Commutators and Companion Matrices over Rings of Stable Rank 1

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ABSTRACT

We consider the group $\mathrm{GL}_n A$ of all invertible n by n matrices over a ring A satisfying the first Bass stable range condition. We prove that every matrix is similar to the product of a lower and upper triangular matrix, and that it is also the product of two matrices each similar to a companion matrix. We use this to show that, when $n \geqslant 3$ and A is commutative, every matrix in $\mathrm{SL}_n A$ is the product of two commutators.

INTRODUCTION

In various situations it is instructive to represent a matrix as a product of matrices of a special nature. A large survey of various results concerning factorization of matrices over a field and bounded operators on a Hilbert space is given in Wu [75]. Given a class of matrices (or operators), one studies products of elements from the class, and asks about the minimal number of factors in a factorization. Classes considered in [75] include the class of normal matrices, the class of involutions, the class of symmetric matrices, and many others. See also [14, 11, 15, 16, 71] for various matrix factorizations.

A classical factorization result is that every orthogonal n by n matrix is the product of at most n reflections [2, Proposition 5, Chapter IX, §6, Section 4]; see [12] for further work on reflections. In linear algebra, one writes an invertible matrix as a product of elementary matrices. One can ask how many elementary matrices (or commutators) are needed to represent any product of elementary matrices (respectively, commutators). In multi-

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plicative simplex methods, one writes an invertible matrix over a field as the product of matrices each of which differs from the identity matrix by one column.

In this paper, A stands for an associative ring with 1. We study invertible matrices over rings A of stable rank 1, and we decompose them into products of triangular matrices, companion matrices, and commutators. As will be seen, our different factorizations are closely related.

Recall [1] that the first Bass stable range condition on A is:

If $a, b \in A$ and $Aa + Ab = A$, then there is $c \in A$ such that $A(a + cb) = A$.

We write $\mathrm{sr}(A) \leqslant 1$ if A satisfies this condition, and we write $\mathrm{sr}(A) = 1$ if $\mathrm{sr}(A) \leqslant 1$ and $A \neq 0$. See [21, 48, 68] for various examples of such rings.

Our first decomposition theorem involves triangular matrices. By Lemma 5 of [7], for any A with $\mathrm{sr}(A) = 1$ and any integer $n \geqslant 1$, every matrix β in $\mathrm{GL}_n A$ is the product of four matrices, each of them either lower or upper triangular. We prove in the next section that β is, in fact, the product of three triangular matrices and is similar to the product of two triangular matrices.

THEOREM 1. *Let $\mathrm{sr}(A) \leqslant 1$ and $n \geqslant 1$. Then every matrix β in $\mathrm{GL}_n A$ can be written as $\lambda\rho\mu$, where λ, μ are lower triangular matrices in $\mathrm{GL}_n A$ and ρ is an upper triangular matrix in $\mathrm{GL}_n A$. Therefore any matrix β in $\mathrm{GL}_n A$ is similar to the product $\rho\mu\lambda^{-1}$ of an upper triangular matrix ρ and a lower triangular matrix $\mu\lambda^{-1}$.*

In the case when A is a field it is well known that every matrix becomes the product of a lower and an upper triangular matrix after conjugation by a permutation matrix (the related decomposition for simple algebraic groups is known as the Bruhat decomposition). This decomposition (when $A = \mathbf{R}$, the reals, or $\mathbf{C}$, the complex numbers) is used to compute eigenvalues of matrices (see, for example, [73] and references there for *LR* or *LU* decomposition).

Next we consider the companion matrices.

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in A[x]$ be a monic polynomial with coefficients a_i in an associative ring A with 1, where x is an indeterminate commuting with A. Then the *companion matrix* $C(a_0, \ldots, a_{n-1})$ of $f(x)$ (see for example, [19, 37]) is the n by n matrix over A with ones along the line just below and parallel to the main diagonal, the elements $-a_0, \ldots, -a_{n-1}$ as the consecutive entries of the last column, and zeros elsewhere.

Note that the companion matrix is invertible if and only if a_0 is invertible. Sometimes [37], the transpose of our companion matrix is called the companion matrix, but this does not make much difference because the companion matrix over any ring A is similar to its transpose; see Proposition 8 below.

