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Fixed point property on symmetric products

Galo Higuera, Alejandro Illanes *

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Cd. Universitaria, México, 04510, D.F., Mexico

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ABSTRACT

For a metric continuum X, let $F_n(X) = \{A \subset X\colon\ A$ is nonempty and has at most n points$\}$. In this paper we show a continuum X such that $F_2(X)$ has the fixed point property while X does not have it.

1. Introduction

A *continuum* is a compact connected metric space with more than one point. Given a continuum X, consider the following hyperspaces of X:

$$2^X = \{A \subset X\colon\ A \text{ is nonempty and closed}\},$$
$$C(X) = \bigl\{A \in 2^X\colon\ A \text{ is connected}\bigr\}, \quad \text{and for each } n \geqslant 1,$$
$$F_n(X) = \bigl\{A \in 2^X\colon\ A \text{ has at most } n \text{ points}\bigr\}.$$

All these hyperspaces are considered with the Hausdorff metric H.

A continuum X is said to have the *fixed point property* (f.p.p.) provided that for each continuous map $f : X \to X$, there exists a point $p \in X$ such that $f(p) = p$.

A discussion on which is known about the f.p.p. of hyperspaces can be found in [8, Chapter VI] and [10, Chapter VII]. We only mention here some facts.

In 1952, B. Knaster posed the following question (see [3, Problem 186]): If X is a continuum with the f.p.p., then does $C(X)$ have the f.p.p.? A fundamental example on the theory of the f.p.p. is the cone over the continuum D_0 (cone(D_0)) which is the union of a circle and a spiral surrounding it. R.J. Knill (see [9]) showed that this cone does not have the f.p.p., in [14] J.T. Rogers, Jr. showed that cone(D_0) and $C(D_0)$ are homeomorphic, thus $C(D_0)$ does not have the f.p.p. Hence D_0 was the first example for which the hyperspace of subcontinua does not have the f.p.p. In [12] S.B. Nadler, Jr. and J.T. Rogers, Jr. answered Knaster's question by showing that $C(D_1)$ (D_1 is the union of D_0 and the disk bounded by its circle) does not have the f.p.p.

* Corresponding author.
E-mail addresses: galoolag@gmail.com (G. Higuera), illanes@matem.unam.mx (A. Illanes).

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Answering an old question by J.T. Rogers, Jr., the second named author has recently shown (see [6]) that, if T_0 is the union of a simple triod and a ray surrounding it, then $C(T_0)$ does not have the f.p.p. The main difference between D_1 and T_0 is that T_0 is tree-like and one-dimensional continuum (with the f.p.p.). Recently, the continuum T_0 has played another important role in the f.p.p. theory on continua. Answering an old question on this topic, the second named author has shown in [5] that cone(T_0) does not have the f.p.p.

The spaces of the form $F_n(X)$ are called *symmetric products*. They were introduced by K. Borsuk and S. Ulam in [1], where they asked if every symmetric product of a continuum with the f.p.p. must have the f.p.p. J. Oledzki (see [13]) constructed a 2-dimensional continuum to answer this question in the negative.

There are only a few results about the f.p.p. on symmetric products. Recently (see [7]), it has been shown that, if X is a chainable continuum then $F_3(X)$ has the f.p.p.

In [11, 8.16, p. 120], S.B. Nadler, Jr. wrote:

> "**8.16.** Is there a continuum X such that $F_n(X)$ has the fixed point property for some $n > 1$ and, yet, $F_m(X)$ does not have the fixed point property for some m?

Comments. We require $n > 1$ in view of Oledzki's example [13] (see comments to Question 8.14). We allow $m < n$ in the question ...."

In this paper we answer Nadler's question by showing a continuum X such that $F_2(X)$ has the f.p.p. while X (and then $F_1(X)$) does not have the f.p.p.

Some other related questions on this topic can be found in [11] and [4].

2. The example

A continuum X is *indecomposable* provided that X cannot be expressed as the union of two of its proper subcontinua. A *Cook continuum* is an indecomposable continuum X with the property that if C is a subcontinuum of X and $f : C \to X$ is a continuous map, then f is constant or $f(p) = p$ for every $p \in C$. The existence of Cook continua was showed by H. Cook in [2, Theorems 8 and 9].

Example. There exists a continuum X such that $F_2(X)$ has the fixed point property while X does not have it.

In order to construct X, take a Cook continuum C. Fix two points $a \neq b$ in C. Let $Z = C \times \{0,1\}$. In Z identify the point $(a,0)$ to the point $(b,1)$ and the point $(a,1)$ to the point $(b,0)$. Consider $\{0,1\}$ with the discrete topology and the sum module 2, which is denoted by $\oplus$. Then, we are identifying each point of the form (a,i) to the point $(b, i \oplus 1)$. The resulting space is denoted by X. Let $\pi : Z \to X$ be the identification map. Since $X = \pi(C \times \{0\}) \cup \pi(C \times \{1\})$, $\pi(C \times \{0\})$ and $\pi(C \times \{1\})$ are connected and $\pi((a,0)) = \pi((b,1)) \in \pi(C \times \{0\}) \cap \pi(C \times \{1\})$, we have that X is connected. Hence X is a continuum. To simplify the notation, for each point $(z,i) \in Z$, we denote $\pi((z,i))$ by $[z,i]$. Let $P = \{[a,0],[b,0]\}$. Note that $P = \{[a,1],[b,1]\} = \{[a,0],[a,1]\} = \{[b,0],[b,1]\}$. Let d be a metric for X. Given a point $[z,i] \in X$, a subset A of X and a positive number ε, let $B(\varepsilon,[z,i])$ be the ε-neighborhood in X, around $[z,i]$ and let $N(\varepsilon, A) = \{q \in X\colon$ there exists $a \in A$ such that $d(q,a) < \varepsilon\}$. Given subsets A and B of X, let $\langle A, B\rangle_2 = \{E \in F_2(X)\colon\ E \subset A \cup B,\ E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset\}$.

