Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Linear Algebra and its Applications 418 (2006) 15–19
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Canonical form of m-by-2-by-2 matrices over a field of characteristic other than two

Genrich Belitskii [a,1], Maxim Bershadsky [b], Vladimir V. Sergeichuk [c,*]

[a] Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
[b] Sapir Academic College P.b., Hof Ashkelon, 79165, Israel
[c] National Academy of Sciences, Institute of Mathematics, Tereshchenkivska St. 3, Kiev, Ukraine

Received 29 June 2005; accepted 19 January 2006
Available online 9 March 2006
Submitted by E. Tyrtyshnikov

Abstract

We give a canonical form of $m\times2\times2$ matrices for equivalence over any field of characteristic not two.

AMS classification: 15A21; 15A69

Keywords: Multidimensional matrices; Tensors; Classification

We give a canonical form of $m\times 2\times 2$ matrices over any field $\mathbb{F}$ of characteristic other than two. Unlike the case $\mathbb{F}=\mathbb{C}$, one of our canonical $2\times 2\times 2$ matrices depends on some parameter that is determined up to multiplication by any z^2, $0\neq z\in\mathbb{F}$. Complex $2\times 2\times 2$ matrices up to equivalence were classified by Schwartz [8] and Duschek [2]. Canonical forms of complex and real $2\times 2\times 2$ matrices for equivalence were given by Oldenburger [5–7]; see also [9, Section IV, Theorem 1.1]. Recently Ehrenborg [3] constructed a canonical form of complex $2\times 2\times 2$ matrices for equivalence basing on a collection of covariants that separates the canonical matrices.

By an $m\times n\times q$ *spatial matrix* over a field $\mathbb{F}$ we mean an array

$$\mathscr{A}=[a_{ijk}]_{i=1}^{m}\,_{j=1}^{n}\,_{k=1}^{q},\quad a_{ijk}\in\mathbb{F};\tag{1}$$

* Corresponding author. The research was done while this author was visiting the Ben-Gurion University of the Negev.
E-mail addresses: genrich@cs.bgu.ac.il (G. Belitskii), maximb@mail.sapir.ac.il (M. Bershadsky), sergeich@imath.kiev.ua (V.V. Sergeichuk).
[1] Partially supported by Israel, Science Foundation, Grant 186/01.

0024-3795/$ - see front matter
doi:10.1016/j.laa.2006.01.012

it is convenient to represent it by the list of its *horizontal slices*

$$\mathscr{A} = ||A_1|\cdots|A_q||,\quad A_1 := [a_{ij1}]_{ij},\ldots,A_q := [a_{ijq}]_{ij}.$$

Two $m \times n \times q$ spatial matrices $\mathscr{A} = [a_{ijk}]$ and $\mathscr{B} = [b_{ijk}]$ are said to be *equivalent* if there exist non-singular

$$S = [s_{ii'}] \in \mathbb{F}^{m\times m},\quad R = [r_{jj'}] \in \mathbb{F}^{n\times n},\quad T = [t_{kk'}] \in \mathbb{F}^{q\times q} \tag{2}$$

such that

$$b_{i'j'k'} = \sum_{ijk} a_{ijk} r_{ii'} s_{jj'} t_{kk'}$$

($\mathscr{A}$ and $\mathscr{B}$ give the same trilinear form $f : \mathbb{F}^m \times \mathbb{F}^n \times \mathbb{F}^q \to \mathbb{F}$ relative to different bases). We can transfer $\mathscr{A}$ to $\mathscr{B}$ as follows: first we produce the simultaneous equivalence transformations with the horizontal slices of $\mathscr{A}$:

$$\mathscr{C} = ||C_1|\cdots|C_q|| := ||R^{\mathrm{T}}A_1S|\cdots|R^{\mathrm{T}}A_qS||, \tag{3}$$

then we produce the non-singular linear operation on the list of horizontal slices of the obtained spatial matrix $\mathscr{C}$:

$$||B_1|\cdots|B_q|| = ||C_1t_{11} + \cdots + C_qt_{q1}|\cdots|C_1t_{1q} + \cdots + C_qt_{qq}||, \tag{4}$$

where S, R, and T are the matrices (2). The last transformation can be realized by a sequence of elementary operations on the list of horizontal slices of $\mathscr{C}$ (interchange any two slides, multiply one slice by a non-zero scalar, and add a scalar multiple of one slice to another one; see [4, Chapter VI, Section 5]).