The companion matrix is the matrix of multiplication by x on the right A-module $A[x]/f(x)A[x]$ relative to the free basis $1, x, \dots, x^{n-1}$.

Let A^n denote the right A-module of all n-columns over A, $M_n A$ the ring of all n by n matrices over A, and $\mathrm{GL}_n A$ the group of all invertible n by n matrices over A. As usual, we identify $M_n A$ with the endomorphism ring of A^n and $\mathrm{GL}_n A$ with the automorphism group of A^n.

A column $v \in A^n$ is called *cyclic* for a matrix $\alpha \in M_n A$, if $v, \alpha v, \dots, \alpha^{n-1} v$ form a free basis for A^n. If such a column exists, the matrix α is also called *cyclic*. In this case, $\gamma^{-1}\alpha\gamma$ is a companion matrix, where $\gamma = (v, \alpha v, \dots, \alpha^{n-1} v) \in \mathrm{GL}_n A$. Conversely, such a column v exists whenever α is similar to a companion matrix.

When A is a field, it is well known that every square matrix over A is similar to a direct sum of companion matrices. We work with companion and cyclic matrices over more general rings. In Section 2, we will use Theorem 1 to prove the following theorem.

THEOREM 2. *Let* $\mathrm{sr}(A) \leqslant 1$ *and* $n \geqslant 1$. *Then every matrix in* $\mathrm{GL}_n A$ *is the product of two cyclic matrices in* $\mathrm{GL}_n A$.

In fact, we will prove that the first factor can be required to be similar to any prescribed invertible companion matrix.

Next we turn our attention to commutators, the main subject of this paper. For any group G, let $c(G)$ be the least integer $s \geqslant 0$ such that every product of commutators is the product of s commutators. If no such s exists, we set $c(G) = \infty$.

Note that $c(H) \leqslant c(G)$ for any group G and any factor group H of G, and that $c(G) \leqslant c([G,G])$ when the commutator subgroup $[G,G]$ of G is perfect (i.e., coincides with its own commutator subgroup).

This number $c(G)$ was extensively studied for various groups G. Note that $c(G) = 0$ if and only if G is commutative. The question whether $c(G) \leqslant 1$, i.e., every element of the commutator subgroup $[G,G]$ is a single commutator, is of particular interest. In Section 5 below, we will survey known results on $c(G)$ for various groups G.

It was proved in [7, Theorem 6] that $c(\mathrm{GL}_n A) \leqslant 5$ for any commutative ring A with $\mathrm{sr}(A) \leqslant 1$ and any $n \geqslant 3$. Using Theorem 2, we improve upon this result, replacing 5 by 2. Namely, in Section 3 below we prove the following theorem.

THEOREM 3. *Let* A *be commutative and* $\mathrm{sr}(A) \leqslant 1$. *Assume that either* $n \geqslant 3$ *or* $n = 2$, *and* 1 *is the sum of two units in* A. *Then* $c(\mathrm{GL}_n A) \leqslant 2$.

In [7, Theorem 2d] it was shown, for any associative ring A with 1 of any finite stable rank, that either $c(\mathrm{GL}_n A) = \infty$ for all sufficiently large n or $c(\mathrm{GL}_n A) \leqslant 6$ for all sufficiently large n. In Section 4, we improve upon this result, replacing 6 by 4.

THEOREM 4. *Let A be an associative ring with* 1 *and* $\mathrm{sr}(A) < \infty$. *Suppose that* $c(\mathrm{GL}_m A) < \infty$ *for some* $m \geqslant \max(\mathrm{sr}(A) + 1, 3)$. *Then* $c(\mathrm{GL}_n A) < \infty$ *for all* $n \geqslant \max(\mathrm{sr}(A) + 1, 3)$ *and* $c(\mathrm{GL}_n A) \leqslant 4$ *for all sufficiently large* n.

By [66], for $n \geqslant \max(\mathrm{sr}(A) + 1, 3)$ the group $[\mathrm{GL}_n A, \mathrm{GL}_n A]$ is perfect and coincides with the subgroup $E_n A$ of $\mathrm{GL}_n A$ generated by all elementary matrices.

NOTATION. We denote by 1_n the identity matrix in $\mathrm{GL}_n A$. For any $a \in A$ and any $i \neq j$, we denote by a^{ij} the elementary matrix which has a at the position i, j and coincides with 1_n elsewhere.