Claim 1. *X does not have the fixed point property.*

Proof. Let $f_0 : X \to X$ be defined by $f_0([z,i]) = [z, i \oplus 1]$. Given $i \in \{1,2\}$, since $f_0([a,i]) = [a, i \oplus 1] = [b, i \oplus 2] = f_0([b, i \oplus 1])$, we have that $f_0([a,i]) = f_0([b, i \oplus 1])$. Thus f_0 is well defined and continuous. Suppose that there exists $[z,i] \in X$ such that $f_0([z,i]) = [z,i]$. Then $[z, i \oplus 1] = [z,i]$. This implies that $z = a$ and $z = b$, a contradiction. We have shown that f_0 does not have fixed points. Therefore, X does not have the fixed point property. $\square$

Claim 2. *Let K be a subcontinuum of C and let $g : K \to X$ be a continuous function. Then g is constant or there exists $i \in \{0,1\}$ such that $g(u) = [u,i]$ for every $u \in K$.*

Proof. We can assume that K is nondegenerate. If $g(K) \subset P$, since P is finite and K is connected we have that $g(K)$ is a one-point set. Hence g is constant. Suppose then that there exists a point $p \in K$ such that $g(p) \notin P$. Let $g(p) = [z,i]$. Since $[z,i] \notin \pi(C \times \{i \oplus 1\})$, there exists $\varepsilon > 0$ such that $B(\varepsilon,[z,i]) \cap \pi(C \times \{i \oplus 1\}) = \emptyset$. By [8, Theorem 14.6], there exists a continuous function $\alpha : [0,1] \to C(K)$ such that $\alpha(0) = \{p\}$, $\alpha(1) = K$ and, if $s < t$, then $\alpha(s) \subsetneq \alpha(t)$. Let $G : C(K) \to C(X)$ be the induced function of g. That is, $G(A) = g(A)$ (the image of A under g). Since α and G are uniformly continuous, there exists $\delta > 0$ such that, if $|s - t| < \delta$, then $H(G(\alpha(t)), G(\alpha(s))) < \varepsilon$. Let $m \geqslant 1$ be such that $\frac{1}{m} < \delta$. Then $H(G(\alpha(\frac{1}{m})), \{[z,i]\}) = H(G(\alpha(\frac{1}{m})), G(\alpha(0))) < \varepsilon$. Thus $G(\alpha(\frac{1}{m})) \subset B(\varepsilon,[z,i]) \subset \pi(C \times \{i\})$. Hence $g|_{\alpha(\frac{1}{m})} : \alpha(\frac{1}{m}) \to \pi(C \times \{i\})$. Since $\pi(C \times \{i\})$ is homeomorphic to the Cook continuum C and $\alpha(\frac{1}{m})$ is a subcontinuum of C, we have that either $g|_{\alpha(\frac{1}{m})}$ is constant or $g(q) = [q,i]$ for every $q \in \alpha(\frac{1}{m})$. We analyze both cases.

Case 1. $g|_{\alpha(\frac{1}{m})}$ is constant.

Since $p \in \alpha(\frac{1}{m})$, $g(\alpha(\frac{1}{m})) = \{g(p)\}$. In this case, we are going to show, by induction, that for each $k \in \{1, \dots, m\}$, $g|_{\alpha(\frac{k}{m})}$ is constant. By hypothesis, this holds for $k = 1$. Now, suppose that $g|_{\alpha(\frac{k}{m})}$ is constant and $k < m$. Since $H(G(\alpha(\frac{k+1}{m})), G(\alpha(\frac{k}{m}))) < \varepsilon$ and $G(\alpha(\frac{k}{m})) = \{g(p)\}$, we have that $g(\alpha(\frac{k+1}{m})) \subset B(\varepsilon, g(p)) \subset \pi(C \times \{i\})$. Since $\pi(C \times \{i\})$ is homeomorphic to C and $\alpha(\frac{k+1}{m})$ is a subcontinuum of C, we have either $g|_{\alpha(\frac{k+1}{m})}$ is constant or $g(q) = [q, i]$ for every $q \in \alpha(\frac{k+1}{m})$. If we take $q \in \alpha(\frac{1}{m}) \setminus \{p\}$, we have $g(q) = g(p)$, thus $g(q) \neq [q, i]$ or $g(p) \neq [p, i]$. Hence, $g|_{\alpha(\frac{k+1}{m})}$ is constant. This completes the induction. In particular, for $k = m$, we obtain $g|K = g$ is constant. This ends the analysis of this case.

Case 2. $g(q) = [q, i]$ for every $q \in \alpha(\frac{1}{m})$.

Let $J = \{t \in [0, 1]\colon\ g(q) = [q, i]$ for every $q \in \alpha(t)\}$ and let $t_0 = \sup J$. Note that $\frac{1}{m} \in J$, so $J \neq \emptyset$ and $\frac{1}{m} \leqslant t_0$. We show that $t_0 \in J$. Let $\{t_k\}_{k=1}^{\infty}$ be a sequence in J such that $\lim t_k = t_0$. Then $\lim \alpha(t_k) = \alpha(t_0)$. Given $q \in \alpha(t_0)$ there exists a sequence $\{q_k\}_{k=1}^{\infty}$ in K such that $q_k \in \alpha(t_k)$, for every $k \geqslant 1$ and $\lim q_k = q$. Then $g(q_k) = [q_k, i]$ for every $k \geqslant 1$. Thus $g(q) = \lim g(q_k) = \lim[q_k, i] = [q, i]$. Hence $t_0 \in J$. In particular, $g(q) \in \pi(C \times \{i\})$ for every $q \in \alpha(t_0)$. Therefore, $g(\alpha(t_0)) \subset \pi(C \times \{i\})$.

