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An approximation theorem for nuclear operator systems

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Abstract

We prove that an operator system $\mathcal{S}$ is nuclear in the category of operator systems if and only if there exist nets of unital completely positive maps $\varphi_\lambda : \mathcal{S} \to M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \to \mathcal{S}$ such that $\psi_\lambda \circ \varphi_\lambda$ converges to $\mathrm{id}_{\mathcal{S}}$ in the point-norm topology. Our proof is independent of the Choi–Effros–Kirchberg characterization of nuclear C^*-algebras and yields this characterization as a corollary. We give an explicit example of a nuclear operator system that is not completely order isomorphic to a unital C^*-algebra.

Keywords: Operator system; Tensor product; Nuclear

1. Introduction

In summary, we prove that an operator system $\mathcal{S}$ has the property that for every operator system $\mathcal{T}$ the *minimal operator system tensor product* $\mathcal{S} \otimes_{\min} \mathcal{T}$ coincides with the *maximal operator system tensor product* $\mathcal{S} \otimes_{\max} \mathcal{T}$ if and only if there is a point-norm factorization of $\mathcal{S}$ through matrices of the type described in the abstract. Our proof of this fact is quite short, direct and independent of the corresponding factorization results of Choi, Effros and Kirchberg for nuclear C^*-algebras. Our proof uses in a key way a characterization of the maximal operator system tensor product given in [7]. We are then able to deduce the Choi–Effros–Kirchberg characterization of nuclear C^*-algebras as an immediate corollary. The proof that one obtains in this

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way of the Choi–Effros–Kirchberg result combines elements of the proofs given in [3] and [15] but eliminates the need to approximate maps into the second dual or to introduce decomposable maps. Finally, we give a fairly simple example of an operator system that is *nuclear* in this sense, but is not completely order isomorphic to any C^*-algebra and yet has second dual completely order isomorphic to $B(\ell^2(\mathbb{N}))$. Earlier, Kirchberg and Wassermann [11] constructed a nuclear operator system that is not even embeddable in any nuclear C^*-algebra.

In [6], Kadison characterized the unital subspaces of a real continuous function algebra on a compact set by observing that the norm of a real continuous function algebra is determined by the unit and the order. As for its noncommutative counterpart, Choi and Effros gave an abstract characterization of the unital involutive subspaces of $\mathcal{B}(\mathcal{H})$ [1]. The observation that the unit and the matrix order in $\mathcal{B}(\mathcal{H})$ determine the matrix norm is key to their characterization. The former is called a real function system or a real ordered vector space with an Archimedean order unit while the latter is termed an operator system.

Although the abstract characterization of an operator system played a key role in the work of Choi and Effros [1] on the tensor products of C^*-algebras, there had not been much attempt to study the categorical aspects of operator systems and their tensor theory until a series of papers [13,14,7,8]. In particular, [7] introduced axioms for tensor products of operator systems and characterized the minimal and maximal tensor products of operator systems.

The positive cone of the minimal tensor product is the largest among all possible positive cones of operator system tensor products while that of the maximal tensor product is the smallest. These extend the minimal tensor product and the maximal tensor product of C^*-algebras. In other words, the minimal (respectively, maximal) operator system tensor product of two unital C^*-algebras is the operator subsystem of their minimal (respectively, maximal) C^*-tensor product.

For the purposes of this paper, a unital C^*-algebra $\mathcal{A}$ will be called *C^*-nuclear* if and only if it has the property that for every unital C^*-algebra $\mathcal{B}$ the minimal C^*-tensor product $\mathcal{A} \otimes_{C^* \min} \mathcal{B}$ is equal to the maximal C^*-tensor product $\mathcal{A} \otimes_{C^* \max} \mathcal{B}$. We say that a C^*-algebra $\mathcal{A}$ has the *completely positive approximation property* (in short, CPAP) if there exists a net of unital completely positive maps $\varphi_\lambda : \mathcal{A} \to \mathcal{A}$ with finite rank which converges to $\mathrm{id}_{\mathcal{A}}$ in the point-norm topology. The Choi–Effros–Kirchberg result is that a C^*-algebra $\mathcal{A}$ is C^*-nuclear if and only if $\mathcal{A}$ has the CPAP if and only if there exist nets of unital completely positive maps $\varphi_\lambda : \mathcal{A} \to M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \to \mathcal{A}$ such that $\psi_\lambda \circ \varphi_\lambda$ converges to $\mathrm{id}_{\mathcal{A}}$ in the point-norm topology [3,9]. For a recent proof which uses operator space methods and the decomposable approximation, we refer the reader to [15, Chapter 12].

An operator system will be called *nuclear* provided that the minimal tensor product of it with an arbitrary operator system coincides with the maximal tensor product. In [7], this property was called (min, max)-nuclear. It is natural to ask whether the approximation theorems of nuclear C^*-algebras [3,9] also hold in the category of operator systems. In Section 3, we show that an operator system $\mathcal{S}$ is nuclear if and only if there exist nets of unital completely positive maps $\varphi_\lambda : \mathcal{S} \to M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \to \mathcal{S}$ such that $\psi_\lambda \circ \varphi_\lambda$ converges to $\mathrm{id}_{\mathcal{S}}$ in the point-norm topology.

