JOURNAL OF NUMBER THEORY 19, 148–175 (1984)

The Theory of Zeta Functions of Several Complex Variables, I

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Communicated by H. Stark

Received April 24, 1978; revised August 8, 1980

The paper introduces a general class of Tate-like zeta functions and proves an analytic continuation and a general formula for the values of such zeta functions at negative integral arguments. In particular these zeta functions discussed include the Shintani zeta functions and the results generalize his formula for the value of such zeta functions at negative integral arguments. © 1984 Academic Press, Inc.

1. INTRODUCTION

In this paper, we develop the elements of a theorem of zeta functions of several complex variables. The zeta functions we consider are generalizations of the usual zeta functions of algebraic number theory. We are able to establish, under rather mild hypotheses, theorems concerning their analytic continuations, behavior at singular points and values at negative integral vectors. The one variable version of the methods we use was described briefly in [4]. By using the methods developed here, one can replace various classical arguments involving contour integration (especially loop integrals of Hankel type) with rather intrinsic arguments which generalize to several variables. Very roughly, our viewpoint is this: Instead of considering the zeta function directly, we consider it as a Mellin transform of a suitable function. We read off the properties of the zeta function from the properties of this function. In particular, we show that the values of our zeta functions at negative integral vectors may be calculated in terms of various derivatives of their inverse Mellin transforms. As a special case, we obtain a new proof of and a generalization of Shintani's recent theorem [5] on the value of a generalized Hurwitz zeta function at negative integers.

* Research supported by National Science Foundation Research Grant MCS77-01279A01 and MCS.

0022-314X/84 $3.00

Let us now describe the zeta functions considered. Let $\mathbb{R}_+$ denote the multiplicative group of positive real numbers, n a positive integer D a discrete subset of

$$\mathbb{R}_+^n = \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \qquad (n \text{ copies}).$$

Denote a typical element of D by

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n).$$

Let $s_1, s_2, \ldots, s_n$ be complex variables and set $\mathbf{s} = (s_i, \ldots, s_n)$. Further, let $a: D \to \mathbb{C}$ be any complex valued function and $\zeta_{D,a}(\mathbf{s})$ a Dirichlet series of the form

$$\zeta_{D,a}(\mathbf{s}) = \sum_{\boldsymbol{\xi} \in D} a(\boldsymbol{\xi})\, \boldsymbol{\xi}^{-\mathbf{s}},$$

where we have employed the symbolic notation

$$\boldsymbol{\xi}^{-\mathbf{s}} = \xi_1^{-s_1} \cdots \xi_n^{-s_n}.$$

The organization of this paper is as follows: In Section 2, we introduce certain Mellin transforms in n complex variables. We study the analytic continuation of these Mellin transforms and their values at nonpositive integral vectors. Section 2 considers only Mellin transforms of Schwartz functions on a certain space $\bar{G}$. Actually, the zeta functions which one meets in practice have Mellin transforms which are not Schwartz functions in that they possess singularities at the origin. In Section 3, we study such Mellin transforms by considering the space of Schwartz functions corresponding to a covering of the space $\bar{G}$. In Section 4, we investigate the properties of $\zeta_{D,a}(\mathbf{s})$. Section 5 is devoted to examples and Section 6 proves an integrality theorem which implies certain arithmetic information about the values of certain Mellin transforms at nonpositive integral arguments. In particular, we derive a formula for Shintani's generalized Hurwitz zeta functions at $-m$ ($m \in \mathbb{Z}$, $m \geqslant 0$) which is somewhat more explicit than given by Shintani.

2. Mellin Transforms and Analytic Continuations

Let $G = \mathbb{R}_+^n$. Then G is a locally compact abelian group, whose Haar measure we normalize to be

$$d^\times \mathbf{y} = \frac{dy_1}{y_1} \times \cdots \times \frac{dy_n}{y_n},$$

where $dy_1, \dots, dy_n$ denote Lebesgue measure on the real line. Let $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0\} = [0, \infty)$ and set $\bar{G} = \bar{\mathbb{R}}_+^n$.

Typically, let $\mathbf{m}$ denote a vector $(m_1, \dots, m_n) \in \mathbb{Z}^n \cap \bar{G}$. Let $\mathscr{D}_{\mathbf{m}}$ denote the monomial differential operator

$$\mathscr{D}_{\mathbf{m}} = \frac{\partial^{m_1}}{\partial_{y_1}^{m_1}} \cdots \frac{\partial^{m_n}}{\partial_{y_n}^{m_n}}$$

acting on the algebra of complex-valued C^∞ functions on $\bar{G}$.

Associated to $\mathbf{y} = (y_1, \dots, y_n) \in \bar{G}$ and $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$, we define the quasi-character $\chi_{\mathbf{s}}(\mathbf{y}) = \mathbf{y}^{\mathbf{s}}$ via

$$\chi_{\mathbf{s}}(\mathbf{y}) = \mathbf{y}^{\mathbf{s}} = y_1^{s_1} \cdots y_n^{s_n}.$$

For a complex-valued function f on G, we define the Mellin transform $\Phi(f, \mathbf{s})$ via

$$\Phi(f, \mathbf{s}) = \int_G f(\mathbf{y})\, \mathbf{y}^{\mathbf{s}} d^\times \mathbf{y},$$

for all values of $\mathbf{s}$ for which the integral converges.

We will say that f is C^∞ on $\bar{G}$ provided that f is C^∞ on G in the usual sense and that f is C^∞-differentiable from the right at all points in $\bar{G} - G$. Note that if f is C^∞ on $\bar{G}$, then $\mathscr{D}_{\mathbf{m}} f$ is also C^∞ on $\bar{G}$. Let $\mathscr{S}(\bar{G})$ denote the Schwartz space of $\bar{G}$. That is, a function $f\colon \bar{G} \to C$ belongs to $\mathscr{S}(\bar{G})$ provided that f is C^∞ on $\bar{G}$ and that for any polynomial P on $\bar{G}$ and any $\mathbf{m}$, the function $P\mathscr{D}_{\mathbf{m}} f$ is bounded on $\bar{G}$. It is clear that $\mathscr{S}(\bar{G})$ is a complex vector space and that $\mathscr{D}_m$ acts as an endomorphism of $\mathscr{S}(\bar{G})$. Furthermore, it is clear that if $f \in \mathscr{S}(\bar{G})$, then

$$\Phi(f, \mathbf{s}) = \int_G f(\mathbf{y})\, \mathbf{y}^{\mathbf{s}} d^\times \mathbf{y}$$

is defined and represents an analytic function of $s_1, \dots, s_n$ provided that $\mathrm{Re}(\mathbf{s}) = (\mathrm{Re}(s_1), \dots, \mathrm{Re}(s_n)) > 0$, where the inequality is to be interpreted coordinate-wise.

If $f \in \mathscr{S}(\bar{G})$, define $\varphi(f, \mathbf{s})$ via

$$\varphi(f, \mathbf{s}) = \frac{\Phi(f, \mathbf{s})}{\Gamma(\mathbf{s})}, \qquad \Gamma(\mathbf{s}) = \Gamma(s_1) \cdots \Gamma(s_n).$$

It is clear from the above discussion that $\varphi(f, \mathbf{s})$ is an analytic function of $s_1, \dots, s_n$ provided that $\mathrm{Re}(\mathbf{s}) > 0$. In most examples of interest, $\varphi(f, \mathbf{s})$ will turn out to be a Dirichlet series in the n complex variables $s_1, \dots, s_n$.

Therefore, it is significant to consider the problem of analytically continuing $\varphi(f, \mathbf{s})$ and determining its values at vectors $\mathbf{s} = -\mathbf{m}$. It is convenient to note that if $f(\mathbf{y}) = e^{-y_1 - \cdots - y_n}$, then $f \in \mathscr{S}(\bar{G})$ a trivial computation shows that $\varphi(f, \mathbf{s}) = 1$.

THEOREM 2.1. *Let* $f \in \mathscr{S}(\bar{G})$.

(a) *If* $\operatorname{Re}(\mathbf{s}) > 0$, *then*

$$\varphi(f, \mathbf{s}) = (-1)^{\operatorname{tr}(\mathbf{m})} \varphi(\mathscr{D}_{\mathbf{m}} f, \mathbf{s} + \mathbf{m}). \tag{2.1}$$

(b) $\varphi(f, \mathbf{s})$ *is an entire function of* $\mathbf{s} = (s_1, \dots, s_n)$.

(c) $$\varphi(f, -\mathbf{m}) = (-1)^{\operatorname{tr}(\mathbf{m})} (\mathscr{D}_{\mathbf{m}} f)(\mathbf{0}), \tag{2.2}$$

where $\operatorname{tr}(\mathbf{m}) = m_1 + \cdots + m_n$.

Proof. (a) Apply repeated integration by parts to the definition of $\Phi(f, \mathbf{s})$ to obtain

$$\begin{aligned}\Phi(f, \mathbf{s}) &= \frac{(-1)^{\operatorname{tr}(\mathbf{m})} \Phi(\mathscr{D}_m f, \mathbf{s} + \mathbf{m})}{\prod_{j=1}^n s_j(s_j + 1) \cdots (s_j + m_j - 1)} \\ &= (-1)^{\operatorname{tr}(\mathbf{m})} \prod_{j=1}^n \frac{\Gamma(s_j)}{\Gamma(s_j + m_j)} \cdot \Phi(\mathscr{D}_{\mathbf{m}} f, \mathbf{s} + \mathbf{m}) \qquad (\operatorname{Re}(\mathbf{s}) > 0).\end{aligned}$$

Note that we can justify differentiation under the integral sign easily from the hypothesis $f \in \mathscr{S}(\bar{G})$. Assertion (a) now follows directly from the definition of $\varphi(f, \mathbf{s})$.

(b) The right side of Eq. (2.1) is defined and analytic for $\operatorname{Re}(\mathbf{s}) > -\mathbf{m}$. Therefore, $\varphi(f, \mathbf{s})$ is defined and analytic for $\operatorname{Re}(\mathbf{s}) > -\mathbf{m}$. Since $\mathbf{m}$ is arbitrary, assertion (b) follows.