Next, we show that $t_0 = 1$. Suppose, to the contrary, that $t_0 < 1$. Let $\lambda > 0$ be such that $B(2\lambda, [a, i]) \cap B(2\lambda, [b, i]) = \emptyset$. Let $U = B(\lambda, [a, i]) \cup B(\lambda, [b, i]) \cup \pi(C \times \{i\})$. Note that U is open in X. Define $\beta : \mathrm{cl}_X(U) \to \pi(C \times \{i\})$ by

$$\beta\big([q, j]\big) = \begin{cases} [q, j], & \text{if } [q, j] \in \pi(C \times \{i\}), \\ [a, i], & \text{if } [q, j] \in \mathrm{cl}_X(B(\lambda, [a, i])) \cap \pi(C \times \{i \oplus 1\}), \\ [b, i], & \text{if } [q, j] \in \mathrm{cl}_X(B(\lambda, [b, i])) \cap \pi(C \times \{i \oplus 1\}). \end{cases}$$

Note that β is well defined and continuous. Since $t_0 < 1$, $g(\alpha(t_0)) \subset \pi(C \times \{i\}) \subset U$ and U is open, there exists $s > t_0$ such that $g(\alpha(s)) \subset U$. If $a \notin \alpha(t_0)$, we can take s with the additional property that $a \notin \alpha(s)$. If $b \notin \alpha(t_0)$, we also ask that $b \notin \alpha(s)$. Thus, if $a \in \alpha(s)$, then $a \in \alpha(t_0)$ and $g(a) = [a, i]$. Also, if $b \in \alpha(s)$, then $b \in \alpha(t_0)$ and $g(b) = [b, i]$.

Since $g(s)$ is a subcontinuum of $K \subset C$, $\pi(C \times \{i\})$ is homeomorphic to C and $(\beta \circ g)|_{\alpha(s)} : \alpha(s) \to \pi(C \times \{i\})$ is continuous, we have that either $(\beta \circ g)|_{\alpha(s)}$ is constant or $(\beta \circ g)|_{\alpha(s)}(q) = [q, i]$ for every $q \in \alpha(s)$. Note that, for every $q \in \alpha(\frac{1}{m})$, $g(q) = [q, i]$, so $\beta(g(q)) = [q, i]$. Thus $\alpha(\frac{1}{m})$ is a nondegenerate subcontinuum of $\alpha(s)$ such that $(\beta \circ g)|_{\alpha(\frac{1}{m})}$ is not constant. Hence $(\beta \circ g)|_{\alpha(s)}$ is not constant. Thus $(\beta \circ g)|_{\alpha(s)}(q) = [q, i]$ for every $q \in \alpha(s)$. That is, $\beta(g(q)) = [q, i]$ for every $q \in \alpha(s)$.

Since $t_0 = \sup J$ and $t_0 < s$, there exists $q \in \alpha(s)$ such that $g(q) \neq [q, i]$. If $g(q) \in \pi(C \times \{i\})$, then $[q, i] = \beta(g(q)) = g(q)$, a contradiction. Hence $g(q) \notin \pi(C \times \{i\})$. Suppose, by example, that $g(q) \in \mathrm{cl}_X(B(\lambda, [a, i])) \setminus \pi(C \times \{i\})$. Then $[q, i] = \beta(g(q)) = [a, i]$. This implies that $q = a$ and $g(a) \neq [a, i]$. Thus $a \in \alpha(s)$ and, by the choice of s, we conclude that $a \in \alpha(t_0)$ and $g(a) = [a, i]$, a contradiction. This completes the proof that $t_0 = 1$.

Hence $g(q) = [q, i]$ for every $q \in \alpha(1) = K$. This ends the analysis for this case. $\square$

We want to show that $F_2(X)$ has the fixed point property. Take then a continuous function $f : F_2(X) \to F_2(X)$.

Given $p \in C$, let $\pm p = \{[p, 0], [p, 1]\}$. Given $Y \subset X$, let $\pm Y = \bigcup\{\pm p\colon\ \pm p \cap Y \neq \emptyset\}$.

Claim 3. *Let $A = \{[x, i], [y, j]\} \in F_2(X)$ be such that $\pm y \cap f(A) = \emptyset$ and $f(A) \notin F_1(X)$. Then $f|_{\langle\{[x,i]\}, \pi(C \times \{j\})\rangle_2}$ is constant.*

Proof. Let $f(A) = \{w_1, w_2\}$. Let $\varepsilon > 0$ be such that $B(\varepsilon, w_1) \cap B(\varepsilon, w_2) = \emptyset$. Let $\delta > 0$ be such that if $B, D \in F_2(X)$ and $H(B, D) < \delta$, then $H(f(B), f(D)) < \varepsilon$. By [8, Theorem 14.6] there exists a continuous function $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = \{[y, j]\}$, $\alpha(1) = \pi(C \times \{j\})$ and $s < t$ implies that $\alpha(s) \subsetneq \alpha(t)$. Let $\lambda > 0$ be such that, if $|s - t| < \lambda$, then $H(\alpha(s), \alpha(t)) < \delta$. Let $m \geqslant 1$ be such that $\frac{1}{m} < \lambda$. For each $k \in \{0, 1, \dots, m\}$, let $\mathcal{E}_k = \langle\{[x, i]\}, \alpha(\frac{k}{m})\rangle_2$.