1. PROOF OF THEOREM 1

We will write matrices β in $\mathrm{GL}_n A$ in the block form

$$\beta = \begin{pmatrix} \eta & v \\ u & a \end{pmatrix},$$

where $\eta \in M_{n-1} A$, $v \in A^{n-1}$ is an $(n-1)$-column over A, u is an $(n-1)$-row over A, and $a \in A$.

Now we take an arbitrary matrix $\beta \in \mathrm{GL}_n A$ and write it as above. Since β is invertible, its last column is unimodular. The first Bass stable range condition implies all higher Bass conditions for A [67]. So there is an $(n-1)$-row u' over A such that $A(a + u'v) = A$.

Since $\mathrm{sr}(A) = 1$, every one-sided unit in A is a unit (a result of Kaplansky; see [68]). So $a' = a + u'v \in \mathrm{GL}_1 A$. Set $u'' = -a'^{-1}(u + u'\eta)$ and $\eta' = \eta + vu''$. Then

$$\begin{pmatrix} 1_{n-1} & 0 \\ u' & 1 \end{pmatrix} \begin{pmatrix} \eta & v \\ u & a \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ u'' & 1 \end{pmatrix} = \begin{pmatrix} \eta' & v \\ 0 & a' \end{pmatrix};$$

hence $h' \in \mathrm{GL}_{n-1} A$.

Proceeding by induction on n (the case $n=1$ is trivial), we can write $\eta' = \lambda'\rho'\mu'$ with lower triangular matrices $\lambda', \mu' \in \mathrm{GL}_{n-1}A$ and an upper triangular matrix $\rho' \in \mathrm{GL}_{n-1}A$. Then

$$\beta = \begin{pmatrix} \eta & v \\ u & a \end{pmatrix} = \begin{pmatrix} 1_{n-1} & 0 \\ -u' & 1 \end{pmatrix} \begin{pmatrix} \eta' & v \\ 0 & a' \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ -u'' & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1_{n-1} & 0 \\ -u' & 1 \end{pmatrix} \begin{pmatrix} \lambda'\rho'\mu' & v \\ 0 & a' \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ -u'' & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda' & 0 \\ -u'\lambda' & 1 \end{pmatrix} \begin{pmatrix} \rho' & (\lambda')^{-1}v \\ 0 & a' \end{pmatrix} \begin{pmatrix} \mu' & 0 \\ -u'' & 1 \end{pmatrix} = \lambda\rho\mu,$$

where

$$\lambda = \begin{pmatrix} \lambda' & 0 \\ -u'\lambda' & 1 \end{pmatrix}, \mu = \begin{pmatrix} \mu' & 0 \\ -u'' & 1 \end{pmatrix} \in \mathrm{GL}_n A$$

are lower triangular, and

$$\rho = \begin{pmatrix} \rho' & (\lambda')^{-1}v \\ 0 & a' \end{pmatrix} \in \mathrm{GL}_n A$$

is upper triangular.

REMARKS. If necessary, we can make all diagonal entries of λ and μ equal to 1, by taking out their diagonal parts and including them in ρ.

It is easy to see that, for any associative ring A with 1 and any integer $n \geqslant 1$, a matrix β in $\mathrm{GL}_n A$ is the product $\rho\lambda$ of an upper triangular matrix ρ and a lower triangular matrix λ, both with all diagonal entries in $\mathrm{GL}_1 A$, if and only if all southeast corner submatrices of β are invertible.

2. PROOF OF THEOREM 2

An upper or lower triangular matrix is called *unit* if all its diagonal entries are 1. (Note that "triangular" in [7] means "unit triangular.")

PROPOSITION 6. *Let A be an associative ring with* 1, $n \geqslant 1$ *an integer, and* $\beta = (\beta_{i,j})$ *a matrix in* $M_n A$ *which coincides with a companion matrix below the main diagonal, i.e.,* $\beta_{i,j} = 1$ *when* $i = j+1$ *and* $\beta_{i,j} = 0$ *when* $i \geqslant j+2$. *Then there is a unit upper triangular matrix* γ *in* $\mathrm{GL}_n A$ *such that* $\gamma^{-1}\beta\gamma$ *is a companion matrix.*

Proof. Let $\{e_1, \dots, e_n\}$ be the standard basis of A^n, i.e., $(e_1, \dots, e_n) = 1_n$. Then the matrix $\gamma = (e_1, \beta e_1, \dots, \beta^{n-1} e_1)$ is unit upper triangular and $\gamma^{-1}\beta\gamma$ is a companion matrix, i.e., $\gamma^{-1}\beta\gamma e_i = e_{i+1}$ for $i = 1, \dots, n-1$. ■

REMARK. The matrix γ in Proposition 6 is in fact unique.