(c) We first prove that if $f \in \mathscr{S}(\bar{G})$ and $f(0) = 0$, then there are functions $f_1, \dots, f_n \in \mathscr{S}(\bar{G})$ such that $f(\mathbf{y}) = y_1 f_1(\mathbf{y}) + \cdots + y_n f_n(\mathbf{y})$. The proof is by induction on n. Let $g(\mathbf{y}) = f(y_1, \dots, y_{n-1}, 0)$. Then $g \in \mathscr{S}(\bar{\mathbb{R}}_+^{n-1})$ and $g(0) = 0$. We assume as induction hypothesis that there are functions $g_1, \dots, g_{n-1} \in \mathscr{S}(\bar{\mathbb{R}}_+^{n-1})$ such that

$$g = y_1 g_1 + \cdots + y_{n-1} g_{n-1}.$$

For $1 \leqslant j \leqslant n - 1$, let

$$f_j(y_1, \dots, y_n) = g_j(y_1, \dots, y_{n-1}) e^{-y_n}.$$

Then $f_j \in \mathscr{S}(\bar{G})$. However, if

$$h(\mathbf{y}) = f(\mathbf{y}) - y_1 f_1(\mathbf{y}) - \cdots - y_{n-1} f_{n-1}(\mathbf{y}),$$

then $h \in \mathscr{S}(\bar{G})$ and $h(y_1,..., y_{n-1}, 0) = 0$. Thus, there exists a C^∞ function f_n on $\bar{G}$ such that $h(\mathbf{y}) = y_n f_n(\mathbf{y})$. Since $h \in \mathscr{S}(\bar{G})$, we have $f_n \in \mathscr{S}(\bar{G})$ and

$$h(\mathbf{y}) = y_n f_n(\mathbf{y}) = f(\mathbf{y}) - y_1 f_1(\mathbf{y}) - \cdots - y_{n-1} f_{n-1}(\mathbf{y}),$$

and the desired decomposition of $f(\mathbf{y})$ has been determined.

Let us now prove assertion (c). By the above argument, there exist functions $f_1,..., f_n \in \mathscr{S}(\bar{G})$ such that

$$f(\mathbf{y}) = f(0)\, e^{-y_1 - \cdots - y_n} + y_1 f_1(\mathbf{y}) + \cdots + y_n f_n(\mathbf{y}).$$

Therefore, applying $\varphi(\cdot, \mathbf{s})$ to both sides of this equation gives

$$\begin{aligned} \varphi(f, \mathbf{s}) &= f(0)\, \varphi(e^{-y_1 - \cdots - y_n}, \mathbf{s}) + \varphi(y_1 f_1, \mathbf{s}) + \cdots + \varphi(y_n f_n, \mathbf{s}) \\ &= f(0) + \varphi(y_1 f_1, \mathbf{s}) + \cdots + \varphi(y_n f_n, \mathbf{s}). \end{aligned} \tag{2.3}$$

Clearly,

$$\Phi(y_i f, \mathbf{s}) = \Phi(f, s_1,..., s_{i-1}, s_i + 1, s_{i+1},..., s_n) \qquad (1 \leqslant i \leqslant n),$$

so that by the definition of φ, we have

$$\begin{aligned} \varphi(y_i f, \mathbf{s}) &= s_i \varphi(f, s_1,..., s_{i-1}, s_i + 1, s_{i+1},..., s_n) \\ &= s_i \varphi(f, \mathbf{s} + \mathbf{e}_i) \qquad (1 \leqslant i \leqslant n), \end{aligned} \tag{2.4}$$

where $\mathbf{e}_i$ is the unit vector

$$\mathbf{e}_i = (0, 0,..., \underset{\substack{\uparrow \\ i}}{1},..., 0).$$

Therefore, by (2.3) and (2.4), we have

$$\varphi(f, \mathbf{s}) = f(\mathbf{0}) + s_1 \varphi(f_1, \mathbf{s} + \mathbf{e}_1) + \cdots + s_n \varphi(f_n, \mathbf{s} + \mathbf{e}_n).$$

Since all terms in the above sum are entire in $\mathbf{s}$ (by (b)), we may set $\mathbf{s} = 0$ to obtain

$$\varphi(f, \mathbf{0}) = f(\mathbf{0}). \tag{2.5}$$

To complete the proof of (c), merely replace $\mathbf{s}$ by $-\mathbf{m}$ in (2.1) and apply (2.5).

3. Mellin Transforms of Functions Which Are Not Regular at the Origin

In the preceding section, our main result evaluates the Mellin transform $\varphi(f, -\mathbf{m})$ of a Schwartz function f on $\bar{G}$ at a nonnegative, integral vector $-\mathbf{m}$. In this section, we extend that result to include functions f with a singularity at the origin. The best known such function is simply $f(y) = 1/(e^y - 1)$ $(y \in \mathbb{R}_+)$, whose Mellin transform $\varphi(f, s)$ is the Riemann zeta function $\zeta(s)$. In this example, the singularity of f at the origin may be removed using a simple device. Namely, introduce $g(y) = yf(y)$. Then $g(y)$ belongs to $\mathscr{S}(\bar{\mathbb{R}}_+)$ and

$$\varphi(g, s) = s\varphi(f, s+1) = s\zeta(s+1).$$

Thus, by Theorem 2.1, $s\zeta(s+1)$ is entire and its values at negative integers $-m$ are given in terms of Bernoulli numbers as follows:

$$-m\zeta(1-m) = B_m, \qquad \text{where} \quad \frac{y}{e^y - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k.$$

Note that at $m = 0$ we recover the residue of $\zeta(s)$ at $s = 1$ as $B_0 = 1$. In the present section, we shall generalize the above example.

Let us begin by defining the class of functions f to which our theory applies. Let

$$K = \{\mathbf{y} \in \bar{G} \mid \operatorname{tr}(\mathbf{y}) = 1\}.$$

Then K is a compact, connected subset of $\bar{G}$. Note that K is a fundamental domain for the action of coordinate-wise multiplication of $\mathbb{R}_+$ on $\bar{G}$; namely for t_1, t_2 distinct nonnegative real numbers, we have

$$t_1 K \cap t_2 K = \varnothing,$$

and

$$\bigcup_{t \in \bar{\mathbb{R}}_+} tK = \bar{G}.$$

Define $G^* \subseteq \bar{\mathbb{R}}_+^{n+1}$ via

$$G^* = \bar{\mathbb{R}}_+ \times K,$$

endowed with the product topology. Let $\pi\colon G^* \to \bar{G}$ be the natural covering

$$\pi(t, \mathbf{k}) = t\mathbf{k}.$$

Then π is 1—1 on $\mathbb{R}_+ \times K$. However, $\pi^{-1}(\mathbf{0}) = \{0\} \times K$. In other words, G^* is a covering of $\bar{G}$ in which the origin is "blown up." Any function f on $\bar{G}$ can be pulled back to a function f^* on G^* via

$$f^*(t, k) = f(t\mathbf{k}).$$

Say that a function g on G^* is C^∞ provided that it is C^∞ in the usual sense on $\mathbb{R}_+ \times K$ and that it is C^∞-differentiable from the right on $\{0\} \times K$. Let $\mathscr{S}(G^*)$ denote the space of Schwartz functions on G^*. That is, $\mathscr{S}(G^*)$ consists of all complex-valued C^∞ functions g on G^* such that any order derivative of g multiplied by any polynomial (in the natural coordinates on G^*) is bounded. Roughly speaking, the requirement that f^* belong to $\mathscr{S}(G^*)$ guarantees that f is well behaved when restricted to lines through the origin.

Note that

$$f \in \mathscr{S}(\bar{G}) \Rightarrow f^* \in \mathscr{S}(G^*).$$

However, observe that there exist functions f on $\bar{G}$ such that $f^* \in \mathscr{S}(G^*)$ but for which $f \notin \mathscr{S}(\bar{G})$. For example, set

$$h(y_1, y_2) = \frac{1}{(\exp(y_1 + 2y_2) - 1)(\exp(3y_1 + y_2) - 1)}, \qquad (y_1, y_2) \in \bar{G} - \{\mathbf{0}\},$$

$$f(y_1, y_2) = y_1 y_2 h(y_1, y_2).$$

It is easy to see that $f^* \in \mathscr{S}(G^*)$ but nevertheless $f \notin \mathscr{S}(\bar{G})$, due to the singularity of f at the origin.

Let r be a real number, f a complex-valued function on $\bar{G} - \{0\}$. We say that f is *r-admissible* provided that

$$\lim_{t \to 0} t^r f(tk)$$

exists for all $k \in K$ and that the function $U_r f$ on G^* defined by

$$\begin{aligned} U_r f(t, k) &= t^r f(tk) && (t \neq 0), \\ &= \lim_{t \to 0} t^r f(tk) && (t = 0), \end{aligned}$$

belongs to $\mathscr{S}(G^*)$. In what follows, we shall study the behavior of the Mellin transform $\varphi(f, \mathbf{s})$ in a neighborhood of $\mathbf{s} = \mathbf{0}$, where f is r-admissible for some r. As we shall see subsequently, the class of r-admissible functions leads to many significant Mellin transforms including the generalized Hurwitz zeta function of Shintani.

Let r now be a nonnegative integer and assume that f is an r-admissible function. Then $U_r f \in \mathscr{S}(G^*)$, so that for $\mathbf{y} \in \bar{G} - \{0\}$, the following derivatives are defined:

$$f_\nu(\mathbf{y}) = \frac{1}{(\nu+r)!}\left(\frac{\partial}{\partial t}\right)^{\nu+r} (t^r f(t\mathbf{y}))\Big|_{t=0} \qquad (\nu = -r, -r+1, -r+2, \ldots).$$

Moreover, f_ν is C^∞ on $\bar{G} - \{0\}$ and is homogeneous of degree ν. We shall call f_ν the *νth homogeneous component of f.* By applying Taylor's theorem to $U_r f$, we derive that for $N \geqslant -r$,

$$f(t\mathbf{k}) = \sum_{\nu=-r}^{N} f_\nu(\mathbf{k})\, t^\nu + o(t^N) \qquad (t \to 0).$$

The constant in the o-term is uniform in k since K is compact. An equivalent formulation to the above is

$$f(\mathbf{y}) = \sum_{\nu=-r}^{N} f_\nu(\mathbf{y}) + o(\mathrm{Tr}(\mathbf{y})^N)$$

as $\mathbf{y} \to 0$ in $\bar{G} - \{0\}$. We refer to the formal sum

$$\sum_{\nu=-r}^{\infty} f_\nu(\mathbf{y})$$

as the (formal) *Laurent expansion of f about* 0.