We are going to show, inductively, that $f|_{\mathcal{E}_k}$ is constant. For $k = 0$, note that $\mathcal{E}_0 = \langle\{[x, i]\}, \alpha(0)\rangle_2 = \langle\{[x, i]\}, \{[y, j]\}\rangle_2 = \{A\}$ is a one point set, so $f|_{\mathcal{E}_0}$ is constant. Suppose now that $k \in \{0, 1, \dots, m-1\}$ and $f|_{\mathcal{E}_k}$ is constant. Since $A \in \mathcal{E}_0 \subset \mathcal{E}_k$, we have $f(D) = f(A)$ for every $D \in \mathcal{E}_k$. Given $B \in \mathcal{E}_{k+1}$, B is of the form $B = \{[x, i], [u, j]\}$, where $[u, j] \in \alpha(\frac{k+1}{m})$. Since $H(\alpha(\frac{k+1}{m}), \alpha(\frac{k}{m})) < \delta$, there exists $[w, j] \in \alpha(\frac{k}{m})$ such that $d([w, j], [u, j]) < \delta$. Thus $H(\{[x, i], [u, j]\}, \{[x, i], [w, j]\}) < \delta$. Hence $H(f(B), f(\{[x, i], [w, j]\})) < \varepsilon$. Since $\{[x, i], [w, j]\} \in \mathcal{E}_k$, we have $f(\{[x, i], [w, j]\}) = f(A) = \{w_1, w_2\}$. Thus $H(f(B), \{w_1, w_2\}) < \varepsilon$. Hence $f(B)$ has exactly one element in each one of the sets $B(\varepsilon, w_1)$ and $B(\varepsilon, w_2)$.

Given $s \in \{1, 2\}$, it is easy to show that the function $g_s : \mathcal{E}_{k+1} \to B(\varepsilon, w_s)$ given by: $g_s(B)$ is the only point in $f(B) \cap B(\varepsilon, w_s)$, is continuous. Define $h_s : \alpha(\frac{k+1}{m}) \to B(\varepsilon, w_s) \subset X$ by $h_s([v, j]) = g_s(\{[x, i], [v, j]\})$. Clearly, h_s is continuous. Since $\pi(C \times \{j\})$ is homeomorphic to C, $\alpha(\frac{k+1}{m})$ is homeomorphic to a subcontinuum of C, so we can apply Claim 2 and obtain that either h_s is constant or there exists $j_s \in \{0, 1\}$ such that $h_s([v, j]) = [v, j_s]$ for every $[v, j] \in \alpha(\frac{k+1}{m})$. Since $[y, j] \in \alpha(0) \subset \alpha(\frac{k+1}{m})$, $\{h_1([y, j]), h_2([y, j])\} = \{g_1(\{[x, i], [y, j]\}), g_2(\{[x, i], [y, j]\})\} = f(\{[x, i], [y, j]\}) = \{w_1, w_2\}$ and $\{w_1, w_2\} \cap \{[y, 0], [y, 1]\} = \emptyset$, we have $h_s([y, j])$ is not of the form $[y, j_s]$, for each $s \in \{0, 1\}$. Hence h_1

and h_2 are constants. Thus, for every $[v, j] \in \alpha(\frac{k+1}{m})$, since $\{w_1, w_2\} = \{h_1([y, j]), h_2([y, j])\} = \{h_1([v, j]), h_2([v, j])\} = \{g_1(\{[x, i], [v, j]\}), g_2(\{[x, i], [v, j]\})\} = f(\{[x, i], [v, j]\})$, we have $f|_{\mathcal{E}_{k+1}}$ is constant. This ends the induction.

In particular, $f|_{\mathcal{E}_m} = f|_{\langle\{[x,i]\},\pi(C\times\{j\})\rangle_2}$ is constant. This completes the proof of Claim 3. □

Claim 4. *If there exists $A \in F_2(X)$ such that $(\pm A \cup P) \cap f(A) = \emptyset$ and $f(A) \notin F_1(X)$, then f has a fixed point.*

Proof. Let $A = \{[x, i], [y, j]\}$ and $f(A) = \{[u, k], [v, l]\}$. Since $\pm y \subset \pm A$ and $\pm A \cap f(A) = \emptyset$, we have $\pm y \cap f(A) = \emptyset$. Then, we can apply Claim 3 and obtain that $f|_{\langle\{[x,i]\},\pi(C\times\{j\})\rangle_2}$ is constant. In particular, $f(A) = f(\{[x, i], [a, j]\}) = f(\{[x, i], [b, j \oplus 1]\})$. Let $A_1 = \{[x, i], [b, j \oplus 1]\}$. Since $\pm b = P$ and $f(A_1) = f(A)$ does not intersect P, we can apply Claim 3 to A_1 and obtain that $f|_{\langle\{[x,i]\},\pi(C\times\{j\oplus 1\})\rangle_2}$ is constant. Since $\langle\{[x, i]\}, \pi(C \times \{j\})\rangle_2 \cup \langle\{[x, i]\}, \pi(C \times \{j \oplus 1\})\rangle_2 = \langle\{[x, i]\}, X\rangle_2$, we have $f|_{\langle\{[x,i]\},X\rangle_2}$ is constant. In particular, $f(\{[x, i], [u, k]\}) = f(A)$.