COROLLARY 7. *Let A be an associative ring with* 1, $n \geqslant 1$ *an integer,* ρ *an upper triangular matrix in* $\mathrm{GL}_n A$ *with diagonal entries* $g_1, g_2, \dots, g_n$ *in* $\mathrm{GL}_1 A$, *and* $\theta = C(a_0, \dots, a_{n-1})$ *a companion matrix. Then:*

(i) $\theta\rho$ *is similar to a companion matrix* $\theta' = C(a'_0, \dots, a'_{n-1})$ *with* $e'_0 = g_n g_{n-1} \cdots g_1 e_0$;

(ii) $\rho\theta$ *is cyclic.*

Proof. (i): Let δ be the diagonal matrix $\mathrm{diag}(g_n g_{n-1} \cdots g_1, g_n g_{n-1} \cdots g_2, \dots, g_n)$. Then we can apply Proposition 6 to $\beta = \delta\theta\gamma\delta^{-1}$. Note that the first row of the matrix $\gamma = (e_1, \beta e_1, \dots, \beta^{n-1} e_1)$ is $(1, 0, \dots, 0)$ and that $\beta_{1,i} = 0$ for $i = 2, \dots, n-1$, so the conjugation by γ does not change the entry $\beta_{1,n} = g_n g_{n-1} \cdots g_1 e_0$.

(ii): Since $\rho\theta$ is similar to $\theta\rho$, (ii) follows from (i). ■

PROPOSITION 8. *Let A be an associative ring with* 1, $n \geqslant 1$ *an integer,* $\theta = C(e_0, \dots, e_{n-1})$ *a companion matrix. Then:*

(i) θ *is similar to its transpose* θ^{T};

(ii) *if* $e_0 \in \mathrm{GL}_1 A$, *then* θ^{-1} *is similar to* $C(a_0^{-1}, a_0^{-1} a_{n-1}, \dots, a_0^{-1} a_1)$.

Proof. Let σ be the permutation matrix in $\mathrm{GL}_n A$ corresponding to the permutation $k \mapsto n-k+1$. Then $\sigma = \sigma^{-1}$, and $\sigma\theta^{\mathrm{T}}\sigma$ has ones along the line below the main diagonal, $(-a_{n-1}, -a_{n-2}, \dots, -a_0)$ as the first row, and zeros elsewhere.

To prove (i), we apply now the proof of Proposition 6 to this matrix $\sigma\theta^{\mathrm{T}}\sigma$. We find that $\rho\sigma\theta^{\mathrm{T}}\sigma\rho^{-1} = \theta$ for the unit diagonal matrix $\rho = (\rho_{i,j})$ with $\rho_{i,j} = e_{n+i-j}$ for all $i < j$.

To prove (ii), observe that $\sigma\theta^{-1}\sigma = C(a_0^{-1}, a_{n-1}a_0^{-1}, \dots, a_1 a_0^{-1})$. ■

Proposition 9. *Let A be an associative ring with 1, $n \geqslant 1$ an integer, $c_0 \in \mathrm{GL}_1 A$, $c_i \in A$ for $i = 2, \dots, n-1$, ρ an upper triangular matrix in $GL_n A$, and λ a lower triangular matrix in $\mathrm{GL}_n A$. Assume that all diagonal entries of both λ and ρ belong to $\mathrm{GL}_1 A$. Then there is a matrix $\varphi \in \mathrm{GL}_n A$ similar to $C(c_0, c_1, \dots, c_{n-1})$ and a cyclic matrix ψ such that $\rho\lambda = \varphi\psi$.*

Proof. Without changing $\rho\lambda$, we can replace ρ and λ by $\rho\delta$ and $\delta^{-1}\lambda$ respectively, where δ is an arbitrary diagonal matrix in $\mathrm{GL}_n A$. So we can prescribe the diagonal entries of ρ. Let them be $-c_0, 1, \dots, 1$.