Let $d^\times k$ be the measure on K such that

$$d^\times \mathbf{k}\, \frac{dt}{t} = d^\times y = \frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n}.$$

Concretely, if K is represented as the image of $\{(k_1, \ldots, k_{n-1}) \in \bar{\mathbb{R}}_+^{n-1} \mid k_1 + \cdots + k_{n-1} \leqslant 1\}$ via the map

$$(u_1, \ldots, u_{n-1}) \mapsto (u_1, \ldots, u_{n-1}, 1 - u_1 - \cdots - u_{n-1}),$$

then

$$d^\times k = du_1 \cdots du_{n-1}/u_1 u_2 \cdots u_{n-1}(1 - u_1 - \cdots - u_{n-1}).$$

Let f be an r-admissible function on $\bar{G}-\{\mathbf{0}\}$ for some nonnegative integer r. Then in particular, $U_r f$ is a bounded function on G^*; $t^r f(tk)$ is a bounded function on $\bar{G}-\{\mathbf{0}\}$. We may write the Mellin transform $\varphi(f, \mathbf{s})$ of f as

$$\begin{aligned}\varphi(f, \mathbf{s}) &= \frac{1}{\Gamma(\mathbf{s})}\int_{\bar{G}} f(\mathbf{y})\,\chi_{\mathbf{s}}(\mathbf{y})\,d^{\times}\mathbf{y}\\ &= \frac{1}{\Gamma(\mathbf{s})}\int_{\mathbb{R}_+}\int_K f(t\mathbf{k})\,\chi_{\mathbf{s}}(t\mathbf{k})\,d^{\times}k\,\frac{dt}{t}.\end{aligned}$$

Thus, since K is compact, we see that $\varphi(f, \mathbf{s})$ is analytic for $\mathrm{Re}(s) > 0$. If we examine the proof of Theorem 2.1a, we see that it applies equally well in case f is only r-admissible. Namely if $\mathrm{Re}(s) > 0$, we have

$$\varphi(f, \mathbf{s}) = (-1)^{\mathrm{tr}(\mathbf{m})}\,\varphi(\mathscr{D}_{\mathbf{m}} f, \mathbf{s}+\mathbf{m}) \tag{3.1}$$

for any $\mathbf{m} \in \mathbb{Z}^n$, $\mathbf{m} \geqslant \mathbf{0}$. In what follows, we shall show that $\varphi(f, \mathbf{s})$ is a meromorphic function of $\mathbf{s}$. Therefore, by analytic continuation, (3.1) will hold for all values of $\mathbf{s}$ for which $\varphi(f, \mathbf{s})$ is holomorphic.

If h is any C^∞ function on K, set

$$\Lambda(h, \mathbf{s}) = \frac{1}{\Gamma(\mathbf{s})}\int_K h(\mathbf{k})\,\chi_{\mathbf{s}}(\mathbf{k})\,d^{\times}\mathbf{k}.$$

We may now state the main result of this section.

THEOREM 3.1. *Let f be r-admissible on $\bar{G}-\{\mathbf{0}\}$, where r is a nonnegative integer.*

(a) *$\varphi(f, \mathbf{s})$ is meromorphic in $\mathbf{s}$. In fact, $\varphi(f, \mathbf{s})$ is holomorphic for* $\mathrm{tr}(\mathbf{s}) \notin \{r, r-1, r-2, \ldots\}$. *Suppose that $\nu \in \{-r, -r+1, -r+2, \ldots\}$. Then*

$$\varphi(f, \mathbf{s}) - \frac{\Lambda(f_\nu, \mathbf{s})}{\mathrm{tr}(\mathbf{s})+\nu}$$

is holomorphic in a neighborhood of the hyperplane $\mathrm{tr}(\mathbf{s}) = -\nu$ *and vanishes on the integral points of the hyperplane.*

(b) *Let* $\mathbf{m} \in \mathbb{Z}^n$, $\mathbf{m} \geqslant 0$. *Then*

$$\varphi(f, \mathbf{s}) - (-1)^{\mathrm{tr}(\mathbf{m})}\,\frac{\Lambda(\mathscr{D}_{\mathbf{m}} f_{\mathrm{tr}(\mathbf{m})}, \mathbf{s}+\mathbf{m})}{\mathrm{tr}(\mathbf{s}+\mathbf{m})}$$

is holomorphic in some neighborhood of the point $\mathbf{s} = -\mathbf{m}$ *and it vanishes at* $\mathbf{s} = -\mathbf{m}$.

(c) *Let* $\boldsymbol{\beta} = (\beta_1,\dots,\beta_n) \in \mathbb{C}^n$, $\operatorname{tr}(\beta) = 1$ *and let* $\mathbf{s}$ *approach* $-\mathbf{m}$ *in such a way that* $(\mathbf{s}+\mathbf{m})/\operatorname{tr}(\mathbf{s}+\mathbf{m})$ *approaches* $\boldsymbol{\beta}$. *Then*

$$\varphi(f, \mathbf{s}) \to (-1)^{\operatorname{tr}(\mathbf{m})} \sum_{j=1}^{n} \beta_j \mathscr{D}_{\mathbf{m}} f_{\operatorname{tr}(\mathbf{m})}(\mathbf{e}_j),$$

where $\mathbf{e}_j$ *is the jth unit vector.*

Remark. It is a consequence of the above theorem that for an r-admissible function f, the Mellin transform $\varphi(f, \mathbf{s})$ has singularities of the form

$$\frac{a_1 s_1 + \cdots + a_n s_n}{s_1 + \cdots + s_n}$$

at $\mathbf{s} = \mathbf{0}$. Therefore, $\varphi(f, \mathbf{s})$ has no single unambiguous value at $\mathbf{s} = \mathbf{0}$. Rather, one must specify how $\mathbf{s}$ approaches $\mathbf{0}$ in order to evaluate

$$\lim_{\mathbf{s} \to \mathbf{0}} \varphi(f, \mathbf{s}).$$

The proof of Theorem 3.1 is most conveniently formulated using two lemmas.

LEMMA 3.2. *Let h be a* C^∞ *function on K.*

(a) $\Lambda(h, \mathbf{s})$ *is an entire function of* $\mathbf{s}$.

(b) *If* $m \in \mathbb{Z}^n$, $\mathbf{m} \geqslant 0$, *then* $\Lambda(h, -\mathbf{m}) = 0$.

(c) *Let* $\mathbf{e}_j$ *denote the jth unit vector. Then*

$$\Lambda(h, \mathbf{e}_j) = h(\mathbf{e}_j) \qquad (j = 1,\dots, n).$$

Proof. (a) Assume first that $\operatorname{Re}(s_n) > 0$. Let

$$C = \{(u_1,\dots, u_{n-1}) \in \overline{\mathbb{R}}_+^{n-1} \mid u_1 + \cdots + u_{n-1} \leqslant 1\}.$$

For $(u_1,\dots, u_{n-1}) \in \overline{\mathbb{R}}_+^{n-1}$, define

$$\begin{aligned} h_1(u_1,\dots, u_{n-1}) &= h(u_1,\dots, u_{n-1}, 1 - u_1 - \cdots - u_{n-1})(1 - u_1 - \cdots - u_{n-1})^{s_n - 1} \\ &\qquad\qquad \text{if} \quad (u_1,\dots, u_{n-1}) \in C, \\ &= 0 \qquad\qquad \text{if} \quad (u_1,\dots, u_{n-1}) \notin C. \end{aligned}$$

Then h_1 is defined on $\overline{\mathbb{R}}_+^{n-1}$ and

$$\Lambda(h, \mathbf{s}) = \varphi(h_1, (s_1,\dots, s_{n-1})).$$

Note that h_1 is not in $\mathscr{S}(\overline{\mathbb{R}}_+^{n-1})$ since it is discontinuous on the boundary of C. However, Theorem 2.1 still applies. Indeed, h_1 has compact support and can be approximated by compactly supported C^∞ functions on $\overline{\mathbb{R}}_+^{n-1}$ which are identical to h_1 near the origin and hence are in $\mathscr{S}(\overline{\mathbb{R}}_+^{n-1})$. Thus, $\varphi(h_1, (s_1,..., s_{n-1}))$ is entire.

The above argument applies if, instead of $\mathrm{Re}(s_n) > 0$, we take $\mathrm{Re}(s_j) > 0$ for any j. Thus, $\Lambda(h, \mathbf{s})$ is holomorphic on $\{\mathbf{s} \in C^n \mid \mathrm{Re}(s_j) > 0 \text{ for some } j\}$. Note now that for any positive integer N and any $\mathbf{u} \in K$,

$$h(\mathbf{u}) = h(\mathbf{u})(u_1 + \cdots + u_n)^N.$$

Expanding $(u_1 + \cdots + u_n)^N$ by the multinomial theorem and applying some identities involving the Γ-function, we find that

$$\Lambda(h, \mathbf{s}) = (-1)^N N! \sum_{\substack{\mathbf{v} \geqslant 0 \\ \mathrm{Tr}(\mathbf{v}) = N}} \binom{-s_1}{v_1} \cdots \binom{-s_n}{v_n} \Lambda(h, \mathbf{s} + \mathbf{v}), \tag{3.2}$$

where the sum is over all $\mathbf{v} \in \mathbb{Z}^n$, $\mathbf{v} \geqslant 0$ such that $\mathrm{Tr}(\mathbf{v}) = v_1 + \cdots + v_n = N$. If N is sufficiently large, then for a given $\mathbf{s} \in \mathbb{C}^n$, we shall have $\mathrm{Re}(s_j + v_j) > 0$ for all j. Thus, each term $\Lambda(h, \mathbf{s} + \mathbf{v})$ of the above sum is holomorphic in a neighborhood of $\mathbf{s}$ and we conclude that $\Lambda(h, \mathbf{s})$ is entire.

(b) Take $N > \mathrm{Tr}(\mathbf{m})$ and apply (3.1), noting that, in each term of the sum some $v_j > m_j$, so that

$$\binom{m_j}{v_j} = 0.$$

(c) Let h_1 be as in part (a) of the proof and set $s_n = 1$. Then by Theorem 2.1c, we have

$$\Lambda(h, \mathbf{e}_n) = \varphi(h_1, \mathbf{0}) = h_1(\mathbf{0}) = h(\mathbf{e}_n).$$

Similarly, we show that

$$\Lambda(h, \mathbf{e}_j) = h(\mathbf{e}_j).$$

Suppose next that g is a function on G^* and that $g \in \mathscr{S}(G^*)$. Further, let $\mathbf{s} \in \mathbb{C}^n$, $\sigma \in \mathbb{C}$. Define the transform

$$\psi(g, \mathbf{s}, \sigma) = \frac{1}{\Gamma(\mathbf{s})\, \Gamma(\sigma)} \int_{G^*} g(t, \mathbf{k})\, \chi_s(\mathbf{k})\, t^\sigma d^\times k \frac{dt}{t}.$$

The properties of this transform are given in the following lemma.

LEMMA 3.3. *Let $g \in \mathscr{S}(G^*)$.*

(a) *$\psi(g, \mathbf{s}, \sigma)$ is an entire function of $\mathbf{s}, \sigma$.*

(b) *If $\mathbf{m} \in \mathbb{Z}^n$, $\mathbf{m} \geqslant 0$, then $\psi(g, -\mathbf{m}, \sigma) = 0$ for all σ.*

(c) *If $m \in \mathbb{Z}$, $m \geqslant 0$, then*

$$\psi(g, \mathbf{s}, \sigma) = (-1)^m \psi\left(\left(\frac{\partial}{\partial t}\right)^m g, \mathbf{s}, \sigma + m\right).$$

Moreover, we have

$$\psi(g, \mathbf{s}, -m) = (-1)^m \Lambda(g_m, \mathbf{s}),$$

where

$$g_m(\mathbf{k}) = \left(\frac{\partial}{\partial t}\right)^m g(t, \mathbf{k})\Big|_{t=0}.$$

(d) *Let $m \in \mathbb{Z}$, $m \geqslant 0$, and define*

$$\rho(\mathbf{s}, \sigma) = \Gamma(\sigma)\, \psi(g, \mathbf{s}, \sigma) - \frac{1}{m!} \frac{\Lambda(g_m, \mathbf{s})}{\sigma + m}.$$

Then $\rho(\mathbf{s}, \sigma)$ is entire in $\mathbf{s}$ and holomorphic in some neighborhood of $\sigma = -m$. Also, $\rho(\mathbf{0}, \sigma) = 0$ for all σ.