Let $A_2 = \{[x, i], [u, k]\}$. Then $f(A_2) = f(A)$. Since $\pm x \subset \pm A$, we have $\pm x \cap f(A_2) = \emptyset$. Thus, applying Claim 3, we obtain that $f|_{\langle\{[u,k]\},\pi(C\times\{i\})\rangle_2}$ is constant. In particular, $f(A_2) = f(\{[u, k], [a, i]\}) = f(\{[u, k], [b, i \oplus 1]\})$. Let $A_3 = \{[u, k], [b, i \oplus 1]\}$. Since $\pm b = P$ and $f(A_3) = f(A_2) = f(A)$ does not intersect P, we can apply Claim 3 to A_3 and obtain that $f|_{\langle\{[u,k]\},\pi(C\times\{i\oplus 1\})\rangle_2}$ is constant. Hence $f|_{\langle\{[u,k]\},X\rangle_2}$ is constant. In particular, $f(\{[u, k], [v, l]\}) = f(A_2) = f(A) = \{[u, k], [v, l]\}$. We have found a fixed point for f. □

Claim 5. *Let $A = \{[x, 0], [y, 1]\} \in F_2(X)$ be such that $\pm A \cap f(A) = \emptyset$ and $f(A) \notin F_1(X)$. Then f has a fixed point.*

Proof. If $P \cap f(A) = \emptyset$, by Claim 4 f has a fixed point and we have finished. Suppose then, by example, that $f(A)$ has the form $f(A) = \{[a, 0], [z, 1]\}$. Since $\pm A \cap f(A) = \emptyset$, we have $\pm y \cap f(A) = \emptyset$. By Claim 3, $f|_{\langle\{[x,0]\},\pi(C\times\{1\})\rangle_2}$ is constant. In particular, $f(\{[x, 0], [z, 1]\}) = f(\{[x, 0], [y, 1]\}) = f(A)$. Let $A_1 = \{[x, 0], [z, 1]\}$. Since $\pm x \cap f(A_1) = \pm x \cap f(A) = \emptyset$, we can apply Claim 3 to A_1 and conclude that $f|_{\langle\{[z,1]\},\pi(C\times\{0\})\rangle_2}$ is constant. In particular, $f(\{[z, 1], [a, 0]\}) = f(\{[z, 1], [x, 0]\}) = f(A_1) = f(A) = \{[a, 0], [z, 1]\}$. Thus, we have found a fixed point for f. □

In Claims 4 and 5, we have shown some particular conditions under which f has a fixed point. From now on, we suppose that f does not have fixed points and we are going to obtain a contradiction. Given $p \in C$, recall that $\pm p = \{[p, 0], [p, 1]\}$. Note that $\pm(\pm p) = \pm p$. By Claim 5, since f does not have fixed points, we have $\pm p \cap f(\pm p) \neq \emptyset$ or $f(\pm p) \in F_1(X)$ for every $p \in C$. So, we can define the function $g : C \to F_1(X)$ by

$$g(p) = \begin{cases} f(\pm p), & \text{if } f(\pm p) \in F_1(X), \\ f(\pm p)\backslash(\pm p), & \text{if } f(\pm p) \notin F_1(X). \end{cases}$$

Claim 6. *The function g is continuous.*

Proof. Note that the function $\varphi : C \to F_2(X)$ given by $\varphi(p) = f(\pm p)$ is continuous. So the set $D = \{p \in C : \varphi(p) \in F_1(X)\}$ is closed in C. In order to see that g is continuous, take a sequence $\{p_k\}_{k=1}^{\infty}$ in C such that $\lim p_k = p \in C$. Consider two cases.

Case 1. $f(\pm p) \notin F_1(X)$.

Since $\pm p \cap f(\pm p) \neq \emptyset$, we may assume that $[p, 0] \in f(\pm p)$. Since $f(\pm p) \neq \pm p$, we have $[p, 1] \notin f(\pm p)$. Then $g(p) = f(\pm p)\backslash\{[p, 0]\}$. Since D is closed and $p \notin D$, we may assume that $p_k \notin D$ and $[p_k, 1] \notin f(\pm p_k)$ for all $k \geqslant 1$. Thus $[p_k, 0] \in f(\pm p_k)$ for every $k \geqslant 1$. Given $k \geqslant 1$, we have $g(p_k) = f(\pm p_k)\backslash\{[p_k, 0]\}$. Since $f(\pm p)$ has two different points: $[p, 0]$ and the unique point $w \in g(p)$ and $\lim f(\pm p_k) = f(\pm p)$, we have that, for each $k \geqslant 1$, we can write $f(\pm p_k) = \{w_k, v_k\}$, where $\lim w_k = w$ and $\lim v_k = [p, 0]$. Since $\lim[p_k, 0] = [p, 0]$, we have $v_k = [p_k, 0]$ for almost all k. Then, for almost all k, $\{w_k\} = g(p_k)$. Hence $\lim g(p_k) = g(p)$.

Case 2. $f(\pm p) \in F_1(X)$.

Since, for every $k \geqslant 1$, $g(p_k) \subset f(\pm p_k)$ and $\lim f(\pm p_k) = f(\pm p)$, we have $\lim g(p_k) = f(\pm p) = g(p)$. □

We have seen that g is a continuous function. Since $F_1(X)$ is naturally homeomorphic to X, we can apply Claim 2 and obtain that g is constant or there exists $i \in \{0, 1\}$ such that $g(p) = \{[p, i]\}$ for every $p \in C$. Note that $f(\pm a) = f(\pm b)$ implies that $g(a) = g(b)$, so g is not one-to-one and then g is constant. Let $[z_0, i_0] \in X$ be such that $\operatorname{Im} g = \{[z_0, i_0]\}$. By the definition of g, $[z_0, i_0] \in f(\pm p)$ for every $p \in C$. Thus we can define a function $h : C \to F_1(X)$ in the following way:

$$h(p) = \begin{cases} \{[z_0, i_0]\}, & \text{if } f(\pm p) = \{[z_0, i_0]\}, \\ f(\pm p)\backslash\{[z_0, i_0]\}, & \text{if } f(\pm p) \neq \{[z_0, i_0]\}. \end{cases}$$

Claim 7. *The function h is continuous.*

Proof. In order to see that h is continuous, take a sequence $\{p_k\}_{k=1}^{\infty}$ in C such that $\lim p_k = p \in C$. Consider two cases.