We then prove, independent of the Choi–Effros–Kirchberg theorem, that a C^*-algebra is C^*-nuclear if and only if it is nuclear as an operator system. Thus, we obtain the Choi–Effros–Kirchberg characterization as a corollary of the factorization result for operator systems.

In contrast, CPAP does not imply nuclearity in the category of operator systems. Let

$$\mathcal{S}_0 = \mathrm{span}\{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}, E_{2,3}, E_{3,2}, E_{3,3}\} \subset M_3.$$

In [7, Theorem 5.18], it is shown that this finite dimensional operator system $\mathcal{S}_0$ is not nuclear.

On the other hand, [7, Theorem 5.16] shows that the minimal and maximal operator system tensor products of $\mathcal{S}_0 \otimes \mathcal{B}$ coincide for every unital C^*-algebra $\mathcal{B}$. Thus, for operator systems, tensoring with C^*-algebras is not sufficient to discern ordinary nuclearity, i.e., (min, max)-nuclearity. However, it is easily seen that the minimal and maximal operator system tensor products of $\mathcal{S} \otimes \mathcal{B}$ coincide for every unital C^*-algebra $\mathcal{B}$ if and only if $\mathcal{S}$ is (min, c)-nuclear, in the sense of [7].

Finally, in Section 4, we construct a nuclear operator system that is not unitally, completely order isomorphic to a unital C^*-algebra. This shows that the theory of nuclear operator systems properly extends the theory of nuclear C^*-algebras. In contrast, by [1], every injective operator system is unitally, completely order isomorphic to a unital C^*-algebra.

2. Preliminaries

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. Following [7], an *operator system structure* on $\mathcal{S} \otimes \mathcal{T}$ is defined as a family of cones $M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ satisfying:

(T1) $(\mathcal{S} \otimes \mathcal{T}, \{M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+\}_{n=1}^\infty, 1_\mathcal{S} \otimes 1_\mathcal{T})$ is an operator system denoted by $\mathcal{S} \otimes_\tau \mathcal{T}$,
(T2) $M_n(\mathcal{S})^+ \otimes M_m(\mathcal{T})^+ \subset M_{mn}(\mathcal{S} \otimes_\tau \mathcal{T})^+$ for all $n, m \in \mathbb{N}$, and
(T3) if $\varphi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are unital completely positive maps, then $\varphi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{mn}$ is a unital completely positive map.

By an *operator system tensor product,* we mean a mapping $\tau : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, such that for every pair of operator systems $\mathcal{S}$ and $\mathcal{T}$, $\tau(\mathcal{S}, \mathcal{T})$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$, denoted $\mathcal{S} \otimes_\tau \mathcal{T}$. We call an operator system tensor product τ *functorial,* if the following property is satisfied:

(T4) For any operator systems $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{T}_1$, $\mathcal{T}_2$ and unital completely positive maps $\varphi : \mathcal{S}_1 \to \mathcal{T}_1$, $\psi : \mathcal{S}_2 \to \mathcal{T}_2$, the map $\varphi \otimes \psi : \mathcal{S}_1 \otimes \mathcal{S}_2 \to \mathcal{T}_1 \otimes \mathcal{T}_2$ is unital completely positive.

An operator system structure is defined on two fixed operator systems, while the functorial operator system tensor product can be thought of as the bifunctor on the category consisting of operator systems and unital completely positive maps.

Given an operator system $\mathcal{R}$ we let $S_n(\mathcal{R})$ denote the set of unital completely positive maps of $\mathcal{R}$ into M_n. For operator systems $\mathcal{S}$ and $\mathcal{T}$, we put

$$M_n(\mathcal{S} \otimes_{\min} \mathcal{T})^+ = \big\{[p_{i,j}]_{i,j} \in M_n(\mathcal{S} \otimes \mathcal{T}) \colon \forall \varphi \in S_k(\mathcal{S}),\ \psi \in S_m(\mathcal{T}), \\ \big[(\varphi \otimes \psi)(p_{i,j})\big]_{i,j} \in M_{nkm}^+\big\}.$$

Then the family $\{M_n(\mathcal{S} \otimes_{\min} \mathcal{T})^+\}_{n=1}^\infty$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$. Moreover, if we let $\iota_\mathcal{S} : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ and $\iota_\mathcal{T} : \mathcal{T} \to \mathcal{B}(\mathcal{K})$ be any unital completely order isomorphic embeddings, then it is shown in [7] that this is the operator system structure on $\mathcal{S} \otimes \mathcal{T}$ arising from the embedding $\iota_\mathcal{S} \otimes \iota_\mathcal{T} : \mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. As in [7], we call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{M_n(\mathcal{S} \otimes_{\min} \mathcal{T})\}_{n=1}^\infty, 1_\mathcal{S} \otimes 1_\mathcal{T})$ the *minimal* tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\min} \mathcal{T}$.