Proof. (a) Let $h_\sigma(\mathbf{k}) = 1/\Gamma(\sigma) \int_0^\infty g(t, \mathbf{k})\, t^{\sigma-1}\, dt$. By Theorem 2.1, $h_\sigma(\mathbf{k})$ is entire in σ. Moreover,

$$\begin{aligned}\psi(g, \mathbf{s}, \sigma) &= \frac{1}{\Gamma(\mathbf{s})} \int_K \left(\frac{1}{\Gamma(\sigma)} \int_0^\infty g(t, \mathbf{k})\, t^\sigma \frac{dt}{t}\right) \chi_{\mathbf{s}}(\mathbf{k})\, d^\times \mathbf{k} \\ &= \Lambda(h_\sigma, \mathbf{s}).\end{aligned}$$

Therefore, by Lemma 3.1a, $\psi(g, \mathbf{s}, \sigma)$ is entire in s.

(b) By Theorem 3.1b, if $\mathbf{m} \in \mathbb{Z}^n$, $\mathbf{m} \geqslant \mathbf{0}$, then

$$\psi(g, -\mathbf{m}, \sigma) = \Lambda(h_\sigma, -\mathbf{m}) = 0.$$

(c) Since $g \in \mathscr{S}(G^*)$, g is in the Schwartz space of $\bar{\mathbb{R}}_+$ when regarded as a function of t and Theorem 2.1 applies to h_σ; namely

$$\psi(g, \mathbf{s}, \sigma) = (-1)^m \psi\left(\left(\frac{\partial}{\partial t}\right)^m g, \mathbf{s}, \sigma + m\right).$$

Now set $\sigma = -m$, to obtain

$$\begin{aligned}\psi(g, \mathbf{s}, -m) &= (-1)^m \,\psi\left(\left(\frac{\partial}{\partial t}\right)^m g, \mathbf{s}, 0\right)\\ &= (-1)^m \Lambda(g_m, \mathbf{s}) \qquad \text{(by (3.2))}.\end{aligned}$$

(d) Note that

$$\begin{aligned}\Gamma(\sigma) &= \frac{\Gamma(\sigma + m + 1)}{\sigma(\sigma + 1) \cdots (\sigma + m)}\\ &= \frac{(-1)^m}{m!} \frac{1}{\sigma + m} + \rho_1(\sigma),\end{aligned}$$

where $\rho_1(\sigma)$ is holomorphic near $\sigma = -m$. Moreover, by parts (b) and (c), we have

$$\psi(g, \mathbf{s}, \sigma) = (-1)^m \Lambda(g_m, \mathbf{s}) + \rho_2(\mathbf{s}, \sigma)(\sigma + m),$$

where $\rho_2(\mathbf{s}, \sigma)$ is an entire function of $\mathbf{s}$ and σ such that $\rho_2(\mathbf{0}, \sigma) = 0$ for all σ. Clearly

$$\begin{aligned}\rho(\mathbf{s}, \sigma) &= \Gamma(\sigma)\, \psi(g, \mathbf{s}, \sigma) - \frac{1}{m!} \frac{\Lambda(g_m, \mathbf{s})}{\sigma + m}\\ &= \left(\frac{(-1)^m}{m!} \frac{1}{\sigma + m} + \rho_1(\sigma)\right) ((-1)^m \Lambda(g_m, \mathbf{s}) + \rho_2(\mathbf{s}, \sigma)(\sigma + m))\\ &\quad - \frac{1}{m!} \frac{\Lambda(g_m, \mathbf{s})}{\sigma + m}\\ &= (-1)^m \rho_1(\sigma) \Lambda(g_m, s) + \frac{(-1)^m}{m!} \rho_2(\mathbf{s}, \sigma) + (\sigma + m)\, \rho_1(\sigma)\, \rho_2(\mathbf{s}, \sigma).\end{aligned}$$

Since $\rho_1(\sigma)$ is holomorphic in a neighborhood of $\sigma = -m$, $\rho_2(\mathbf{s}, \sigma)$ is entire in $\mathbf{s}$ and σ and $\Lambda(g_m, \mathbf{s})$ is entire in $\mathbf{s}$, we see that $\rho(\mathbf{s}, \sigma)$ is entire in $\mathbf{s}$ and holomorphic in a neighborhood of $\sigma = -m$. Moreover, since $\Lambda(g_m, \mathbf{0}) = 0$ and $\rho_2(\mathbf{0}, \sigma) = 0$ for all σ, we see that

$$\rho(\mathbf{0}, \sigma) = 0$$

for all σ. This completes the proof of (d).

The following result gives the relationship between the ψ-transform and the Mellin transform.

LEMMA 3.4. *Let f be an r-admissible function on $\bar{G}-\{0\}$. Then for* $\operatorname{Re}(\mathbf{s})>0$, *we have*

$$\varphi(f,\mathbf{s})=\Gamma(\operatorname{tr}(\mathbf{s})-r)\,\psi(U_r f,\mathbf{s},\operatorname{tr}(\mathbf{s})-r). \tag{3.3}$$

Proof. For $\operatorname{Re}(\mathbf{s})>0$,

$$\begin{aligned}
\varphi(f,\mathbf{s})&=\frac{1}{\Gamma(\mathbf{s})}\int_{\bar{G}} f(\mathbf{y})\,\chi_{\mathbf{s}}(\mathbf{y})\,d^{\times}y\\
&=\Gamma(\operatorname{tr}(\mathbf{s})-r)\frac{1}{\Gamma(\mathbf{s})\,\Gamma(\operatorname{tr}(\mathbf{s})-r)}\int_K\left(\int_0^{\infty} f(t\mathbf{k})\,t^{\operatorname{tr}(\mathbf{s})}\frac{dt}{t}\right)\chi_{\mathbf{s}}(\mathbf{k})\,d^{\times}\mathbf{k}\\
&=\Gamma(\operatorname{tr}(\mathbf{s})-r)\frac{1}{\Gamma(\mathbf{s})\,\Gamma(\operatorname{tr}(\mathbf{s})-r)}\int_K\left(\int_0^{\infty} t^r f(t\mathbf{k})\,t^{\operatorname{tr}(\mathbf{s})-r}\frac{dt}{t}\right)\chi_{\mathbf{s}}(\mathbf{k})\,d^{\times}\mathbf{k}\\
&=\Gamma(\operatorname{tr}(\mathbf{s})-r)\frac{1}{\Gamma(\mathbf{s})\,\Gamma(\operatorname{tr}(\mathbf{s})-r)}\int_{G^*} U_r f(t,k)\,t^{\operatorname{tr}(\mathbf{s})-r}\,\chi_{\mathbf{s}}(\mathbf{k})\frac{dt}{t}\,d^{\times}k\\
&=\Gamma(\operatorname{tr}(\mathbf{s})-r)\,\psi(U_r f,\mathbf{s},\operatorname{tr}(\mathbf{s})-r).
\end{aligned}$$

Proof of Theorem 3.1. (a) Since f is r-admissible, $U_r f\in\mathscr{S}(G^*)$, so that $\psi(U_r f,\mathbf{s},\sigma)$ is an entire function of s and σ. Thus, by Lemma 3.4, $\varphi(f,\mathbf{s})$ is meromorphic for all $\mathbf{s}$ and is actually holomorphic provided that $\operatorname{tr}(\mathbf{s})-r\notin\{0,-1,-2,\ldots\}$. Moreover, we have

$$\varphi(f,\mathbf{s})=\Gamma(\operatorname{tr}(\mathbf{s})-r)\,\psi(U_r f,\mathbf{s},\operatorname{tr}(\mathbf{s})-r).$$

Therefore by setting $\sigma=\operatorname{tr}(s)-r$, $g=U_r f$ in Lemma 3.3d, we see that for $m\geqslant 0$, $m\in\mathbb{Z}$,

$$\varphi(f,\mathbf{s})-\frac{1}{m!}\frac{\Lambda(f_m,\mathbf{s})}{\operatorname{tr}(\mathbf{s})-r+m}$$

is holomorphic in a neighborhood of the hyperplane $\operatorname{tr}(s)=r-m$. Again referring to Lemma 3.3d, we deduce that the above difference vanishes on the integral points of the hyperplane.

(b) Let $\mathbf{m}\in\mathbb{Z}^n$, $\mathbf{m}\geqslant 0$. Then if $\nu\in\{-r,-r+1,-r+2,\ldots\}$,

$$(\mathscr{D}_{\mathbf{m}}f)_\nu=\mathscr{D}_{\mathbf{m}}(f_{\nu+\operatorname{tr}(\mathbf{m})}). \tag{3.4}$$

Furthermore, the fact that $U_r f\in\mathscr{S}(G^*)$ implies that

$$U_{r+\operatorname{tr}(\mathbf{m})}(\mathscr{D}_{\mathbf{m}}f)\in\mathscr{S}(G^*).$$

Thus, $\mathscr{D}_m f$ is $r + \operatorname{tr}(\mathbf{m})$ admissible. Apply part (a) to the function $\mathscr{D}_m f$, and set $\nu = 0$, to derive that

$$\varphi(\mathscr{D}_m f, \mathbf{s}) - \frac{A(\mathscr{D}_{\mathbf{m}} f_{\operatorname{tr}(\mathbf{m})}, \mathbf{s})}{\operatorname{tr}(\mathbf{s})}$$

is holomorphic in a neighborhood of the hyperplane $\operatorname{tr}(\mathbf{s}) = 0$ and vanishes on the integral points of the hyperplane. Now from (3.1), we have

$$\varphi(f, \mathbf{s}) = (-1)^{\operatorname{tr}(\mathbf{m})} \varphi(\mathscr{D}_m f, \mathbf{s} + \mathbf{m}).$$

Therefore,

$$\varphi(f, \mathbf{s}) - (-1)^{\operatorname{tr}(\mathbf{m})} \frac{A(\mathscr{D}_{\mathbf{m}} f_{\operatorname{tr}(\mathbf{m})}, \mathbf{s} + \mathbf{m})}{\operatorname{tr}(\mathbf{s} + \mathbf{m})}$$

is holomorphic in a neighborhood of the hyperplane $\operatorname{tr}(\mathbf{s}) = -\mathbf{m}$ and vanishes there.

