theorem 2 of [1, p. 229] are satisfied, the algorithm has quadratic convergence.

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