Set $\pi = C(-1, 0, \dots, 0)$, a permutation matrix. By Proposition 6, there is a unit upper triangular matrix γ such that $\gamma^{-1}\rho\pi\gamma = C(c_0, a_1, \dots, a_{n-1})$. Set $\varepsilon = \prod_{i=1}^{n-1}(a_i - c_i)^{i,n}$. Then $\gamma^{-1}\rho\pi\gamma\varepsilon = C(c_0, c_1, \dots, c_{n-1})$, and $\gamma\varepsilon\gamma^{-1}$ has the form $\prod_{i=1}^{n-1}(b_i)^{i,n}$ with $b_i \in A$.

Set $\varphi = \rho\pi\gamma\varepsilon\gamma^{-1}$ and $\psi = \gamma\varepsilon^{-1}\gamma^{-1}\pi^{-1}\lambda$. Then $\rho\lambda = \varphi\psi$ and $\gamma^{-1}\varphi\gamma = \gamma^{-1}\rho\pi\gamma\varepsilon = C(c_0, c_1, \dots, c_{n-1})$. So it remains to show that $\psi = \gamma\varepsilon^{-1}\gamma^{-1}\pi^{-1}\lambda$ is similar to a companion matrix. To do this, we just apply Proposition 6 to the matrix $\sigma\psi\sigma$, where $\sigma = \sigma^{-1}$ is as in the proof of Proposition 8 above. ■

Putting Proposition 9 and Theorem 1 together, we obtain the following stronger version of Theorem 2:

Theorem 10. *Let $\mathrm{sr}(A) \leqslant 1$ and $n \geqslant 1$. Then for any matrix β in $\mathrm{GL}_n A$ and any companion matrix θ in $\mathrm{GL}_n A$ there are matrices φ, ψ in $\mathrm{GL}_n A$ such that $\beta = \varphi\psi$, φ is similar to the given companion matrix θ, and ψ is cyclic.*

3. PROOF OF THEOREM 3

Lemma 11. *For any associative ring A with 1 and any integer $n \geqslant 2$ there exists an invertible matrix $\beta_n \in E_n A$ such that the matrix $\beta_n - 1_n \in E_n A$.*

Proof. When $n = 2$, we can take

$$\beta_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

When $n = 3$, we can take

$$\beta_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $n \geqslant 4$, we can write β_n as the direct sum of the above matrices β_2 and β_3. For example, $\beta_4 = \beta_2 \oplus \beta_2$ is the required matrix in $E_4 A$, $\beta_5 = \beta_3 \oplus \beta_2$ is the required matrix in $E_5 A$, and so on. ■

COROLLARY 12. *For any associative ring A with* 1, *any integer* $n \geqslant 2$, *and any column* $(a_i) \in A^n$, *the matrix* $\Pi(a_i)^{i,n}$ *is a commutator in* $E_{n+1}A$.

Proof. We have $\Pi(a_i)^{i,n} = [\beta_n \oplus 1, \Pi(b_i)^{i,n}]$, where $b = (b_i) = (\beta_n - 1_n)^{-1} a \in A^n$. ■

Now we are ready to prove Theorem 3. Under the conditions of Theorem 3, let β be an arbitrary matrix in $\mathrm{SL}_n A$. By Theorem 2 we can write $\beta = \chi\kappa$ with cyclic matrices χ and κ.

By Proposition 8, κ^{-1} is cyclic. We write

$$\alpha\chi\alpha^{-1} = C(a_0, a_1, \ldots, a_{n-1}) \quad \text{and} \quad \varepsilon\kappa^{-1}\varepsilon^{-1} = C(b_0, b_1, \ldots, b_{n-1})$$

with $a_i, b_i \in A$ and $\alpha, \varepsilon \in \mathrm{GL}_n A$. Since $\det \beta = 1$, we have $a_0 = b_0$.

Set $\tau = \Pi_{i=1}^{n-1}(b_i - a_i)^{i,n}$ (so τ differs from 1_n only in the last column). Then $\alpha\chi\alpha^{-1} = \varepsilon\kappa^{-1}\varepsilon^{-1}\tau$; hence $\chi = \alpha^{-1}\varepsilon\kappa^{-1}\varepsilon^{-1}\tau\alpha$. So $\beta = \chi\kappa = \alpha^{-1}\varepsilon\kappa^{-1}\varepsilon^{-1}\tau\alpha\kappa = \alpha^{-1}\tau\alpha[\alpha^{-1}\tau^{-1}\varepsilon, \kappa^{-1}]$.