(c) Assume that $\mathbf{s}$ tends to $-\mathbf{m}$ in such a way that

$$\frac{\mathbf{s} + \mathbf{m}}{\operatorname{tr}(\mathbf{s} + \mathbf{m})} \to \boldsymbol{\beta}.$$

Then set

$$\frac{\mathbf{s} + \mathbf{m}}{\operatorname{tr}(\mathbf{s} + \mathbf{m})} = \sum_{j=1}^{n} \gamma_j \mathbf{e}_j, \qquad \gamma_j = \frac{s_j + m_j}{\operatorname{tr}(\mathbf{s} + \mathbf{m})}.$$

Our assumption implies that $\gamma_j \to \beta_j$, where β_j denotes the jth component of $\boldsymbol{\beta}$ with respect to the standard coordinate system in $\mathbb{R}^n$. By part (b), we have

$$\begin{aligned} \varphi(f, \mathbf{s}) &\to (-1)^{\operatorname{tr}(\mathbf{m})} \sum_{j=1}^{n} \beta_j A(\mathscr{D}_{\mathbf{m}} f_{\operatorname{tr}(\mathbf{m})}, \mathbf{e}_j) \\ &= (-1)^{\operatorname{tr}(\mathbf{m})} \sum_{j=1}^{n} \beta_j \mathscr{D}_{\mathbf{m}} f_{\operatorname{tr}(\mathbf{m})}(\mathbf{e}_j) \end{aligned}$$

by Lemma 3.2c. This completes the proof of Theorem 3.1.

COROLLARY 3.5. *Let* $m \in \mathbb{Z}^n$, $\mathbf{m} \geqslant 0$, $\boldsymbol{\alpha} \in \mathbb{C}^n$, $\operatorname{tr}(\boldsymbol{\alpha}) \neq 0$. *Furthermore, let f be an r-admissible function on $\bar{G}$ for some nonnegative integers. Then the following limit exists*

$$\lim_{s \to 0} \varphi(f, -m + s\boldsymbol{\alpha}).$$

Moreover, the value of this limit equals

$$\frac{(-1)^{\operatorname{tr}(\mathbf{m})}}{\operatorname{tr}(\boldsymbol{\alpha})} \sum_{j=1}^{n} \alpha_j \mathscr{D}_{\mathbf{m}} f_{\operatorname{tr}(\mathbf{m})}(\mathbf{e}_j).$$

Remarks. (1) Our choice of K as $\{y \in \overline{\mathbb{R}}_+^n \mid \operatorname{tr}(\boldsymbol{\alpha}) = 1\}$ is certainly not the only possible choice for a fundamental domain for the multiplicative action of $\mathbb{R}_+$ on $\overline{\mathbb{R}}_+^n$. In effect, Shintani uses

$$K = \bigcup_{j=1}^{n} K_j,$$

where

$$K_j = \{\mathbf{y} \in \overline{\mathbb{R}}_+^n \mid y_j = 1, y_i \leqslant 1, i = 1,..., n\}.$$

(2) To facilitate a direct comparison (in Section 6) of our formulation with Shintani's result, we note that

$$(m!)^{-1} \mathscr{D}_{\mathbf{m}} f_{\operatorname{tr}(\mathbf{m})}(\mathbf{e}_j)$$

is the coefficient of

$$t^{r+\operatorname{tr}(\mathbf{m})} y_1^{m_1} \cdots y_{j-1}^{m_{j-1}} y_{j+1}^{m_{j+1}} \cdots y_n^{m_n}$$

in the Taylor expansion at the origin of

$$t^r f(ty_1,..., ty_{j-1}, t, ty_{j+1},..., ty_n).$$

(3) One advantage of the formulation of Theorem 3.1 is that it displays clearly certain linear and multilinear relations among the special values of $\varphi(f, \mathbf{s})$ for various functions f. For example, if f and g are both admissible, then so is fg. Moreover, the homogeneous pieces of fg are related to those of f and g by

$$(fg)_\nu = \sum f_\mu g_{\nu-\mu}, \tag{3.5}$$

where the sum is clearly finite.

On the other hand, we have by Leibniz' rule,

$$\mathscr{D}_{\mathbf{m}}(fg)_\nu = \sum_\mu \sum_{0 \leqslant \boldsymbol{\lambda} \leqslant \mathbf{m}} \binom{\mathbf{m}}{\boldsymbol{\lambda}} \mathscr{D}_{\boldsymbol{\lambda}} f_\mu \mathscr{D}_{\mathbf{m}-\boldsymbol{\lambda}} g_{\nu-\mu}, \tag{3.6}$$

where

$$\binom{\mathbf{m}}{\boldsymbol{\lambda}} = \prod \binom{m_i}{\lambda_i}.$$

Thus, by Theorem 3.1c, the special values of $\varphi(fg, \mathbf{s})$ depend bilinearly over $\mathbb{Z}$ on $\{\mathscr{D}_\lambda f_\mu(e_i)\}$ and $\{\mathscr{D}_\lambda g_\mu(e_i)\}$. One important example of this phenomenon occurs when $f(\mathbf{y}) = e^{-\mathbf{x}\cdot\mathbf{y}}$, where $\mathbf{x} \in \mathbb{R}^n$. Then $\varphi(e^{-\mathbf{x}\cdot\mathbf{y}}g(\mathbf{y}), \mathbf{s})$ depends on both $\mathbf{s}$ and $\mathbf{x}$. But $f_\nu = (\mathbf{x}\cdot\mathbf{y})^\nu/\nu!$ for $\nu \geqslant 0$. Thus the special values are polynomials in $(x_1,..., x_n)$ with coefficients in the $\mathbb{Z}$-module generated by $\{\mathscr{D}_\lambda f_\mu(e_i)\}$. The simplest case of this phenomenon occurs when one expresses the ordinary Bernoulli polynomials as polynomials whose coefficients are ordinary Bernoulli numbers.

4. The Values of Zeta Functions at Integral Arguments

In this section, we apply the theory of the Mellin transform $\varphi(f, s)$ to study the zeta function $\zeta_{D,a}(s)$. Specifically, under very general hypotheses, we obtain an analytic continuation of $\zeta_{D,a}(\mathbf{s})$ as a meromorphic function of several complex variables and we obtain a formula for its values at vectors $-\mathbf{m}$, $\mathbf{m} \geqslant \mathbf{0}$, $\mathbf{m} \in \mathbb{Z}^n$.

Suppose that we are given a discrete subset D of $\mathbb{R}^n_+$ and a function $a: D \to \mathbb{C}$ such that the series

$$\zeta_{D,a}(\mathbf{s}) = \sum_{\boldsymbol{\xi}\in D} a(\boldsymbol{\xi})\, \boldsymbol{\xi}^{-\mathbf{s}}$$

converges absolutely and uniformly on compact subsets of the product of half planes $\mathrm{Re}(\mathbf{s}) > \boldsymbol{\sigma}_0$. Then $\zeta_{D,a}(\mathbf{s})$ is an analytic function of $\mathbf{s}$ for $\mathrm{Re}(\mathbf{s}) > \boldsymbol{\sigma}_0$.

Suppose that $g \in \mathscr{S}(\bar{G})$. Consider the function $f: G \to \mathbb{C}$ defined by

$$f(\mathbf{y}) = f_{D,a,g}(\mathbf{y}) = \sum_{\xi\in D} a(\xi)\, g(\xi y).$$

Since g is rapidly decreasing at ∞, the series converges absolutely and uniformly on compact subsets of $\bar{G} - \{0\}$. In fact, f is C^∞ on $\bar{G} - \{0\}$. Note, however, that f may not be C^∞ at $\mathbf{0}$. For example, if $D = \{1, 2, 3,...\}$, $a(\boldsymbol{\xi}) = 1$ for all $\boldsymbol{\xi}$, $g(y) = e^{-y}$ $(y \in \mathbb{R}_+)$, then

$$f(y) = \sum_{\xi=1}^{\infty} e^{-\xi y} = \frac{e^{-y}}{1 - e^{-y}},$$

which is C^∞ on $\mathbb{R}_+$, but is not differentiable at $0 \in \bar{G}$. This example is fairly typical of what occurs in situations of interest. (It corresponds to the case of the Riemann zeta function.) Note that in this special case, $yf(y)$ is differentiable at 0, so that $f(y)$ is r-admissible for $r = 1$. This suggests that we impose the following restriction on f.

ASSUMPTION $A_{g,r}$. *The function defined by*

$$f(\mathbf{y}) = \sum_{\xi \in D} a(\xi)\, g(\xi \mathbf{y}) \qquad (\mathbf{y} \in \bar{G} - \{\mathbf{0}\}).$$

is r-admissible for some nonnegative integer r.

Assumption $A_{g,r}$ allows us to draw many conclusions about $\zeta_{D,a}(\mathbf{s})$ using the Mellin transform results of the preceding section. Let us begin then by computing the Mellin transform $\Phi(f, \mathbf{s})$. Directly from the definitions, we derive that

$$\Phi(f, \mathbf{s}) = \sum_{\xi \in \mathbf{D}} a(\xi) \int_G g(\xi \mathbf{y})\, \mathbf{y}^{\mathbf{s}} d^\times \mathbf{y} \qquad (\mathrm{Re}(\mathbf{s}) > \sigma_0),$$

where the interchange of summation and integration is justified since g is rapidly decreasing at ∞. In the last formula, make the change of variable

$$\mathbf{y} \to \xi^{-1} \mathbf{y},$$

under which the measure $d^\times \mathbf{y}$ is invariant, to obtain

$$\begin{aligned} \Phi(f, \mathbf{s}) &= \zeta_{D,a}(\mathbf{s}) \int_G g(\mathbf{y})\, \mathbf{y}^{\mathbf{s}} d^\times \mathbf{y} \\ &= \zeta_{D,a}(\mathbf{s})\, \Phi(g, \mathbf{s}) \qquad (\mathrm{Re}(\mathbf{s}) > \sigma_0). \end{aligned}$$

Therefore, we see that

$$\zeta_{D,a}(\mathbf{s}) = \frac{\varphi(f, \mathbf{s})}{\varphi(g, \mathbf{s})} \qquad (\mathrm{Re}(\mathbf{s}) > \sigma_0). \tag{4.1}$$

Since f is r-admissible and $g \in \mathscr{S}(\bar{G})$, Theorems 3.1 and 2.1, respectively, imply that $\zeta_{D,a}(\mathbf{s})$ is a meromorphic function of $\mathbf{s}$ and that the formula for (4.1) is valid at all $\mathbf{s}$ for which the functions $\zeta_{D,a}(\mathbf{s})$, $\varphi(f, \mathbf{s})$, and $\varphi(g, \mathbf{s})$ are simultaneously holomorphic. Moreover, applying Theorem 3.1, we deduce the following result.