Case 1. $f(\pm p) \neq \{[z_0, i_0]\}$.

Let $w \in X$ be such that $f(\pm p) = \{w, [z_0, i_0]\}$. Then $h(p) = \{w\}$. Since $\lim f(\pm p_k) = f(\pm p)$, we have that there exists a sequence $\{w_k\}_{k=1}^{\infty}$ in X such that, for every $k \geqslant 1$, $w_k \in f(\pm p_k)$ and $\lim w_k = w$. Since $w \neq [z_0, i_0]$, we may assume that $w_k \neq [z_0, i_0]$ for every $k \geqslant 1$. Thus, for each $k \geqslant 1$, $f(\pm p_k) = \{w_k, [z_0, i_0]\}$ and $h(p_k) = w_k$. Therefore, $\lim h(p_k) = h(p)$.

Case 2. $f(\pm p) = \{[z_0, i_0]\}$.

Since, for every $k \geqslant 1$, $h(p_k) \subset f(\pm p_k)$ and $\lim f(\pm p_k) = f(\pm p) \in F_1(X)$, we have $\lim h(p_k) = f(\pm p) = h(p)$. □

Claim 8. $f(\pm p) = \{[z_0, i_0]\}$, *for every* $p \in C$.

Proof. We have seen that h is a continuous function. Since $F_1(X)$ is naturally homeomorphic to X, we can apply Claim 2 and conclude that either h is constant or there exists $i \in \{0, 1\}$ such that $h(p) = \{[p, i]\}$ for every $p \in C$. Note that $f(\pm a) = f(\pm b)$ implies that $h(a) = h(b)$, so h is not one-to-one and then h is not constant. Suppose that $\operatorname{Im} h = \{\{[w_0, j_0]\}\}$. If $[w_0, j_0] \neq [z_0, i_0]$, then $f(\pm p) = \{[w_0, j_0], [z_0, i_0]\}$ for every $p \in C$. Given $p \in C$, $f(\pm p) \notin F_1(X)$. Since we are assuming that f does not have fixed points, by Claim 5, $\pm p \cap f(\pm p) \neq \emptyset$. Thus $\pm p \cap \{[w_0, j_0], [z_0, i_0]\} \neq \emptyset$. But if we take a point $p_0 \in C \setminus \{w_0, z_0, a, b\}$, we have $[p_0, 0], [p_0, 1] \notin \{[w_0, j_0], [z_0, i_0]\}$, a contradiction. This shows that $[w_0, j_0] = [z_0, i_0]$. Hence $\operatorname{Im} h = \{\{[z_0, i_0]\}\}$. Thus, for each $p \in C$, $h(p)$ cannot have elements in $f(\pm p) \setminus \{[z_0, i_0]\}$. Therefore, $f(\pm p) = \{[z_0, i_0]\}$. □

Claim 9. *Let K be a composant of C such that K does not contain the points* z_0, a *and* b. *Let* $p \in K$. *Then* $f(\{[p, i_0], [z_0, i_0]\}) = \{[z_0, i_0]\}$.

Proof. By Claim 8 we know that $f(\pm p) = \{[z_0, i_0]\}$. By [8, Theorem 14.6] there exists a continuous function $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = \{[p, i_0 \oplus 1]\}$, $\alpha(1) = \pi(C \times \{i_0 \oplus 1\})$ and if $s < t$, then $\alpha(s) \subsetneq \alpha(t)$. Let $J = \{t \in [0, 1]\colon\ f(\{[p, i_0], w\}) = \{[z_0, i_0]\}$ for every $w \in \alpha(t)\}$. Since the unique element in $\alpha(0)$ is $[p, i_0 \oplus 1]$ and $f(\{[p, i_0], [p, i_0 \oplus 1]\}) = f(\pm p) = \{[z_0, i_0]\}$, we have $0 \in J$. Thus it makes sense to define $t_0 = \sup J$. It is easy to check that $t_0 \in J$.

We want to prove that $t_0 = 1$. Suppose to the contrary that $t_0 < 1$. Then $\alpha(t_0)$ is a proper subcontinuum of $\pi(C \times \{i_0 \oplus 1\})$ and this space is homeomorphic to C, therefore it is indecomposable and $\pi(K \times \{i_0 \oplus 1\})$ is one of its composants. Thus $\alpha(t_0) \subset \pi(K \times \{i_0 \oplus 1\})$. Since $[a, i_0 \oplus 1]$, $[b, i_0 \oplus 1]$, $[z_0, i_0 \oplus 1]$ do not belong to $\pi(K \times \{i_0 \oplus 1\})$, we have $[z_0, i_0] \notin \pi(K \times \{i_0 \oplus 1\})$. Hence $[z_0, i_0], [z_0, i_0 \oplus 1] \notin \alpha(t_0)$ and $\{[z_0, i_0], [z_0, i_0 \oplus 1]\} \cap \alpha(t_0) = \emptyset$. This implies that $[z_0, i_0] \notin \pm\alpha(t_0)$. Thus there exists $\varepsilon > 0$ such that $N(\varepsilon, \{[z_0, i_0]\}) \cap N(\varepsilon, \pm\alpha(t_0)) = \emptyset$. Let $\delta > 0$ be such that $\delta < \varepsilon$ and if $A, B \in F_2(X)$ and $H(A, B) < \delta$, then $H(f(A), f(B)) < \varepsilon$. We can also ask that δ has the property that if $w, w_0 \in X$ and $d(w, w_0) < \delta$, then $H(\pm w, \pm w_0) < \varepsilon$. Let $\lambda > 0$ be such that, if $|t - s| < \lambda$, then $H(\alpha(s), \alpha(t)) < \delta$.