When $n \geqslant 3$, τ is a commutator by Corollary 12. When $n = 2$, τ is also a commutator by the condition of the theorem; see [1, Lemma 1.6]. So $\alpha^{-1}\tau\alpha$ is a commutator and κ is the product of two commutators.

4. PROOF OF THEOREM 4

LEMMA 13. *Let A be an associative ring with* 1, *and* $n \geqslant 2$ *an integer. Assume that either* $n \geqslant 3$ *or* $n = 2$, *and* 1 *is the sum of two units in A. If* $\beta = \lambda\rho \in \mathrm{GL}_n A$ *is a product of a lower triangular matrix* λ *and an upper triangular matrix* ρ *in* $\mathrm{GL}_n A$, *then* β *is a product of two commutators.*

Proof. As in the proof of Proposition 9 above, we set $\pi = C(-1,0,\dots,0)$. By Proposition 6, $\chi = \lambda\pi$ is similar to a companion matrix $C(-1,a_1,\dots,a_{n-1})$. Also by Proposition 6, $\kappa^{-1} = \rho^{-1}\pi$ is similar to a companion matrix $C(-1,b_1,\dots,b_{n-1})$. We have $\beta = \chi\kappa$. Now we proceed as in the proof of Theorem 3.

We write

$$\alpha\chi\alpha^{-1} = C(-1,a_1,\dots,a_{n-1}) \text{ and } \varepsilon\kappa^{-1}\varepsilon^{-1} = C(-1,b_1,\dots,b_{n-1})$$

$$\text{with} \quad \alpha,\varepsilon \in \mathrm{GL}_n A.$$

Set $\tau = \prod_{i=1}^{n-1}(b_i - a_i)^{i,n}$ (so τ differs from 1_n only in the last column). Then $\alpha\chi\alpha^{-1} = \varepsilon\kappa^{-1}\varepsilon^{-1}\tau$; hence $\chi = \alpha^{-1}\varepsilon\kappa^{-1}\varepsilon^{-1}\tau\alpha$. So $\beta = \chi\kappa = \alpha^{-1}\varepsilon\kappa^{-1}\varepsilon^{-1}\tau\alpha\kappa = \alpha^{-1}\tau\alpha[\alpha^{-1}\tau^{-1}\varepsilon,\kappa^{-1}]$.

When $n \geqslant 3$, τ is a commutator by Corollary 12. When $n = 2$, τ is also a commutator by the condition of the theorem. So $\alpha^{-1}\tau\alpha$ is a commutator, and κ is the product of two commutators. ■

COROLLARY 14. *Under the conditions of Lemma* 13, *let* $t \geqslant 2$ *be an integer. If a matrix* β *is a product of* t *unit triangular matrices* (*upper and lower*), *then* β *is a product of* $1+[t/2]$ *commutators.*

Proof. If t is odd, we can eliminate a factor by either combining two adjoint triangular factors of the same type, or doing this after conjugation by the first factor (thus passing to a matrix similar to β). So it suffices to prove our conclusion for even t. The case $t = 2$ was stated as Lemma 13. When $t = 2s \geqslant 4$, we proceed by induction on s as in the proof of Corollary 14 in [7]. Namely, we write β as a product of $2s$ triangular factors and switch two factors, to replace two triangular matrices by one commutator. ■

REMARK. By Corollary 14 of [7], β is a product of $3+[t/2]$ commutators when $n \geqslant 3$.

Now we can complete our proof of Theorem 4. By [7, Lemma 9 and Corollary 12], every matrix in $E_nA = [\mathrm{GL}_nA, \mathrm{GL}_nA]$, $n \geqslant \max(\mathrm{sr}(A)+1,3)$, is a product of a bounded number of elementary matrices; hence $c(\mathrm{GL}_nA) \leqslant c(E_nA) < \infty$. By [7, Lemma 9, Corollary 12, and Theorem 20b] every matrix

β in E_nA is a product of six unit triangular matrices, for sufficiently large n. By Corollary 14 with $t=6$, β is then a product of four commutators.