PROPOSITION 4.1. *Assume that* $A_{g,r}$ *holds. Then* $\zeta_{D,a}(\mathbf{s})$ *may be analytically continued to a meromorphic function of* $\mathbf{s}$ *whose only possible singularities are at those* $\mathbf{s}$ *for which either* $\varphi(g, \mathbf{s}) = 0$ *or* $\mathrm{tr}(\mathbf{s}) \in \{r, r-1, r-2, \ldots\}$. *Moreover, if* $m \in \mathbb{Z}^n$, $\mathbf{m} \geqslant \mathbf{0}$, *and if* $\varphi(g, -\mathbf{m}) \neq 0$, *then*

$$\zeta_{D,a}(\mathbf{s}) - (-1)^{\mathrm{tr}(\mathbf{m})} \frac{1}{\varphi(g, \mathbf{s})} \frac{\Lambda(\mathscr{D}_{\mathbf{m}} f_{\mathrm{tr}(\mathbf{m})}, \mathbf{s} + \mathbf{m})}{\mathrm{tr}(\mathbf{s} + \mathbf{m})}$$

is holomorphic in a neighborhood of the hyperplane $\mathrm{tr}(\mathbf{s}) = -\mathbf{m}$ *and vanishes on this hyperplane.*

By applying Theorem 2.1c and Theorem 3.1c, we now easily deduce

THEOREM 4.2. *Let* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$, $\mathrm{tr}(\boldsymbol{\alpha}) \neq 0$, *and let* $\mathbf{s}$ *approach* $-\mathbf{m}$ *in such a way that* $(\mathbf{s} + \mathbf{m})/\mathrm{tr}(\mathbf{s} + \mathbf{m})$ *approaches* $\boldsymbol{\alpha}$. *Furthermore, assume that* $A_{g,r}$ *holds and that* $(\mathscr{D}_{\mathbf{m}}\, g(0)) \neq 0$. *Then*

$$\zeta_{D,a}(\mathbf{s}) \to \frac{1}{\mathrm{tr}(\boldsymbol{\alpha})} \sum_{j=1}^{n} \alpha_j \mathscr{D}_{\mathbf{m}} f_{\mathrm{tr}(\mathbf{m})}(\mathbf{e}_j) \Big/ (\mathscr{D}_{\mathbf{m}}\, g)(\mathbf{0}).$$

The above formula assumes a particularly elegant shape in case g equals the function

$$g_0(\mathbf{y}) = e^{-y_1 - y_2 - \cdots - y_n},$$

since in this case, $\varphi(g, s) = 1$. In this case, we derive

THEOREM 4.3. *Let* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$, $\mathrm{tr}(\boldsymbol{\alpha}) \neq 0$, *and let* $\mathbf{s}$ *approach* $-\mathbf{m}$ *in such a way that* $(\mathbf{s} + \mathbf{m})/\mathrm{tr}(\mathbf{s} + \mathbf{m})$ *approaches* $\boldsymbol{\alpha}$. *Suppose that the function*

$$f(\mathbf{y}) = \sum_{\boldsymbol{\xi} \in D} a(\boldsymbol{\xi})\, e^{-\boldsymbol{\xi} \cdot \mathbf{y}}$$

is r-admissible for some nonnegative integer r. Then

$$\zeta_{D,a}(\mathbf{s}) \to \frac{(-1)^{\mathrm{tr}(\mathbf{m})}}{\mathrm{tr}(\boldsymbol{\alpha})} \sum_{j=1}^{n} \alpha_j \mathscr{D}_{\mathbf{m}} f_{\mathrm{tr}(\mathbf{m})}(\mathbf{e}_j).$$

COROLLARY 4.4. *Let* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r) \in \mathbb{C}^n$, $\mathrm{tr}(\boldsymbol{\alpha}) \neq 0$. *Suppose that the function*

$$f(\mathbf{y}) = \sum_{\boldsymbol{\xi} \in D} a(\boldsymbol{\xi})\, e^{-\boldsymbol{\xi} \cdot \mathbf{y}}$$

is r-admissible for some nonnegative integer r. Then

$$\lim_{s \to 0} \zeta_{D,a}(-\mathbf{m} + \boldsymbol{\alpha} s) = \frac{(-1)^{\mathrm{tr}(\mathbf{m})}}{\mathrm{tr}(\boldsymbol{\alpha})} \sum_{j=1}^{n} \alpha_j \mathscr{D}_{\mathbf{m}} f_{\mathrm{tr}(\mathbf{m})}(\mathbf{e}_j).$$

Remark. In the formulas of Theorem 4.3 and Corollary 4.4, no dependence on r is explicitly stated. However, note that the definition of

$$f_{\mathrm{tr}(\mathbf{m})}(\mathbf{y})$$

involves r.

5. Some Examples

In this section, we apply Corollary 4.4 to evaluate a number of zeta functions at $-\mathbf{m}$, $\mathbf{m} \geqslant 0$.

Example 5.1 (Riemann Zeta Function). Let $n=1$, $D=\{1,2,3,...\}$, $a(\xi)=1$ for all $\xi \in D$. Then $\zeta_{D,a}(s)$ is just the Riemann zeta function $\zeta(s)$. In this case,

$$f(y)=\sum_{\xi=1}^{\infty} a(\xi)\, e^{-\xi y}=\frac{1}{e^{y}-1},$$

which we have seen is 1-admissible. Therefore, for $m=-1,0,1,...,$ we have

$$\begin{aligned} f_m(y) &= \frac{1}{(m+1)!}\left(\frac{\partial}{\partial t}\right)^{m+1} (tf(ty))_{t=0} \\ &= \frac{B_{m+1}}{(m+1)!}\, y^m, \end{aligned}$$

where B_n denotes the nth Bernoulli number, defined as

$$\frac{y}{e^y-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}\,y^n.$$

Therefore, by Corollary 4.4, we have

$$\begin{aligned} \zeta(-m) &= (-1)^m\, \mathscr{D}_m f_m(1) \\ &= (-1)^m \frac{B_{m+1}}{m+1}, \end{aligned}$$

a well-known result.

Example 5.2 (Dirichlet L-Functions). Let n and D be as in Example 1. Let χ be a primitive Dirichlet character modulo the conductor γ. Set $a(\xi)=\chi(\xi)$ $(\xi \in D)$. Then $\zeta_{D,a}(s)=L(s,\chi)$, the Dirichlet L-series associated to χ. The generalized Bernoulli numbers $B_{n,\chi}$ in the sense of Kubota and Leopoldt [2] are defined as

$$f(y)=\sum_{a=1}^{\gamma-1}\bar{\chi}(a)\,\frac{e^{ay}}{e^{\gamma y}-1}=\sum_{n=0}^{\infty}\frac{B_{n,\chi}}{n!}\,y^n.$$

It is clear that $f(y)$ is 1-admissible and that

$$f_m(y) = \frac{B_{m+1,\chi}}{(m+1)!} y^m \qquad (m = -1, 0, 1, \ldots).$$

Therefore, Corollary 4.4 yields

$$\begin{aligned} L(-m, \chi) &= (-1)^m \mathscr{D}_m f_m(1) \\ &= (-1)^m \frac{B_{m+1,\chi}}{m+1}. \end{aligned}$$

EXAMPLE 5.3 (Shintani's Generalized Hurwitz Zeta Functions). Let

$$A = (a_{jk})_{1 \leqslant j \leqslant r, 1 \leqslant k \leqslant n}$$

be a matrix of positive real numbers for which $r \leqslant n$. Let $\chi = (\chi_1, \ldots, \chi_r)$, where χ_j is a complex number such that $|\chi_j| = 1$. Let $\mathbf{x} = (x_1, \ldots, x_r)$, $x_j > 0$ and let

$$C_+ = \mathbb{Z}^n \cap \overline{\mathbb{R}}^n_+.$$

Define a zeta function of n complex variables $\mathbf{s} = (s_1, \ldots, s_n)$ via

$$\zeta(\mathbf{s}, A, \mathbf{x}, \chi) = \sum_{\mathbf{z} \in C_+} \frac{\chi^{\mathbf{z}}}{A^*(\mathbf{z} + \mathbf{x})^{\mathbf{s}}},$$

where A^* denotes the transpose of A. More explicitly,

$$\zeta(\mathbf{s}, A, \mathbf{x}, \chi) = \sum_{z_1, \ldots, z_r = 0}^{\infty} \prod_{k=1}^{r} \chi_k^{z_k} \prod_{j=1}^{n} \left(\prod_{l=1}^{r} a_{lj}(z_l + x_l) \right)^{-s_j}.$$

The special case of these zeta functions for which $s_1 = s_2 = \cdots = s_n$ was studied by Shintani [5], who derived analytic continuations and the value at nonpositive integers m by using the method of Hankel's contour integrals. We have adopted his notation to facilitate comparison. Using our results, we can extend his results to the several variable case.

Set $D = \{A^*(\mathbf{z} + \mathbf{x}) \mid \mathbf{z} \in C_+\}$. For $\xi = A^*(\mathbf{z} + \mathbf{x}) \in D$, set $a(\xi) = \chi^{\mathbf{z}}$. Then

$$\begin{aligned} f(\mathbf{y}) &= \sum_{\xi \in D} a(\xi) e^{-\xi \cdot \mathbf{y}} \\ &= \sum_{\mathbf{z} \in C_+} \chi^{\mathbf{z}} e^{-A^*(\mathbf{z}+\mathbf{x})\mathbf{y}} \qquad (\mathbf{y} \in \bar{G} - \{0\}). \end{aligned}$$

Let $L_j(\mathbf{y})$ $(1 \leqslant j \leqslant r)$ denote the linear form

$$L_j(\mathbf{y}) = \sum_{k=1}^{n} a_{jk} y_k.$$

Then a simple computation shows that

$$f(\mathbf{y}) = \prod_{j=1}^{r} \frac{\exp\{L_j(\mathbf{y})(1-x_j)\}}{\exp\{L_j(\mathbf{y})\} - \chi_j}. \tag{5.1}$$

If we let $Q(\mathbf{y})$ be defined as

$$Q(\mathbf{y}) = \prod_{j=1}^{r} L_j(\mathbf{y}),$$

and $P(\mathbf{y}) = f(\mathbf{y})\, Q(\mathbf{y})$, then we see that $\mathbf{P}(\mathbf{y}) \in \mathscr{S}(\bar{G})$ and thus is 0-admissible, whereas $Q(\mathbf{y})$ is homogeneous of degree r and does not vanish on $\bar{G} - \{0\}$ since all a_{ij} are positive. Therefore, by Proposition 4.1, $f(\mathbf{y})$ is r-admissible and $\zeta(\mathbf{s}, A, \mathbf{x}, \boldsymbol{\chi})$ is a meromorphic function of $\mathbf{s}$ whose only possible singularities occur when $\mathrm{tr}(\mathbf{s}) \in \{r, r-1, r-2,...\}$.