The mapping $\min : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ sending $(\mathcal{S}, \mathcal{T})$ to $\mathcal{S} \otimes_{\min} \mathcal{T}$ is an injective, associative, symmetric and functorial operator system tensor product. The positive cone of the minimal tensor product is the largest among all possible positive cones of operator system tensor products [7, Theorem 4.6]. For C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we have the completely order isomorphic inclusion

$$\mathcal{A} \otimes_{\min} \mathcal{B} \subset \mathcal{A} \otimes_{\mathrm{C}^* \min} \mathcal{B}$$

[7, Corollary 4.10].

For operator systems $\mathcal{S}$ and $\mathcal{T}$, we put

$$D_n^{\max}(\mathcal{S}, \mathcal{T}) = \bigl\{\alpha(P \otimes Q)\alpha^*\colon\ P \in M_k(\mathcal{S})^+,\ Q \in M_l(\mathcal{T})^+,\ \alpha \in M_{n,kl},\ k, l \in \mathbb{N}\bigr\}.$$

Then it is a matrix ordering on $\mathcal{S} \otimes \mathcal{T}$ with order unit $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$. Let $\{M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+\}_{n=1}^{\infty}$ be the Archimedeanization of the matrix ordering $\{D_n^{\max}(\mathcal{S}, \mathcal{T})\}_{n=1}^{\infty}$. Then it can be written as

$$M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+ = \bigl\{X \in M_n(\mathcal{S} \otimes \mathcal{T})\colon\ \forall \varepsilon > 0,\ X + \varepsilon I_n \otimes 1_{\mathcal{S}} \otimes 1_{\mathcal{T}} \in D_n^{\max}(\mathcal{S}, \mathcal{T})\bigr\}.$$

We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+\}_{n=1}^{\infty}, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ the *maximal* operator system tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\max} \mathcal{T}$.

The mapping $\max : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ sending $(\mathcal{S}, \mathcal{T})$ to $\mathcal{S} \otimes_{\max} \mathcal{T}$ is an associative, symmetric and functorial operator system tensor product. The positive cone of the maximal tensor product is the smallest among all possible positive cones of operator system tensor products [7, Theorem 5.5]. For C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we have the completely order isomorphic inclusion

$$\mathcal{A} \otimes_{\max} \mathcal{B} \subset \mathcal{A} \otimes_{\mathrm{C}^* \max} \mathcal{B}$$

[7, Theorem 5.12].

3. An approximation theorem for nuclear operator systems

We prove the main theorem of this paper which generalizes the Choi–Effros–Kirchberg approximation theorem. The proof is quite simple compared to the original one. In particular, the proof does not depend on the Kaplansky density theorem.

Theorem 3.1. *Suppose that $\Phi : \mathcal{S} \to \mathcal{T}$ is a unital completely positive map for operator systems $\mathcal{S}$ and $\mathcal{T}$. The following are equivalent*:

(i) *the map*

$$\mathrm{id}_{\mathcal{R}} \otimes \Phi : \mathcal{R} \otimes_{\min} \mathcal{S} \to \mathcal{R} \otimes_{\max} \mathcal{T}$$

is completely positive for any operator system $\mathcal{R}$;

(ii) *the map*

$$\mathrm{id}_E \otimes \Phi : E \otimes_{\min} \mathcal{S} \to E \otimes_{\max} \mathcal{T}$$

is completely positive for any finite dimensional operator system E;

(iii) *there exist nets of unital completely positive maps* $\varphi_\lambda : \mathcal{S} \to M_{n_\lambda}$ *and* $\psi_\lambda : M_{n_\lambda} \to \mathcal{T}$ *such that* $\psi_\lambda \circ \varphi_\lambda$ *converges to the map* Φ *in the point-norm topology.*

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "\Phi"] \arrow[dr, "\varphi_\lambda"'] & & \mathcal{T} \\ & M_{n_\lambda} \arrow[ur, "\psi_\lambda"'] & \end{tikzcd}$$

Proof. Clearly, (i) implies (ii).

(iii) $\Rightarrow$ (i). For any operator system $\mathcal{R}$ and any $n \in \mathbb{N}$, if we identify $M_k(M_n \otimes \mathcal{R}) = M_{nk} \otimes \mathcal{R}$ in the usual manner, then a somewhat tedious calculation shows that $D_k^{\max}(M_n, \mathcal{R}) = M_{nk}(\mathcal{R})^+ = M_k(M_n \otimes_{\min} \mathcal{R})^+$. This gives an independent verification that $\mathcal{R} \otimes_{\max} M_n = \mathcal{R} \otimes_{\min} M_n$, i.e., that the two operator system structures are identical. Alternatively, this fact follows from [7, Corollary 6.8], which they point out is obtained independently of the Choi–Effros–Kirchberg theorem. From the maps

$$\mathcal{R} \otimes_{\min} \mathcal{S} \xrightarrow{\mathrm{id}_{\mathcal{R}} \otimes \varphi_\lambda} \mathcal{R} \otimes_{\min} M_{n_\lambda} = \mathcal{R} \otimes_{\max} M_{n_\lambda} \xrightarrow{\mathrm{id}_{\mathcal{R}} \otimes \psi_\lambda} \mathcal{R} \otimes_{\max} \mathcal{T},$$

we see that the map

$$\mathrm{id}_{\mathcal{R}} \otimes \psi_\lambda \circ \varphi_\lambda : \mathcal{R} \otimes_{\min} \mathcal{S} \to \mathcal{R} \otimes_{\max} \mathcal{T}$$