Choose $t_1 > t_0$ such that $t_1 < t_0 + \lambda$ and $t_1 < 1$. We are going to obtain a contradiction by showing that $t_1 \in J$. Take $w \in \alpha(t_1)$. Since $|t_1 - t_0| < \lambda$, there exists $w_0 \in \alpha(t_0)$ such that $d(w, w_0) < \delta$. Thus $H(\pm w, \pm w_0) < \varepsilon$ and $H(f(\{[p, i_0], w\}), f(\{[p, i_0], w_0\})) < \varepsilon$. Since $w_0 \in \alpha(t_0)$, we have $\pm w_0 \subset \pm\alpha(t_0)$ and $f(\{[p, i_0], w_0\}) = \{[z_0, i_0]\}$. Hence $\pm w \subset N(\varepsilon, \pm\alpha(t_0))$ and $f(\{[p, i_0], w\}) \subset B(\varepsilon, [z_0, i_0])$. Thus $\pm w \cap N(\varepsilon, \{[z_0, i_0]\}) = \emptyset$. We have shown that $\pm w \cap f(\{[p, i_0], w\}) = \emptyset$. Since $[p, i_0 \oplus 1] \in \alpha(0) \subset \alpha(t_0)$, $\pm[p, i_0 \oplus 1] \subset \pm\alpha(t_0)$, then $\pm[p, i_0 \oplus 1] \cap N(\varepsilon, \{[z_0, i_0]\}) = \emptyset$ and $\pm[p, i_0 \oplus 1] \cap f(\{[p, i_0], w\}) = \emptyset$. Let $A = \{[p, i_0], w\}$. We have seen that $\pm A \cap f(A) = \emptyset$. Since $w \in \alpha(t_1) \subset \pi(C \times \{i_0 \oplus 1\})$ and $[p, i_0] \in \pi(C \times \{i_0\})$, we can apply Claim 5 and conclude that $f(A) \in F_1(X)$ (remember that we are assuming that f does not have fixed points).

We have proven that, for each $w \in \alpha(t_1)$, $f(\{[p, i_0], w\}) \in F_1(X)$. If we define $\psi(w) = f(\{[p, i_0], w\})$ we have a continuous function $\psi : \alpha(t_1) \to F_1(X)$. Since $\alpha(t_1)$ is a subcontinuum of $\pi(C \times \{i_0 \oplus 1\})$, this continuum is homeomorphic to C and $F_1(X)$ is naturally homeomorphic to X, we can apply Claim 2 and obtain that either ψ is constant or there exists $j \in \{0, 1\}$ such that, for every $w = [u, i_0 \oplus 1] \in \alpha(t_1)$, $f(\{[p, i_0], w\}) = \{[u, j]\}$. In the second case, since $[p, i_0 \oplus 1] \in \alpha(t_1)$, we have $f(\{[p, i_0], [p, i_0 \oplus 1]\}) = \{[p, j]\}$. But, by Claim 8, $f(\{[p, i_0], [p, i_0 \oplus 1]\}) = \{[z_0, i_0]\}$. Thus $[p, j] = [z_0, i_0]$. This implies that $p \in \{a, b, z_0\}$. This is a contradiction since $p \in K$. We have shown that ψ is constant. Since $[p, i_0 \oplus 1] \in \alpha(t_1)$, we have $f(\{[p, i_0], w\}) = \{[z_0, i_0]\}$, for every $w \in \alpha(t_1)$. Hence $t_1 \in J$, a contradiction. This completes the proof that $t_0 = 1$.

We have proven that $f(\{[p, i_0], w\}) = \{[z_0, i_0]\}$ for every $w \in \pi(C \times \{i_0 \oplus 1\})$.

In the case that $z_0 = a$, we have $[z_0, i_0] = [b, i_0 \oplus 1] \in \pi(C \times \{i_0 \oplus 1\})$, so $f(\{[p, i_0], [z_0, i_0]\}) = \{[z_0, i_0]\}$ and we finish. The case $z_0 = b$ is similar. Therefore, we may assume that $z_0 \notin \{a, b\}$. In particular, $[z_0, i_0] \notin P$.

By [8, Theorem 14.6], there exists a continuous function $\beta : [0, 1] \to C(X)$ such that $\beta(0) = \{[a, i_0]\}$, $\beta(1) = \pi(C \times \{i_0\})$ and that satisfies that, if $s < t$, then $\beta(s) \subsetneq \beta(t)$. Let $L = \{t \in [0, 1]\colon\ f(\{[p, i_0], w\}) = \{[z_0, i_0]\}$ for every $w \in \beta(t)\}$. Since $[a, i_0] = [b, i_0 \oplus 1] \in \pi(C \times \{i_0 \oplus 1\})$, by what we have shown, $f(\{[p, i_0], [a, i_0]\}) = \{[z_0, i_0]\}$, so $0 \in L$. Then it has sense to define $s_0 = \sup L$. It is easy to check that $s_0 \in L$.