Let $\mathscr{A}$ be the spatial matrix (1), and let

$$A_k := [a_{ijk}]_{ij},\quad \widetilde{A}_j := [a_{ijk}]_{ik},\quad \widetilde{\widetilde{\mathrm{A}}}_i := [a_{ijk}]_{jk}.$$

We say that $\mathscr{A}$ is *regular* if each of the sets of matrices

$$\mathscr{S} = \{A_1,\ldots,A_q\},\quad \widetilde{\mathscr{S}} = \{\widetilde{A}_1,\ldots,\widetilde{A}_n\},\quad \widetilde{\widetilde{\mathscr{S}}} = \{\widetilde{\widetilde{\mathrm{A}}}_1,\ldots,\widetilde{\widetilde{\mathrm{A}}}_m\} \tag{5}$$

is linearly independent.

Let $\mathscr{A}$ be non-regular, and let q', n', and m' be the ranks of the sets (5). Let us make the first q' matrices in $\mathscr{S}$ linearly independent and the others zero by elementary operations. Then reduce the "new" $\widetilde{\mathscr{S}}$ and $\widetilde{\widetilde{\mathscr{S}}}$ in the same way. We obtain a spatial matrix $\mathscr{B} = [b_{ijk}]$, whose $m' \times n' \times q'$ submatrix

$$\mathscr{B}' = [b_{ijk}]_{i=1}^{m'}{}_{j=1}^{n'}{}_{k=1}^{q'}$$

is regular and whose entries outside of $\mathscr{B}'$ are zero; the submatrix $\mathscr{B}'$ is called a *regular part of* $\mathscr{A}$. Two spatial matrices of the same size are equivalent if and only if their regular parts are equivalent [1, Lemma 4.7]. Hence, it suffices to give a canonical form of a regular spatial matrix.

Theorem 1. *Over a field $\mathbb{F}$ of characteristic other than two, each regular $m \times n \times q$ spatial matrix $\mathscr{A}$ with $n \leqslant 2$ and $q \leqslant 2$ is equivalent to exactly one of the following spatial matrices:*

$$\left\|1\right\| \quad (1\times 1\times 1), \tag{6}$$

$$\left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| \quad (2\times 2\times 1), \tag{7}$$

$$\left\|\begin{matrix}1\\ 0\end{matrix}\;\middle|\;\begin{matrix}0\\ 1\end{matrix}\right\| \quad (2\times 1\times 2), \tag{8}$$

$$\left\|\begin{array}{cc|cc}1 & 0 & 0 & 1\end{array}\right\| \quad (1\times 2\times 2), \tag{9}$$

$$\left\|\begin{array}{cc|cc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{array}\right\| \quad (3\times 2\times 2), \tag{10}$$

$$\left\|\begin{array}{cc|cc}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right\| \quad (3\times 2\times 2), \tag{11}$$

$$\left\|\begin{array}{cc|cc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right\| \quad (4\times 2\times 2), \tag{12}$$

$$\mathscr{K}(\tilde{a}) := \left\|\begin{array}{cc|cc}1 & 0 & 0 & a\\ 0 & 1 & 1 & 0\end{array}\right\| \quad (a = 0 \textit{ or } \tilde{a} \in \mathbb{F}^*/\mathbb{F}^{*2},\ 2\times 2\times 2), \tag{13}$$

where $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ *is the multiplicative group of* $\mathbb{F}$, $\mathbb{F}^*/\mathbb{F}^{*2}$ *is its factor group by* $\mathbb{F}^{*2} := \{z^2 \mid 0 \neq z \in \mathbb{F}\}$, *and a is any fixed element of the coset* $\tilde{a}$; *changing to another* $a \in \tilde{a}$ *gives an equivalent spatial matrix* $\mathscr{K}(\tilde{a})$.