is completely positive for any operator system $\mathcal{R}$. Since $\|\cdot\|_{\mathcal{R} \otimes_{\max} \mathcal{T}}$ is a cross norm, $\mathrm{id}_{\mathcal{R}} \otimes (\psi_\lambda \circ \phi_\lambda)(z)$ converges to $\mathrm{id}_{\mathcal{R}} \otimes \Phi(z)$ for each $z \in \mathcal{R} \otimes \mathcal{S}$. It follows that $z \in (\mathcal{R} \otimes_{\min} \mathcal{S})^+$ implies $\mathrm{id}_{\mathcal{R}} \otimes \Phi(z) \in (\mathcal{R} \otimes_{\max} \mathcal{T})^+$.

(ii) $\Rightarrow$ (iii). Let E be a finite dimensional operator subsystem of $\mathcal{S}$. There exists a state ω_1 on E which plays a role of the non-canonical Archimedean order unit on the dual space E^* [1, Corollary 4.5]. In other words, (E^*, ω_1) is an operator system. We can regard the inclusion $\iota : E \subset \mathcal{S}$ as an element in $(E^* \otimes_{\min} \mathcal{S})^+$ [8, Lemma 8.4]. The restriction $\Phi|_E : E \to \mathcal{T}$ can be identified with the element $(\mathrm{id}_{E^*} \otimes \Phi)(\iota)$. By assumption, it belongs to $(E^* \otimes_{\max} \mathcal{T})^+$. We consider the directed set

$$\Omega = \big\{(E, \varepsilon)\colon\ E \text{ is a finite dimensional operator subsystem of } \mathcal{S},\ \varepsilon > 0\big\}$$

with the standard partial order. Let $\lambda = (E, \varepsilon)$. For any $\varepsilon > 0$, the restriction $\Phi|_E$ can be written as

$$\Phi|_E + \varepsilon\omega_1 \otimes 1_{\mathcal{T}} = \alpha f \otimes Q \alpha^*$$

for $\alpha \in M_{1,n_\lambda m}$, $f \in M_{n_\lambda}(E^*)^+$ and $Q \in M_m(\mathcal{T})^+$. The map $f : E \to M_{n_\lambda}$ is completely positive and the matrix $f(1_{\mathcal{S}})$ is positive semi-definite. Let P be the support projection of $f(1_{\mathcal{S}})$. For $x \in \mathcal{S}^+$, we have

$$0 \leqslant f(x) \leqslant \|x\| f(1_{\mathcal{S}}) \leqslant \|x\| \big\| f(1_{\mathcal{S}}) \big\| P.$$

Since every element in $\mathcal{S}$ can be written as a linear combination of positive elements in $\mathcal{S}$, the range of f is contained in $PM_{n_\lambda}P$. The positive semi-definite matrix $f(1_{\mathcal{S}})$ is invertible in $PM_{n_\lambda}P$. We denote by $f(1_{\mathcal{S}})^{-1}$ its inverse in $PM_{n_\lambda}P$. Put $p = \operatorname{rank} P$ and let $U^*PU = I_p \oplus 0$ be the diagonalization of P. Since we can write

$$\alpha f \otimes Q\alpha^* = \alpha\left(f(1_{\mathcal{S}})^{\frac{1}{2}}U\begin{pmatrix} I_p \\ 0\end{pmatrix} \otimes I_m\right)\cdot\left[\begin{pmatrix} I_p & 0\end{pmatrix}U^*f(1_{\mathcal{S}})^{-\frac{1}{2}}ff(1_{\mathcal{S}})^{-\frac{1}{2}}U\begin{pmatrix} I_p \\ 0\end{pmatrix}\otimes Q\right]$$
$$\cdot\left(f(1_{\mathcal{S}})^{\frac{1}{2}}U\begin{pmatrix} I_p \\ 0\end{pmatrix} \otimes I_m\right)^*\alpha^*,$$

we may assume that $f : E \to M_{n_\lambda}$ is a unital completely positive map. By the Arveson extension theorem, $f : E \to M_{n_\lambda}$ extends to a unital completely positive map $\varphi_\lambda : \mathcal{S} \to M_{n_\lambda}$. We define a completely positive map $\psi'_\lambda : M_{n_\lambda} \to \mathcal{T}$ by

$$\psi'_\lambda(A) = \alpha A \otimes Q\alpha^*, \quad A \in M_{n_\lambda}.$$

For $x \in E$, we have

$$\left\|\Phi(x) - \psi'_\lambda \circ \varphi_\lambda(x)\right\| = \left\|\Phi(x) - \alpha f(x) \otimes Q\alpha^*\right\| = \varepsilon\left\|\omega_1(x)1_{\mathcal{T}}\right\| \leqslant \varepsilon\|x\|.$$