If $[z_0, i_0] \in \beta(s_0)$, then $f(\{[p, i_0], [z_0, i_0]\}) = \{[z_0, i_0]\}$ and we finish. Suppose then that $[z_0, i_0] \notin \beta(s_0)$. This implies that $s_0 < 1$. We are going to obtain a similar contradiction as that we obtained when we supposed that $t_0 < 1$.

If $[z_0, i_0 \oplus 1] \in \beta(s_0) \subset \pi(C \times \{i_0\})$, then $z_0 \in \{a, b\}$, which is absurd. Hence $\pm z_0 = \{[z_0, i_0], [z_0, i_0 \oplus 1]\}$ does not intersect $\beta(s_0)$. This implies that $[z_0, i_0] \notin \pm\beta(s_0)$. Then, there exists $\varepsilon > 0$ such that $N(\varepsilon, \{[z_0, i_0]\}) \cap N(\varepsilon, (\pm\beta(s_0)) \cup P) = \emptyset$. Since $p \notin \{z_0, a, b\}$, we have $\{[p, i_0], [p, i_0 \oplus 1]\} \cap \{[z_0, i_0], [z_0, i_0 \oplus 1]\} = \emptyset$, so we can also ask that $N(\varepsilon, \pm p) \cap N(\varepsilon, \pm z_0) = \emptyset$. Let $\delta > 0$ be such that $\delta < \varepsilon$ and satisfies that, if $A, B \in F_2(X)$ and $H(A, B) < \delta$, then $H(f(A), f(B)) < \varepsilon$. We also ask that δ has the property that, if $w, w_0 \in X$ and $d(w, w_0) < \delta$, then $H(\pm w, \pm w_0) < \varepsilon$. Let $\lambda > 0$ be such that if $|t - s| < \lambda$, then $H(\beta(s), \beta(t)) < \delta$.

Choose $s_1 > s_0$ such that $s_1 < s_0 + \lambda$ and $s_1 < 1$. We are going to obtain a contradiction by showing that $s_1 \in L$. Take $w \in \beta(s_1)$. Since $|s_1 - s_0| < \lambda$, there exists $w_0 \in \beta(s_0)$ such that $d(w, w_0) < \delta$. Thus $H(\pm w, \pm w_0) < \varepsilon$ and $H(f(\{[p, i_0], w\}), f(\{[p, i_0], w_0\})) < \varepsilon$. Since $w_0 \in \beta(s_0)$, we have $\pm w_0 \subset \pm\beta(s_0)$ and $f(\{[p, i_0], w_0\}) = \{[z_0, i_0]\}$. Hence $\pm w \subset N(\varepsilon, \pm\beta(s_0))$ and $f(\{[p, i_0], w\}) \subset B(\varepsilon, [z_0, i_0])$. Thus $\pm w \cap N(\varepsilon, \{[z_0, i_0]\}) = \emptyset$. This shows that $\pm w \cap f(\{[p, i_0], w\}) = \emptyset$ and $P \cap f(\{[p, i_0], w\}) = \emptyset$. Since $f(\{[p, i_0], w\}) \subset N(\varepsilon, \pm z_0)$, we have $\pm p \cap f(\{[p, i_0], w\}) = \emptyset$. Let $A_1 = \{[p, i_0], w\}$. We have seen that $\pm A_1 \cap f(A_1) = \emptyset$ and $f(A_1) \cap P = \emptyset$. We can apply Claim 4 and obtain that $f(A) \in F_1(X)$.

We have shown that, for each $w \in \beta(s_1)$, $f(\{[p, i_0], w\}) \in F_1(X)$. If we define $\eta(w) = f(\{[p, i_0], w\})$ we have a continuous function $\eta : \beta(s_1) \to F_1(X)$. Since $\beta(s_1)$ is a subcontinuum of $\pi(C \times \{i_0\})$ and this continuum is homeomorphic to C and $F_1(X)$ is naturally homeomorphic to X, we can apply Claim 2 and obtain that either η is constant or there exists $j \in \{0, 1\}$ such that, for every $w = [u, i_0] \in \beta(s_1)$, $f(\{[p, i_0], w\}) = \{[u, j]\}$. In the second case, since $[a, i_0] \in \beta(s_1)$, we have $f(\{[p, i_0], [a, i_0]\}) = \{[a, j]\}$. But we know that $f(\{[p, i_0], [a, i_0]\}) = f(\{[p, i_0], [b, i_0 \oplus 1]\}) = \{[z_0, i_0]\}$. Thus $[a, j] = [z_0, i_0]$, this implies that $z_0 \in \{a, b\}$, which is contrary to our assumption. With this, we have shown that η is constant. Since $[a, i_0] \in \beta(s_1)$, we have $f(\{[p, i_0], w\}) = \{[z_0, i_0]\}$, for every $w \in \beta(s_1)$. Hence $s_1 \in L$, a contradiction. This completes the proof that $[z_0, i_0] \in \beta(s_0)$ and $f(\{[p, i_0], [z_0, i_0]\}) = \{[z_0, i_0]\}$. □

Let K be as in Claim 9. Since K is dense in C, there exists a sequence $\{p_k\}_{k=1}^{\infty}$ of points of K such that $\lim p_k = z_0$. Since $\lim f(\{[p_k, i_0], [z_0, i_0]\}) = f(\{[z_0, i_0], [z_0, i_0]\})$ and $f(\{[p_k, i_0], [z_0, i_0]\}) = \{[z_0, i_0]\}$, for every $k \geqslant 1$, we have $f(\{[z_0, i_0], [z_0, i_0]\}) = \{[z_0, i_0]\}$. Hence $\{[z_0, i_0]\}$ is a fixed point of f. This contradicts what we are supposing and finishes the proof that $F_2(X)$ has the fixed point property.

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