Proof. Let $\mathscr{A}$ be a regular $m \times n \times q$ spatial matrix with $n \leqslant 2$ and $q \leqslant 2$.

Step 1. First we prove that $\mathscr{A}$ is equivalent to at least one of the spatial matrices (6)–(13).

Let $\mathscr{A}$ be $m \times n \times 1$. Since $\mathscr{A} = \|A\|$ is regular, it can be reduced by transformations (3) to (6) or (7).

Let $\mathscr{A}$ be $1 \times 2 \times 2$. We reduce $\mathscr{A} = \|A \mid B\|$ to the form $\|1\ \ 0 \mid b_1\ \ b_2\|$ by transformations (3). Since $\mathscr{A}$ is regular, $b_2 \neq 0$; we make $b_2 = 1$ multiplying the second columns of A and B by b_2^{-1}. Then we make $b_1 = 0$ by adding a multiple of b_2 and obtain (9). Similarly, if $\mathscr{A}$ is $2 \times 1 \times 2$, then it reduces to (8).

It remains to consider $\mathscr{A}$ of size $m \times 2 \times 2$ with $m \geqslant 2$. Since $\mathscr{A} = \|A \mid B\|$ is regular, $A \neq 0$, $B \neq 0$, and the rows of the $m \times 4$ matrix $[A\ \ B]$ are linearly independent; that is, $\operatorname{rank}[A\ \ B] = m \leqslant 4$. Interchanging A and B if necessary, we make

$$\operatorname{rank} A \geqslant \operatorname{rank} B. \tag{14}$$

If $m = 4$, then $[A\ \ B]$ is a non-singular 4×4 matrix; we reduce $\mathscr{A} = \|A \mid B\|$ by elementary row-operations to the form (12).

If $m = \operatorname{rank}[A\ \ B] = 3$, then $\operatorname{rank} A = 2$ by (14); we reduce consecutively $\mathscr{A} = \|A \mid B\|$ by transformations (3) as follows:

$$\mathscr{A} \to \left\|\begin{array}{cc|cc}* & * & * & *\\ * & * & * & *\\ 0 & 0 & * & *\end{array}\right\| \to \left\|\begin{array}{cc|cc}* & * & * & *\\ * & * & * & *\\ 0 & 0 & 0 & 1\end{array}\right\| \to \left\|\begin{array}{cc|cc}1 & 0 & b_{11} & b_{12}\\ 0 & 1 & b_{21} & b_{22}\\ 0 & 0 & 0 & 1\end{array}\right\|.$$

Replacing B by $B - b_{11}A$, we make $b_{11} = 0$, then make zero the $(1, 2)$ and $(2, 2)$ entries of B by adding the third row. So $\mathscr{A}$ is equivalent to (10) or (11).

Let $m = \operatorname{rank}[A\ \ B] = 2$. If A is singular, then by (14) $\operatorname{rank} A = \operatorname{rank} B = 1$ and we reduce $\mathscr{A}$ by transformations (3) and (4):

$$\left\|\begin{array}{cc|cc}1 & 0 & * & *\\ 0 & 0 & * & *\end{array}\right\| \to \left\|\begin{array}{cc|cc}1 & 0 & 0 & b_{12}\\ 0 & 0 & b_{21} & b_{22}\end{array}\right\|.$$

Since $\mathscr{A}$ is regular, both (b_{21}, b_{22}) and (b_{12}, b_{22}) are non-zero. Because rank $B = 1$, $b_{21} = 0$ or $b_{12} = 0$. We replace A by the non-singular $A + B$.

So we can suppose that A is non-singular and reduce $\mathscr{A}$ to the form

$$\left\| \begin{array}{cc|cc} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{array} \right\|.$$

Since $\mathscr{A}$ is regular, $B \neq 0$. Preserving $A = I_2$, we reduce B by similarity transformations as follows. If $b_{21} = 0$, then we make $b_{21} \neq 0$ using

$$\begin{bmatrix} 1 & 0 \\ -\varepsilon & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix}, \quad \varepsilon = 1 \quad \text{or} \quad \varepsilon = -1.$$

Multiplying the first row of B by b_{21} and its first column by b_{21}^{-1}, we obtain $b_{21} = 1$. Replacing B by $B - \alpha A = B - \alpha I_2$ with $\alpha := (b_{11} + b_{22})/2$, we make $b_{11} = -b_{22}$. Finally, we reduce B to the form

$$\begin{bmatrix} 1 & -b_{11} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ 1 & -b_{11} \end{bmatrix} \begin{bmatrix} 1 & b_{11} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & * \\ 1 & 0 \end{bmatrix}$$

and obtain (13).