5. HISTORICAL SURVEY ON COMMUTATORS

It seems that the question of whether a matrix α can be written in the form $\alpha=[\beta,\gamma]=\beta\gamma\beta^{-1}\gamma^{-1}$ was first considered by Shoda [54]. He showed that for an algebraically closed field F, a matrix α in GL_nF is a commutator $[\beta,\gamma]$ in GL_nF if and only if $\det\alpha=1$, i.e., $\alpha\in\mathrm{SL}_nF$. Since $[\mathrm{GL}_nF,\mathrm{GL}_nF]=\mathrm{SL}_nF$ for any infinite field F, this means that $c(\mathrm{GL}_nF)\leqslant 1$ for an algebraically closed field F. He also showed that $c(\mathrm{GL}_nF)\leqslant 2$ for any real closed field F (later [55], he returned to the problem and showed, in particular, that $c(\mathrm{SL}_nF)\leqslant n$ for any infinite field F).

Tôyama [64] obtained similar results for some types of compact simple Lie groups. Gotô [23] extended this, proving that $c(G)=1$ for any connected compact topological group G such that the commutator subgroup is dense. Such groups G include all connected compact semisimple Lie groups G. See also [47, 50].

Ore [46, Theorem 1] proved that $c(G)\leqslant 1$ for the symmetric group G of any set (finite or infinite; in the infinite case, $G=[G,G]$). He conjectured (p. 313) that $c(G)\leqslant 1$ for every finite simple group (later [65, 33] this was proved for some G). He also claimed [46, Theorem 7] that $c(A_n)=1$ for the alternating group $A_n=[S_n,S_n]$ for $n\geqslant 5$. Ito [36] proved this claim independently (this was proved again in [33, 34]).

Honda [31] characterized the commutators in any finite group G in terms of its group algebra over the complex numbers.

Fan [18] proved that $c(\mathrm{U}(n))=1$ for the (compact) unitary groups $\mathrm{U}(n)$. Generalizing a result of Taussky [58] (see also [57, p. 802]), he also observed that given any elements g,h of any group G and any integer $m\geqslant 0$, then gh^{-1} is a product of m commutators if and only if there are $f_1,\dots,f_{2m+1}\in G$ such that $x=f_1f_2\cdots f_{2m+1}$ and $y=f_{2m+1}f_{2m}\cdots f_1$.

Griffiths [24] studied commutators in the free product $G=G_1^*\cdots *G_n$ of finitely presented groups G_i. He showed that $c(G)\geqslant n$ provided that $[G_i,G_i]$ is not trivial for all i. Later [20] it was shown that in fact $c(G)\geqslant\Sigma c(G_i)$.

Thompson [59–62] showed that $c(\mathrm{GL}_nF)\leqslant 1$, $c(\mathrm{SL}_nF)\leqslant 2$, and $c(\mathrm{PSL}_nF)\leqslant 1$ for any field F and any $n\geqslant 1$. For some A and n, $c(\mathrm{SL}_nA)=2$, which gives examples of finite perfect groups G with $c(G)=2$. Special cases of his results were treated in [25, 56, 72].

MacDonald [41] originated the study of commutators in groups G with cyclic $[G,G]$. His results imply that $c(G)\leqslant m/2$ when G is nilpotent and $[G,G]$ is cyclic of m elements, and that there are finite groups G with cyclic

$[G,G]$ and an arbitrarily large $c(G)$. He also showed that $c(G)>1$ for the nilpotent group G of class 2 on four generators.

Rodney [52] showed that if $[G,G]$ is cyclic and either G is nilpotent or $[G,G]$ is infinite, then $c(G)=1$. He also showed [53] that $c(G)=1$ when $[G,G]$ is central and finite, and generated by three elements. Liebeck [40] showed that $c(G)=1$ when $[G,G]$ is central and can be generated by two elements.

Isaacs [35] used wreath products to produce finite groups G with $c(G)\neq 1$. He noted that no simple group G is known with $c(G)>1$.

In [22] finite groups G with cyclic $[G,G]$ of m elements are studied. A necessary and sufficient condition on m was found in order that $c(G)=1$ for any such G. For example, $c(G)=1$ when m is a power of a prime. It is claimed that the smallest order of G with cyclic $[G,G]$ and $c(G)\neq 1$ is 240 (then $m=60$), and that the smallest order of G with $c(G)\neq 1$ is 96, and its $[G,G]$ is nonabelian of 32 elements. (See [28] for a proof.)