Define generalized Bernoulli numbers indexed by nonnegative integer vectors $\mathbf{m}$ by

$$B_{\mathbf{m}+\mathbf{1}}(A, \mathbf{1} - \mathbf{x}, \boldsymbol{\chi})^{(j)} = (m_1 + 1) \cdots (m_n + 1)\, \mathscr{D}_{\mathbf{m}} f_{\mathrm{tr}(\mathbf{m})}(\mathbf{e}_j) \qquad j = 1,..., m,$$

where f is defined by (6.1). The second remark at the end of Section 3 shows that if $\mathbf{m} = (m-1,..., m-1)$ for some positive integer m, then this definition coincides with Shintani's [5, Proposition 1]. The peculiar factor of $(m_1 + 1) \cdots (m_n + 1)$ is due to the indexing shift caused by defining the ordinary Bernoulli numbers B_k as coefficients of $y^k/k!$ in the Taylor series of $y/(e^y - 1)$ rather than in the Laurent series of $1/(e^y - 1)$.

By Corollary 4.4, if $\boldsymbol{\alpha} \in \mathbb{C}^n$, $\mathrm{tr}(\boldsymbol{\alpha}) \neq 0$, then

$$\lim_{s \to 0} \zeta(-\mathbf{m} + s\boldsymbol{\alpha}, A, x, \chi)$$
$$= \frac{(-1)^{\mathrm{tr}(\mathbf{m})}}{\mathrm{tr}(\boldsymbol{\alpha})} (m_1 + 1) \cdots (m_n + 1) \sum_{j=1}^{n} \alpha_j B_{\mathbf{m}}(A, \mathbf{1} - \mathbf{x}, \boldsymbol{\chi})^{(j)}.$$

In particular, set $s_1 = s_2 = \cdots = s_n = s$ corresponding to $\alpha_1 = \cdots = \alpha_n = 1$ and define

$$\zeta_0(s, A, \mathbf{x}, \boldsymbol{\chi}) = \zeta((s, s,..., s), A, \mathbf{x}, \boldsymbol{\chi}).$$

Then $\zeta_0(s)$ is holomorphic at $s=-m$ and

$$\zeta_0(1-m, A, \mathbf{x}, \chi) = (-1)^{n(m-1)}\, m^{-n} \sum_{j=1}^{n} B_m(A, 1-\mathbf{x}, \chi)^{(j)}/n,$$

where we have written B_m for $B_{(m,m,\ldots,m)}$. This is precisely the formula obtained by Shintani [5, Proposition 1].

6. An Integrality Theorem

Theorem 3.1c is well suited to statements about integrality of special values of L-functions. In this section we apply it to a situation which contains Shintani's generalized Hurwitz zeta function as a special case. Namely, we let $A = (a_{ij})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n)$ be an $r \times n$ matrix of complex numbers. Define the associated linear forms

$$L_i = L_i(\mathbf{y}) = \sum_{j=1}^{n} a_{ij} y_{ij} \qquad (1 \leqslant i \leqslant r).$$

Let $g(\mathbf{u})$ be a function of r variables $u_1, \ldots, u_r$ such that

$$f(\mathbf{y}) = g(L_1(\mathbf{y}), \ldots, L_r(\mathbf{y}))$$

is r-admissible for some nonnegative integer r. Suppose g has a Laurent expansion at the origin:

$$g(\mathbf{u}) = \sum_{\operatorname{tr}(\boldsymbol{\eta}) \geqslant -\mathbf{q}} c(\eta)\, \mathbf{u}^{\boldsymbol{\eta}},$$

where $\mathbf{q} \in \mathbb{Z}^r$, $\mathbf{q} \geqslant 0$, $\operatorname{tr}(\mathbf{q}) = q$. Then f is, in fact, q-admissible. Furthermore, the νth homogeneous piece of f is given by

$$f_\nu(\mathbf{y}) = \sum_{\substack{\operatorname{tr}(\boldsymbol{\eta}) = \nu \\ \boldsymbol{\eta} \geqslant -q}} c(\boldsymbol{\eta})\, L_1(\mathbf{y})^{\eta_1} \cdots L_r(\mathbf{y})^{\eta_r}.$$

In order to differentiate f_ν with respect to $y_1, \ldots, y_n$ we express $\partial/\partial y_j$ in terms of the $\partial/\partial L_i$. Note that

$$\frac{\partial L_i}{\partial y_i} = a_{ij},$$

and thus,

$$\frac{\partial}{\partial y_j} = \sum_{i=1}^{r} a_{ij} \frac{\partial}{\partial L_i} \qquad (j = 1, \ldots, n).$$

Let $\boldsymbol{\lambda}$, $\boldsymbol{\kappa} \in \mathbb{Z}^n$, $\boldsymbol{\rho}$, $\boldsymbol{\mu} \in \mathbb{Z}^r$, $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$, $\boldsymbol{\rho}$, $\boldsymbol{\mu} \geqslant 0$ and $\mathrm{Tr}(\boldsymbol{\lambda}) = \mathrm{Tr}(\kappa) = \mathrm{Tr}(\rho) = \mathrm{Tr}(\boldsymbol{\mu}) = b$, where b is some fixed nonnegative integer. The notation $\sum_{\boldsymbol{\lambda}}$ will refer to summation over all $\boldsymbol{\lambda} \in \mathbb{Z}^n$, $\boldsymbol{\lambda} \geqslant 0$, $\mathrm{Tr}(\boldsymbol{\lambda}) = b$. Similar summation conventions will be observed with respect to $\boldsymbol{\kappa}$, $\boldsymbol{\rho}$, $\boldsymbol{\mu}$. Set

$$D_{\boldsymbol{\kappa}} = \left(\frac{\partial}{\partial y_1}\right)^{\kappa_1} \cdots \left(\frac{\partial}{\partial y_n}\right)^{\kappa_n}, \qquad E_{\boldsymbol{\mu}} = \left(\frac{\partial}{\partial L_1}\right)^{\mu_1} \cdots \left(\frac{\partial}{\partial L_i}\right)^{\mu_r}.$$

Define the $\alpha(A, \boldsymbol{\kappa}, \boldsymbol{\mu})$ via

$$D_{\boldsymbol{\kappa}} = \sum_{\boldsymbol{\rho}} \alpha(A, \boldsymbol{\kappa}, \boldsymbol{\rho})\, E_{\boldsymbol{\rho}}.$$

It is clear that

$$\begin{aligned} D_{\boldsymbol{\kappa}}(\mathbf{y}^{\boldsymbol{\lambda}}) &= \boldsymbol{\kappa}! && \text{if} \quad \boldsymbol{\lambda} = \boldsymbol{\kappa}, \\ &= 0 && \text{if} \quad \boldsymbol{\lambda} \neq \boldsymbol{\kappa}, \\ E_{\boldsymbol{\mu}}(\mathbf{L}^{\boldsymbol{\rho}}) &= \boldsymbol{\mu}! && \text{if} \quad \boldsymbol{\rho} = \boldsymbol{\mu}, \\ &= 0 && \text{if} \quad \boldsymbol{\rho} \neq \boldsymbol{\mu}. \end{aligned}$$

Set

$$\mathbf{L}^{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda}} \beta(A, \boldsymbol{\mu}, \boldsymbol{\lambda})\, \mathbf{y}^{\boldsymbol{\lambda}}.$$

Then

$$\begin{aligned} D_{\boldsymbol{\kappa}}(\mathbf{L}^{\boldsymbol{\mu}}) &= D_{\boldsymbol{\kappa}} \left(\sum_{\boldsymbol{\lambda}} \beta(A, \boldsymbol{\mu}, \boldsymbol{\lambda})\, \mathbf{y}^{\boldsymbol{\lambda}} \right) \\ &= \kappa!\, \beta(A, \boldsymbol{\mu}, \boldsymbol{\kappa}) \end{aligned}$$

and

$$\begin{aligned} D_{\boldsymbol{\kappa}}(\mathbf{L}^{\boldsymbol{\mu}}) &= \sum_{\boldsymbol{\rho}} \alpha(A, \boldsymbol{\kappa}, \boldsymbol{\rho})\, E_{\boldsymbol{\rho}} \mathbf{L}^{\boldsymbol{\mu}} \\ &= \mu!\, \alpha(A, \boldsymbol{\kappa}, \boldsymbol{\mu}). \end{aligned}$$

Comparing the last two equations yields

$$\alpha(A, \boldsymbol{\kappa}, \boldsymbol{\mu}) = \frac{\boldsymbol{\kappa}!}{\boldsymbol{\mu}!} \beta(A, \boldsymbol{\mu}, \boldsymbol{\kappa}).$$

To compute $\beta(\mathbf{A}, \boldsymbol{\mu}, \boldsymbol{\kappa})$, we apply the multinomial theorem:

$$\prod_{i=1}^{r} L_i(\mathbf{y})^{\mu_i} = \sum_{\substack{\boldsymbol{\lambda} \geqslant 0 \\ \mathrm{Tr}(\boldsymbol{\lambda}) = \mathrm{Tr}(\boldsymbol{\mu})}} \sum_{\mathbf{K}}{}^{*} \frac{\mu!}{\prod_{i,j} k_{ij}!} \prod_{i,j} a_{ij}^{k_{ij}} \mathbf{y}^{\boldsymbol{\lambda}},$$

where $\sum_{\mathbf{K}}^{*}$ denotes a sum over all integer matrices $\mathbf{K}$ with nonnegative entries such that the ith row $\mathbf{K}_i$ and jth column satisfy, respectively, $\mathrm{tr}(\mathbf{K}_i) = \mu_i$, $\mathrm{tr}(\mathbf{K}^j) = \lambda_j$. Let us write symbolically

$$\mathbf{K}! = \prod_{i,j} k_{ij}!$$

$$\mathbf{A}^{\mathbf{K}} = \prod_{i,j} a_{ij}^{k_{ij}}.$$

Then the definition of $\beta(\mathbf{A}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ implies that

$$\beta(\mathbf{A}, \boldsymbol{\mu}, \boldsymbol{\kappa}) = \sum_{\substack{\mathbf{K} \geqslant \mathbf{0} \\ \mathrm{tr}(\mathbf{K}_i) = \mu_i \\ \mathrm{tr}(\mathbf{K}^j) = \kappa_j}} \frac{\boldsymbol{\mu}!}{\mathbf{K}!} \mathbf{A}^{\mathbf{K}}. \tag{6.1}$$

Thus, we have

$$\alpha(\mathbf{A}, \boldsymbol{\mu}, \mathbf{v}) = \sum_{\substack{\mathbf{K} \geqslant 0 \\ \mathrm{tr}(\mathbf{K}_i) = \mu_i \\ \mathrm{tr}(\mathbf{K}^j) = \kappa_j}} \frac{\mathbf{v}!}{\mathbf{K}!} \mathbf{A}^{\mathbf{K}}. \tag{6.2}$$

Note that since $\mathbf{K} \geqslant \mathbf{0}$ and $\mathrm{tr}(\mathbf{K}^j) = \kappa_j$, we see that $\mathbf{K}^j!$ divides $v_j!$. (The quotient is a multinomial coefficient.) Therefore we observe that

$$\alpha(\mathbf{A}, \boldsymbol{\mu}, \boldsymbol{\kappa}) \in \mathbb{Z}[a_{ij}]_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant n}}. \tag{6.3}$$