Hence, we can take nets of unital completely positive maps $\varphi_\lambda : \mathcal{S} \to M_{n_\lambda}$ and completely positive maps $\psi'_\lambda : M_{n_\lambda} \to \mathcal{T}$ such that $\psi'_\lambda \circ \varphi_\lambda$ converges to the map Φ in the point-norm topology. Since each φ_λ is unital, $\psi'_\lambda(I_{n_\lambda})$ converges to $1_{\mathcal{T}}$. Let us choose a state ω_λ on M_{n_λ} and set

$$\psi_\lambda(A) = \frac{1}{\|\psi'_\lambda\|}\psi'_\lambda(A) + \omega_\lambda(A)\left(1_{\mathcal{T}} - \frac{1}{\|\psi'_\lambda\|}\psi'_\lambda(I_{n_\lambda})\right).$$

Then $\psi_\lambda : M_{n_\lambda} \to \mathcal{T}$ is a unital completely positive map such that $\psi_\lambda \circ \varphi_\lambda$ converges to the map Φ in the point-norm topology. □

Putting $\mathcal{S} = \mathcal{T}$ and $\Phi = \mathrm{id}_{\mathcal{S}}$, we obtain the following corollary.

Corollary 3.2. *Let $\mathcal{S}$ be an operator system. The following are equivalent*:

(i) *$\mathcal{S}$ is nuclear*;
(ii) *we have*

$$E \otimes_{\min} \mathcal{S} = E \otimes_{\max} \mathcal{S}$$

for any finite dimensional operator system E;
(iii) *there exist nets of unital completely positive maps $\varphi_\lambda : \mathcal{S} \to M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \to \mathcal{S}$ such that $\psi_\lambda \circ \varphi_\lambda$ converges to* $\mathrm{id}_{\mathcal{S}}$ *in the point-norm topology.*

Corollary 3.3 *(Choi–Effros–Kirchberg theorem). Let $\mathcal{A}$ be a unital C^*-algebra. Then $\mathcal{A}$ is C^*-nuclear if and only if there exist nets of unital completely positive maps $\varphi_\lambda : \mathcal{A} \to M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \to \mathcal{A}$ such that $\psi_\lambda \circ \varphi_\lambda$ converges to* $\mathrm{id}_{\mathcal{A}}$ *in the point-norm topology.*

Proof. It will be enough to prove that if $\mathcal{A}$ is C^*-nuclear, then for every operator system $\mathcal{T}$, the minimal and maximal operator system tensor products coincide on $\mathcal{A} \otimes \mathcal{T}$. Again this fact follows from [7, Corollary 6.8] which is independent of the Choi–Effros–Kirchberg theorem.

Since the notation is somewhat different in [7] and their result relies on several earlier results, we repeat the argument below.

Let $C_u^*(\mathcal{T})$ be the universal C^*-algebra generated by the operator system $\mathcal{T}$ as defined in [7]. Since $\mathcal{A}$ is C^*-nuclear, we have that $\mathcal{A} \otimes_{C^* \min} C_u^*(\mathcal{T}) = \mathcal{A} \otimes_{C^* \max} C_u^*(\mathcal{T})$. But we have that $\mathcal{A} \otimes_{\min} \mathcal{T} \subseteq \mathcal{A} \otimes_{C^* \min} C_u^*(\mathcal{T})$ completely order isomorphically, by [7, Corollary 4.10]. Also, by [7, Theorem 6.4] the inclusion of the *commuting* tensor product $\mathcal{A} \otimes_{\mathrm{c}} \mathcal{T} \subseteq \mathcal{A} \otimes_{C^* \max} C_u^*(\mathcal{T})$ is a complete order isomorphism.

Thus, the fact that $\mathcal{A}$ is C^*-nuclear implies that $\mathcal{A} \otimes_{\min} \mathcal{T} = \mathcal{A} \otimes_{\mathrm{c}} \mathcal{T}$ completely order isomorphically. Finally, the result follows from the fact [7, Theorem 6.7], that for any C^*-algebra $\mathcal{A}$, $\mathcal{A} \otimes_{\mathrm{c}} \mathcal{T} = \mathcal{A} \otimes_{\max} \mathcal{T}$, completely order isomorphically. □

Remark 3.4. Suppose that we call an operator system $\mathcal{S}$ *C^*-nuclear* if $\mathcal{S} \otimes_{\min} \mathcal{B} = \mathcal{S} \otimes_{\max} \mathcal{B}$ for every unital C^*-algebra $\mathcal{B}$. Then it follows by [7, Theorem 6.4], that an operator system $\mathcal{S}$ is C^*-nuclear if and only if $\mathcal{S} \otimes_{\min} \mathcal{T} = \mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$ for every operator system $\mathcal{T}$. In the terminology of [7], this latter property is the definition of $(\min, \mathrm{c})$-nuclearity. Thus, an operator system is C^*-nuclear if and only if it is $(\min, \mathrm{c})$-nuclear. A complete characterization of such operator systems is still unknown.

By a result of Choi and Effros [2], a C^*-algebra $\mathcal{A}$ is nuclear if and only if its enveloping von Neumann algebra $\mathcal{A}^{**}$ is injective. We wish to extend this result to nuclear operator systems. In the next section we produce an example of a nuclear operator system that is not completely order isomorphic to any C^*-algebra.