Guralnick [26] constructed many groups G with an arbitrarily large $c(G)$ whose $[G,G]$ is a direct product of many groups. For example, he showed that for any integer $n\geqslant 1$ and any finitely generated abelian group G' of rank $\geqslant n^2$ there is a group G such that $[G,G]$ is central in G and isomorphic to G' and $c(G)=n$. He also determined all pairs (m,n) for which there is a group G with $[G,G]$ cyclic of order n and $c(G)>m$ (see [27]). It was shown in [28] that for any abelian group G' of even order which is a direct product of four nontrivial factors, there is a group G with $[G,G]$ isomorphic to G' and $c(G)\neq 1$.

Brenner [3] observed that $c(G)=1$ provided that there is a conjugacy class C in G such that $CC=G$. He proved the existence of such a class C for many groups G (see [3] and references there; see also [32]).

Djoković [10] studied $c(G)$ for real semisimple Lie groups G. In particular, he proved that $c(\mathrm{SL}_n\mathbf{H})=1$, where $\mathbf{H}$ is the quaternions. The number $c(\mathrm{GL}_n A)$ for division rings A was studied by Kursov [38, 39], Draxl [13], Rehmann [51], and Cohn [6]. See [32, 9, 49, 33] for results about other matrix groups.

For groups G of operators, $c(G)$ was studied by Harpe and Skandalis [29, 30] and by Brown and Pearcy [4].

Commutators in groups G of diffeomorphisms were studied in McDuff [44], Mather [42, 43], and Epstein [17]. For example, it was observed in [44] that if $G=\mathrm{Diff}_\Omega \mathbf{R}^n$ is the group of all diffeomorphisms of $\mathbf{R}^n$ preserving a volume form Ω and $n\geqslant 3$, then $c(G)\leqslant 28$ in the case $\mathrm{vol}_\Omega \mathbf{R}^n=\infty$ and $c(G)\leqslant 10{,}206$ otherwise. Here $\mathbf{R}$ stands for the reals.

Wood [74] showed that $c(G)=\infty$ for the universal covering group G of $\mathrm{SL}_2\mathbf{R}$. For rings A of continuous functions on topological spaces X, the groups $G=\mathrm{SL}_n A$ and other gauge groups were studied in [63, 69, 70]. For example, by [69], when A is the ring of all continuous functions $\mathbf{R} \to \mathbf{R}$, we have $c(\mathrm{SL}_n A) < \infty$ when $n \geqslant 3$, and $c(\mathrm{SL}_2 A) = \infty$.

Newman [45] proved $c(\mathrm{SL}_n A) \leqslant (2 \log n)/\log \frac{3}{2} + c(\mathrm{SL}_3 A)$ for any commutative principal ideal ring A and any integer $n > 3$. So, if $c(\mathrm{SL}_3 A)$ is finite, the number $c(\mathrm{SL}_n A)$ cannot grow too fast as a function of n. By [5], $c(\mathrm{SL}_3 A) < \infty$ for any ring A of algebraic integers. Newman [45] asked whether $c(\mathrm{SL}_3 A) < \infty$ for all such A.

Dennis and Vaserstein [7] obtained the negative answer to this question. For example, $c(\mathrm{SL}_n A) = \infty$ for all $n \geqslant 2$ when $A = \mathbf{C}[x]$, $\mathbf{C}$ the complex numbers. However, for any commutative principal ideal ring A, we have $c(\mathrm{SL}_n A) \leqslant 5 + c(\mathrm{SL}_3 A)$ when $n > 3$ and $c(\mathrm{SL}_n A) \leqslant 6$ for all sufficiently large n, if $c(\mathrm{SL}_3 A)$ is finite. So, if $c(\mathrm{SL}_3 A)$ is finite, $c(\mathrm{SL}_n A)$ is bounded as a function of $n \geqslant 3$. Similar results were obtained in [7] for all rings A of finite stable rank.

In [8], it was shown that $c(G) \leqslant 2$ for many "infinite" groups G, such as the limit of the automorphism groups of direct powers of an object in a category.

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Received 14 June 1989; final manuscript accepted 26 February 1990