Similarly, since $\mathrm{tr}(\mathbf{K}_i) = \mu_i$, Eq. (6.1) implies that

$$\beta(\mathbf{A}, \boldsymbol{\mu}, \boldsymbol{\kappa}) \in \mathbb{Z}[a_{ij}]_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant n}}. \tag{6.4}$$

Let us now compute the derivatives of the homogeneous part $f_v(\mathbf{y})$: Clearly $f_v(\mathbf{y}) = g_v(L_1(\mathbf{y}),..., L_r(\mathbf{y}))$, so that for $v \geqslant -q$, we have

$$f_v(\mathbf{y}) = \sum_{\substack{\mathrm{tr}(\boldsymbol{\eta}) = v \\ \boldsymbol{\eta} \geqslant -\mathbf{q}}} c_{\boldsymbol{\eta}} L_1(\mathbf{y})^{\eta_1} \cdots L_r(\mathbf{y})^{\eta_r}.$$

Thus,

$$\mathscr{D}_{\mathbf{m}} f_\nu(\mathbf{y}) = \sum_{\substack{\mathrm{tr}(\boldsymbol{\eta})=\nu \\ \boldsymbol{\eta} > -\mathbf{q}}} c_{\boldsymbol{\eta}} \sum_{\substack{\boldsymbol{\rho} > 0 \\ \mathrm{tr}(\boldsymbol{\rho})=\mathrm{tr}(\mathbf{m})}} \alpha(\mathbf{A}, \mathbf{m}, \boldsymbol{\rho})\, E_{\boldsymbol{\rho}}(L_1^{\eta_1} \cdots L_r^{\eta_r}).$$

Therefore,

$$\begin{aligned} \mathscr{D}_{\mathbf{m}} f_{\mathrm{tr}(\mathbf{m})}(\mathbf{y}) &= \sum_{\substack{\mathrm{tr}(\boldsymbol{\eta})=\mathrm{tr}(\mathbf{m}) \\ \boldsymbol{\eta} > -\mathbf{q}}} c_{\boldsymbol{\eta}} \sum_{\substack{f > 0 \\ \mathrm{tr}(\boldsymbol{\rho})=\mathrm{tr}(\mathbf{m})}} \alpha(\mathbf{A}, \mathbf{m}, \boldsymbol{\rho})\, E_{\boldsymbol{\rho}}(L_1^{\eta_1} \cdots L_r^{\eta_r}) \\ &= \sum_{\substack{\mathrm{tr}(\boldsymbol{\eta})=\mathrm{tr}(\mathbf{m}) \\ \boldsymbol{\eta} > -\mathbf{q}}} c_{\boldsymbol{\eta}} \sum_{\substack{\boldsymbol{\rho} > 0 \\ \mathrm{tr}(\boldsymbol{\rho})=\mathrm{tr}(\mathbf{m})}} \boldsymbol{\rho}! \binom{\boldsymbol{\eta}}{\boldsymbol{\rho}} \alpha(\mathbf{A}, \mathbf{m}, \boldsymbol{\rho})\, \mathbf{L}^{\boldsymbol{\eta}-\boldsymbol{\rho}}, \end{aligned}$$

where

$$\binom{\boldsymbol{\eta}}{\boldsymbol{\rho}} = \prod_{i=1}^{r} \binom{\eta_i}{\rho_i}.$$

Therefore, for $1 \leqslant j \leqslant n$,

$$\mathscr{D}_{\mathbf{m}} f_{\mathrm{tr}(\mathbf{m})}(\mathbf{e}_j) = \sum_{\substack{\mathrm{tr}(\boldsymbol{\eta})=\mathrm{tr}(\mathbf{m}) \\ \boldsymbol{\eta} > -\mathbf{q}}} c_{\boldsymbol{\eta}} \sum_{\substack{\boldsymbol{\rho} > 0 \\ \mathrm{tr}(\boldsymbol{\rho})=\mathrm{tr}(\mathbf{m})}} \boldsymbol{\rho}! \binom{\boldsymbol{\eta}}{\boldsymbol{\rho}} \times \alpha(\mathbf{A}, \mathbf{m}, \boldsymbol{\rho}) \prod_{i=1}^{r} a_{ij}^{\eta_i - \rho_i}.$$

Thus, finally we conclude:

THEOREM 6.1. *Let all notations be as above and let* $\boldsymbol{\alpha} \in \mathbb{C}^n$, $\mathrm{tr}(\boldsymbol{\alpha}) \neq 0$. *Then*

$$\lim_{s \to 0} \varphi(f, -\mathbf{m} + \boldsymbol{\alpha} s) = \frac{(-1)^{\mathrm{tr}(\mathbf{m})}}{\mathrm{tr}(\boldsymbol{\alpha})} \sum_{j=1}^{r} \alpha_j \sum_{\substack{\boldsymbol{\eta}, \boldsymbol{\rho} \\ \boldsymbol{\eta} > -\mathbf{q}, \boldsymbol{\rho} > 0 \\ \mathrm{tr}(\boldsymbol{\eta})=\mathrm{tr}(\boldsymbol{\rho})=\mathrm{tr}(\mathbf{m})}} c_{\boldsymbol{\eta}} \boldsymbol{\rho}! \binom{\boldsymbol{\eta}}{\boldsymbol{\rho}} \times a(\mathbf{A}, \mathbf{m}, \boldsymbol{\rho}) \prod_{i=1}^{r} a_{ij}^{\eta_i - \rho_i}.$$

Theorem 6.1 is a source of integrality facts needed for, say, p-adic interpolation. For clearly,

$$\left(\prod_{i=1}^{r} a_{ij} \right)^{\mathrm{tr}(\mathbf{m})} \varphi(f, -\mathbf{m})$$

is a homogeneous polynomial in the a_{ij} with coefficients in the module $\mathbb{Z}[c_{\boldsymbol{\eta}}]_{\mathrm{tr}\boldsymbol{\eta} = \mathrm{tr}\, m}$.

Let us illustrate Theorem 6.1 in case of the zeta function

$$\zeta(\mathbf{s}, A, \mathbf{x}, \mathbf{1}) = \sum_{\mathbf{z} \in C_+} \frac{1}{A^*(\mathbf{z}+\mathbf{x})^{\mathbf{s}}},$$

where $\mathbf{1} = (1, 1, \ldots, 1)$. In this case, we recall from Section 5 that

$$f(\mathbf{y}) = g(L_1(\mathbf{y}), \ldots, L_r(\mathbf{y})),$$

where

$$g(\mathbf{u}, \mathbf{x}) = g(\mathbf{u}) = \prod_{i=1}^{r} \frac{e^{u_i(1-x_i)}}{e^{u_i}-1} = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} B_{\mathbf{k}}(\mathbf{1}-\mathbf{x}) \frac{\mathbf{u}^{\mathbf{k}-1}}{\mathbf{k}!},$$

where $B_{\mathbf{k}}(\mathbf{x})$ is generalized Bernoulli polynomial defined as

$$B_{\mathbf{k}}(\mathbf{x}) = B_{k_1}(x_1)\, B_{k_2}(x_2) \cdots B_{k_r}(x_r).$$

Therefore,

$$g(\mathbf{u}) = \sum_{\boldsymbol{\eta} \geq -1} \frac{B_{\boldsymbol{\eta}+1}(1-x)}{(\boldsymbol{\eta}+1)!} u^{\boldsymbol{\eta}}.$$

From Theorem 6.1 we now deduce

COROLLARY 6.2. *Let* $\boldsymbol{\alpha} \in \mathbb{C}^n$, $\operatorname{tr}(\alpha) \neq 0$. *Then*

$$\lim_{s \to 0} \zeta(-\mathbf{m}, +\boldsymbol{\alpha} s, \mathbf{A}, \mathbf{x}, \mathbf{1})$$

$$= \frac{(-1)^{\operatorname{tr}(\mathbf{m})}}{\operatorname{tr}(\boldsymbol{\alpha})} \sum_{j=1}^{n} \alpha_j \sum_{\substack{\boldsymbol{\eta}, \boldsymbol{\rho} \\ \boldsymbol{\eta} \geq -1, \boldsymbol{\rho} \geq 0 \\ \operatorname{tr}(\boldsymbol{\eta}) = \operatorname{tr}(\boldsymbol{\rho}) = \operatorname{tr}(\mathbf{m})}} \frac{B_{\boldsymbol{\eta}+1}(1-\mathbf{x})}{(\boldsymbol{\eta}+1)!} \rho! \binom{\boldsymbol{\eta}}{\boldsymbol{\rho}}$$

$$\times\, \alpha(\mathbf{A}, \mathbf{m}, \boldsymbol{\rho}) \prod_{i=1}^{r} a_{ij}^{\eta_i - \rho_i}.$$

In particular, for $s_1 = s_2 = \cdots = s_r$, we have a more explicit version of Shintani's Corollary to Proposition 1.

COROLLARY 6.3. *For* $m = 0, 1, 2, \ldots$, *we have*

$$\zeta_0(-m, \mathbf{A}, \mathbf{x}, \mathbf{1}) = \frac{(-1)^{nm}}{n} \sum_{\substack{\boldsymbol{\eta}, \boldsymbol{\rho} \\ \boldsymbol{\eta} \geq -1, \boldsymbol{\rho} \geq 0 \\ \operatorname{tr}(\boldsymbol{\eta}) = \operatorname{tr}(p) = nm}} \frac{B_{\boldsymbol{\eta}+1}(\mathbf{1}-\mathbf{x})}{(\boldsymbol{\eta}+1)!} \rho! \binom{\boldsymbol{\eta}}{\boldsymbol{\rho}}$$

$$\times\, \alpha(\mathbf{A}, \mathbf{m}, \boldsymbol{\rho}) \prod_{i=1}^{r} a_{ij}^{\eta_i - \rho_i}$$

where $\mathbf{m} = (m, m, \ldots, m)$.

REFERENCES

1. R. M. DAMERELL, L-functions of elliptic curves with complex multiplication, *Acta Arith.* **17** (1970), 287–301.
2. K. IWASAWA, "Lectures on p-adic L-Functions," Ann. Math. Stud. No. 74, Princeton Univ. Press, Princeton, N.J., 1972.
3. T. KUBOTA AND H. LEOPOLDT, Eine p-adische Theorie der Zetawerte, I, *J. Reine Angew. Math.* **214/215** (1964), 328–339.
4. M. RAZAR, Dirichlet series and Eichler cohomology, to appear.
5. T. SHINTANI, On evaluation of zeta functions of totally real algebraic number fields at non-positive integers, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **23** (1976), 393–417.
6. E. C. TITCHMARSH, "The Theory of the Riemann Zeta Function," Oxford Univ. Press, London/New York, 1951.
7. D. ZAGIER, A Kronecker limit formula for real quadratic fields, *Math. Ann.* **213** (1975), 153–184.
8. D. ZAGIER, Valeurs des fonctions zêta des corps quadratiques réales aux entiers négatifs, to appear.