An operator space X is called *nuclear* provided that there exist nets of complete contractions $\varphi_\lambda : X \to M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \to X$ such that $\psi_\lambda \circ \varphi_\lambda$ converges to id_X in the point-norm topology. Kirchberg [10] gives an example of an operator space X that is not nuclear, but such that the bidual X^{**} is completely isometric to an injective von Neumann algebra. A later theorem of Effros, Ozawa and Ruan [5, Theorem 4.5] implies that Kirchberg's operator space X is also not locally reflexive. See [4] for further details on local reflexivity.

These pathologies do not occur for operator systems. This follows from the works of Kirchberg [10] and of Effros, Ozawa and Ruan [5]. The following summarizes their results.

Theorem 3.5. *Let $\mathcal{S}$ be an operator system. Then the following are equivalent*:

(i) *$\mathcal{S}$ is a nuclear operator system*;
(ii) *$\mathcal{S}$ is a nuclear operator space*;
(iii) *$\mathcal{S}^{**}$ is unitally completely order isomorphic to an injective von Neumann algebra.*

Proof. Clearly, (i) implies (ii) by Theorem 3.1.

For (ii) $\Rightarrow$ (iii), combine [5, Theorem 4.5] and [1, Theorem 3.1] and Sakai's theorem.

Finally, the proof that (iii) implies (i), is due to Kirchberg [10, Lemma 2.8(ii)]. □

Smith's characterization of nuclear C^*-algebras [16, Theorem 1.1] follows from (ii) $\Rightarrow$ (i). We now see another contrast between operator spaces and operator systems.

Corollary 3.6. *Let $\mathcal{S}$ be an operator system. If $\mathcal{S}^{**}$ is unitally completely order isomorphic to an injective von Neumann algebra, then $\mathcal{S}$ is a locally reflexive operator space.*

Proof. By the above result, $\mathcal{S}$ is a nuclear operator space and hence by [5, Theorem 4.4], $\mathcal{S}$ is locally reflexive. □

Corollary 3.7. *Every finite dimensional nuclear operator system is unitally completely order isomorphic to the direct sum of matrix algebras.*

Proof. Let $\mathcal{S}$ be a finite dimensional operator system. Then $\mathcal{S} = \mathcal{S}^{**}$, which by the above result is unitally completely order isomorphic to a finite dimensional C^*-algebra. □

Remark 3.8. Kirchberg [10, Theorem 1.1] proves that every nuclear separable operator system is unitally completely isometric to a quotient of the CAR-algebra by a hereditary C^*-subalgebra and that conversely, every such quotient gives rise to a nuclear separable operator system.

4. A nuclear operator system that is not a C^*-algebra

Kirchberg and Wassermann [11] constructed a remarkable example of a nuclear operator system that has no unital complete order embedding into any nuclear C^*-algebra. So, in particular, they give an example of a nuclear operator system that is not unitally completely order isomorphic to a C^*-algebra. In this section we provide a very concrete example of this latter phenomena.

Let $\mathcal{K}_0 \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ denote the norm closed linear span of $\{E_{i,j}\colon (i,j) \neq (1,1)\}$, where $E_{i,j}$ are the standard matrix units and let

$$\mathcal{S}_0 = \{\lambda I + K_0\colon \lambda \in \mathbb{C},\ K_0 \in \mathcal{K}_0\} \subseteq \mathcal{B}\big(\ell^2(\mathbb{N})\big)$$

denote the operator system spanned by $\mathcal{K}_0$ and the identity operator. The goals of this section are to show that $\mathcal{S}_0$ is a nuclear operator system that it is not unitally completely order isomorphic to any C^*-algebra and that $\mathcal{S}_0^{**}$ is unitally completely order isomorphic to $\mathcal{B}(\ell^2(\mathbb{N}))$.

Let $V_n : \mathbb{C}^n \to \ell^2(\mathbb{N})$ be the isometric inclusion defined by $V_n(e_j) = e_j$, $1 \leqslant j \leqslant n$ and let $Q_n \in \mathcal{B}(\ell^2(\mathbb{N}))$ be the projection onto the orthocomplement of $V_n(\mathbb{C}^n)$. Finally, define unital completely positive maps, $\varphi_n : \mathcal{B}(\ell^2(\mathbb{N})) \to M_n$ and $\psi_n : M_n \to \mathcal{B}(\ell^2(\mathbb{N}))$ by

$$\varphi_n(X) = V_n^* X V_n \quad \text{and} \quad \psi_n(Y) = V_n Y V_n^* + y_{1,1} Q_n, \quad Y = (y_{i,j}).$$

Proposition 4.1. *The following hold*:

(i) $\psi_n(M_n) \subseteq \mathcal{S}_0$;
(ii) *for any $m \in \mathbb{N}$ and $(X_{i,j}) \in M_m(\mathcal{S}_0)$, $\|(X_{i,j}) - (\psi_n \circ \varphi_n(X_{i,j}))\| \to 0$ as $n \to +\infty$*;
(iii) *$\mathcal{S}_0$ is a nuclear operator system.*

Proof. Given $Y \in M_n$, we have that $\psi_n(Y - y_{1,1} I_n) \in \mathcal{K}_0$, and hence $\psi_n(Y) \in \mathcal{S}_0$ and (i) follows.