Step 2. Now we prove that $\mathscr{A}$ is equivalent to exactly one of the spatial matrices (6)–(13). Let two distinct spatial matrices of the form (6)–(12) be equivalent. Then they have the same size, and so they are (10) and (11), or $\|I_2 \mid B(a)\|$ and $\|I_2 \mid B(b)\|$. If $\|A \mid B\|$ is (11), then rank $(\alpha A + \beta B) \neq 1$ for all $\alpha, \beta \in \mathbb{F}$, hence (10) is not equivalent to (11).

Let

$$\|I_2 \mid B(a)\| := \left\| \begin{array}{cc|cc} 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 \end{array} \right\|, \quad 0 \neq a \in \mathbb{F},$$

be equivalent to $\|I_2 \mid B(b)\|$. By (3) and (4), there is a non-singular matrix

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

such that the matrices I_2 and $B(b)$ are simultaneously equivalent to

$$\alpha I_2 + \beta B(a) \quad \text{and} \quad \gamma I_2 + \delta B(a),$$

which are simultaneously equivalent to

$$I_2 \quad \text{and} \quad (\alpha I_2 + \beta B(a))^{-1}(\gamma I_2 + \delta B(a)).$$

Then the last matrix is similar to $B(b)$. Hence, the matrices

$$\begin{bmatrix} \alpha & -\beta a \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} \gamma & \delta a \\ \delta & \gamma \end{bmatrix} = \begin{bmatrix} \alpha\gamma - \beta\delta a & (\alpha\delta - \beta\gamma)a \\ \alpha\delta - \beta\gamma & \alpha\gamma - \beta\delta a \end{bmatrix}, \quad (\alpha^2 - \beta^2 a) \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix}$$

are similar. Equating their traces and determinants, we obtain

$$\alpha\gamma - \beta\delta a = 0, \quad (\alpha\gamma - \beta\delta a)^2 - (\alpha\delta - \beta\gamma)^2 a = (\alpha^2 - \beta^2 a)^2(-b).$$

Therefore, $a = bz^2$, where $z = (\alpha^2 - \beta^2 a)/(\alpha\delta - \beta\gamma)$.

Conversely, if $a = bz^2$ and $0 \neq z \in \mathbb{F}$, then $\|I_2 \mid B(b)\|$ is equivalent to $\|I_2 \mid zB(b)\|$, which is equivalent to $\|I_2 \mid B(a)\|$ since

$$\begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & bz \\ z & 0 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & bz^2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}. \qquad \square$$

References

[1] G.R. Belitskii, V.V. Sergeichuk, Complexity of matrix problems, Linear Algebra Appl. 361 (2003) 203–222.
[2] A. Duschek, Eine abbildung der binarën trilinearform, Jahresber. Deutsch. Math. Verein. 32 (1923) 234–239.
[3] R. Ehrenborg, Canonical forms of two by two by two matrices, J. Algebra 213 (1999) 195–244.
[4] S. Mac Lane, G. Birkhoff, Algebra, AMS Chelsea Publishing, 1999.
[5] R. Oldenburger, On canonical binary trilinear forms, Bull. Amer. Math. Soc. 38 (1932) 385–387.
[6] R. Oldenburger, On arithmetic invariants of binary cubic and binary trilinear forms, Bull. Amer. Math. Soc. 42 (1936) 871–873.
[7] R. Oldenburger, Real canonical binary trilinear forms, Amer. J. Math. 59 (1937) 427–435.
[8] E. Schwartz, Ueber binäre trilineare formen, Math. Z. 12 (1922) 18–35.
[9] N.P. Sokolov, Spatial matrices and their applications, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960 (in Russian).