If $X \in \mathcal{K}_0$, then the first $n \times n$ matrix entries of $\psi_n \circ \varphi_n(X)$ agree with those of X and the remaining entries are 0. Since X is compact, $\|X - \psi_n \circ \varphi_n(X)\| \to 0$ and since both maps are unital, we have that (ii) holds for the case $m = 1$. The case $m > 1$ follows similarly.

Statement (iii) follows by (ii) and Theorem 3.1. □

Theorem 4.2. *The nuclear operator system $\mathcal{S}_0$ is not unitally completely order isomorphic to a C^*-algebra.*

Proof. Assume to the contrary that $\mathcal{A}$ is a unital C^*-algebra and that $\gamma : \mathcal{A} \to \mathcal{S}_0$ is a unital, complete order isomorphism. Then γ is also a completely isometric isomorphism. Use the Stinespring representation [12, Theorem 4.1] to write $\gamma(a) = P\pi(a)P$, where $\pi : \mathcal{A} \to \mathcal{B}(\ell^2(\mathbb{N}) \oplus \mathcal{H})$ is a unital $*$-homomorphism and $P : \ell^2(\mathbb{N}) \oplus \mathcal{H} \to \ell^2(\mathbb{N})$ denotes the orthogonal projection.

Let $a_{i,j}$, $(i, j) \neq (1, 1)$ denote the unique elements of $\mathcal{A}$, satisfying $\gamma(a_{i,j}) = E_{i,j}$. Relative to the decomposition $\ell^2(\mathbb{N}) \oplus \mathcal{H}$, we have that

$$\pi(a_{i,j}) = \begin{pmatrix} E_{i,j} & B_{i,j} \\ C_{i,j} & D_{i,j} \end{pmatrix},$$

where $B_{i,j} : \mathcal{H} \to \ell^2(\mathbb{N})$, $C_{i,j} : \ell^2(\mathbb{N}) \to \mathcal{H}$ and $D_{i,j} : \mathcal{H} \to \mathcal{H}$ are bounded operators.

By choosing an orthonormal basis $\{u_t\}_{t\in T}$ we may regard $B_{i,j}$ as an $\mathbb{N} \times T$ matrix and $C_{i,j}$ as a $T \times \mathbb{N}$ matrix. Since $\|\pi(a_{i,j})\| = \|E_{i,j}\| = 1$, we must have that the i-th row of $B_{i,j}$ is 0 and the j-th column of $C_{i,j}$ is 0.

If $k \neq i$, then

$$1 = \big\|(E_{i,j}, E_{k,k+1})\big\| = \big\|\big(\pi(a_{i,j}), \pi(a_{k,k+1})\big)\big\| \geqslant \big\|(E_{i,j}, B_{i,j}, E_{k,k+1}, B_{k,k+1})\big\|$$

from which it follows that the k-th row of $B_{i,j}$ is also 0. This proves that $B_{i,j} = 0$ for all $(i, j) \neq (1, 1)$.

A similar argument using the fact that $\big\|\big(\begin{smallmatrix} E_{i,j} \\ E_{k+1,k} \end{smallmatrix}\big)\big\| = 1$ for $k \neq j$ yields that $C_{i,j} = 0$ for all $(i, j) \neq (1, 1)$.

Since $\mathcal{A}$ is the closed linear span of $a_{i,j}$, $(i, j) \neq (1, 1)$ and the identity it follows that for any $a \in \mathcal{A}$,

$$\pi(a) = \begin{pmatrix} \gamma(a) & 0 \\ 0 & \rho(a) \end{pmatrix},$$

for some linear map $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$.

But since π is a unital $*$-homomorphism, it follows that $\gamma : \mathcal{A} \to \mathcal{B}(\ell^2(\mathbb{N}))$ is a unital $*$-homomorphism and, consequently, that $\mathcal{S}_0$ is a C^*-subalgebra of $\mathcal{B}(\ell^2(\mathbb{N}))$. But $E_{1,2}, E_{2,1} \in \mathcal{S}_0$, while $E_{1,1} = E_{1,2}E_{2,1} \notin \mathcal{S}_0$. This contradiction completes the proof. □

By Theorem 3.5, we know that $\mathcal{S}_0^{**}$ is an injective von Neumann algebra, so it is interesting to identify the precise algebra.

Theorem 4.3. *$\mathcal{S}_0^{**}$ is unitally completely order isomorphic to $\mathcal{B}(\ell^2(\mathbb{N}))$.*

Proof. We only prove that $\mathcal{S}_0^{**}$ is unitally order isomorphic to $\mathcal{B}(\ell^2(\mathbb{N}))$. To this end, let $\mathcal{S} = \{\lambda I + K\colon \lambda \in \mathbb{C},\ K \in \mathcal{K}(\ell^2(\mathbb{N}))\}$, denote the unital C^*-algebra spanned by the compact operators $\mathcal{K}(\ell^2(\mathbb{N}))$ and the identity. Thus, $\mathcal{S}_0 \subseteq \mathcal{S}$ is a codimension 1 subspace.

As vector spaces, we have that $\mathcal{S} = \mathbb{C} \oplus \mathcal{K}(\ell^2(\mathbb{N}))$, so that $\mathcal{S}^* = \mathbb{C} \oplus \mathcal{T}(\ell^2(\mathbb{N}))$, where this latter space denotes the trace class operators.

We let $\delta_{i,j} : \mathcal{K}(\ell^2(\mathbb{N})) \to \mathbb{C}$ denote the linear functional satisfying

$$\delta_{i,j}(E_{k,l}) = \begin{cases} 1 & i = k,\ j = l, \\ 0 & \text{otherwise} \end{cases}$$

so that every element of $\mathcal{K}(\ell^2(\mathbb{N}))^*$ is of the form $\sum_{i,j} t_{i,j}\delta_{i,j}$ for some trace class matrix $T = (t_{i,j})$. We identify $\mathcal{S}^* = \mathbb{C} \oplus \mathcal{T}(\ell^2(\mathbb{N}))$ where

$$\langle (\beta, T), \lambda I + K \rangle = \beta\lambda + \sum_{i,j} t_{i,j}k_{i,j} = \beta\lambda + \operatorname{tr}(T^t K)$$

with $K = (k_{i,j})$.

The functional (β, T) is positive if and only if T is a positive operator and $\beta \geqslant \operatorname{tr}(T)$. If (β, T) is a positive functional on $\mathcal{S}$, then we have

$$0 \leqslant \langle (\beta, T), K \rangle = \operatorname{tr}(T^t K) \quad \text{and} \quad 0 \leqslant \langle (\beta, T), I - I_n \rangle = \beta - \operatorname{tr}(T^t I_n)$$

for all positive compact operators K and $n \in \mathbb{N}$. Let $\lambda I + K$ be a positive operator. Since K is compact, we have $\lambda \geqslant 0$. The converse follows from

$$\langle (\beta, T), \lambda I + K \rangle = \beta\lambda + \operatorname{tr}(T^t K) \geqslant \operatorname{tr}(T^t(\lambda I + K)) \geqslant 0.$$

Identify $\mathcal{S}_0^*$ with $\mathbb{C} \oplus \mathcal{T}_0$ where $\mathcal{T}_0$ denotes the trace class operators $T_0 = (t_{i,j})$ with $t_{1,1} = 0$. Since every positive functional on $\mathcal{S}_0$ extends to a positive functional on $\mathcal{S}$ by the Krein theorem, we have that (β, T_0) defines a positive functional if and only if there exists $\alpha \in \mathbb{C}$, such that $T = T_0 + \alpha E_{1,1}$ is positive and $\beta \geqslant \operatorname{tr}(T_0) + \alpha$. That is if and only if $\beta \geqslant \operatorname{tr}(T)$, where T is some positive trace class operator equal to T_0 modulo the span of $E_{1,1}$.

In a similar fashion we may identify $\mathcal{S}_0^{**}$ as the vector space $\mathbb{C} \oplus \mathcal{B}_0$, where $X_0 = (x_{i,j}) \in \mathcal{B}_0$ if and only if X_0 is bounded and $x_{1,1} = 0$. Moreover, (μ, X_0) will define a positive element of $\mathcal{S}_0^{**}$ if and only if

$$\mu\beta + \sum_{(i,j)\neq(1,1)} x_{i,j}t_{i,j} \geqslant 0,$$

for every positive linear functional (β, T_0).

We claim that (μ, X_0) is positive if and only if $\mu I + X_0 \in \mathcal{B}(\ell^2(\mathbb{N}))$ is a positive operator. This will show that the bijection

$$(\mu, X_0) \in \mathcal{S}_0^{**} \mapsto \mu I + X_0 \in \mathcal{B}(\ell^2(\mathbb{N}))$$

is an order isomorphism. Also, note that the identity of $\mathcal{S}_0^{**}$ is $(1, 0)$, so that this map is unital.

To see the claim, first let $(\mu, X_0) \in \mathcal{S}_0^{**}$ be positive. Given any $T = T_0 + \alpha E_{1,1}$ a positive trace class operator, let $\beta = \alpha + \operatorname{tr}(T_0) = \operatorname{tr}(T)$. Then (β, T_0) is positive in $\mathcal{S}_0^*$ and, hence

$$0 \leqslant \mu\beta + \sum_{(i,j)\neq(1,1)} x_{i,j}t_{i,j} = \mu \operatorname{tr}(T) + \operatorname{tr}(X_0^t T) = \operatorname{tr}((\mu I + X_0)^t T).$$

Since T was an arbitrary trace class operator, this shows that $\mu I + X_0$ is a positive operator in $\mathcal{B}(\ell^2(\mathbb{N}))$.

Conversely, if $\mu I + X_0$ is a positive operator, then for any positive $(\beta, T_0) \in \mathcal{S}_0^*$, pick α as above and set $T = \alpha E_{1,1} + T_0$. We have that

$$\mu\beta + \sum_{(i,j)\neq(1,1)} x_{i,j}t_{i,j} \geqslant \mu\,\mathrm{tr}(T) + \mathrm{tr}\big(X_0^t T\big) = \mathrm{tr}\big((\mu I + X_0)^t T\big) \geqslant 0,$$

since both operators are positive.

This completes the proof of the claim and of the theorem. □

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