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On non-abelian C-minimal groups

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Abstract

We investigate the structure of C-minimal valued groups that are not abelian-by-finite. We prove among other things that they are nilpotent-by-finite.

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1. Introduction and preliminaries

1.1. Introduction

Macpherson and Steinhorn introduced *C-minimality* in [5] as a variant of the notion of o-minimality. In a C-minimal structure, a ternary relation, with some specific properties, the C-relation, plays the role analogous to the order in an o-minimal structure: any parameter-definable subset is quantifier-free definable with formulae using just the C-relation and the equality. Less developed than o-minimality for the moment, this notion leads already to some promising results (see [5,2]), applies to expansions of algebraically closed valued fields [4], may have in some ways a development analogous to o-minimality (see [2]). Some of the tools of stability can be developed in this context [1,3]. One of the main interests was to provide a natural setting for studying algebraically closed valued fields and some groups with a chain of uniformly definable normal subgroups with trivial intersection. For example in any valued field (F, v) a

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C-relation can be defined in a natural way by

$$C(x;y,z) \quad \text{iff} \quad v(z-x) < v(z-y)$$

and this relation is preserved by the addition and the multiplication by non-zero elements. The valued field (F,v) is bi-interpretable with the C-field (F,C) obtained by expanding the field structure by the C-relation. If (F,v) is algebraically closed, then, by quantifier elimination results, the definable subsets can be described, modulo a finite set, as finite Boolean combinations of additive cosets of fractional ideals

$$A_\mu = \{x \in F \mid v(x) \geqslant \mu\} \quad \text{and} \quad M_\mu = \{x \in F \mid v(x) > \mu\},$$

where μ belongs to the group of valuations vF. These cosets can be defined by an atomic formula or a negation of an atomic formula using the C-relation and parameters from F: the structure (F,C) is C-minimal. Conversely, the authors of [2] proved that the C-minimal fields are exactly the algebraically closed valued fields. The situation is somehow analogous to the o-minimal context, where o-minimal fields are real closed.

The situation is more complicated for C-minimal groups and they are far less understood than the o-minimal ones: we do not know which groups can be endowed with a C-minimal structure. On the other hand we have many examples of abelian and even non-abelian C-minimal groups. For instance, from results of [8] we have that the additive group of any valued field is C-minimal. In [9] we gave examples of C-minimal groups that are not virtually abelian (i.e. abelian-by-finite). In all these examples the C-relation comes from a valuation, that is a map v from the group G to a chain I with a last element ∞ with no immediate predecessor, satisfying $v^{-1}(\infty)=1_G$, $v(xy^{-1}) \geqslant \min\{v(x),v(y)\}$ and $v(x)<v(y) \rightarrow v(x^z)<v(y^z)$ for every $x,y,z \in G$. The C-relation is defined by

$$C(x;y,z) \quad \text{iff} \quad v(xz^{-1}) < v(yz^{-1}).$$

In [5] some structure theory of C-minimal groups was developed. The authors divide them in three classes. In the first one the C-relation derives from a group valuation in the sense above (in this case we speak of *C-minimal valued groups*), while in the two others it is not the case. Nevertheless, the description given there suggests that C-minimal groups not belonging to the first class somehow mix o-minimal totally ordered groups and C-minimal valued groups. It seems necessary to study the valued case before the two others; also we believe that for many questions the general case can be derived from the valued case without too much work. Moreover we can use some of the familiar machinery about valuations. In a C-minimal valued group $\mathbb{G}=(G,+,v)$ we have a uniformly definable family of normal subgroups:

$$K_\mu = \{x \in G \mid v(x) \geqslant \mu\} \quad \text{and} \quad H_\mu = \{x \in G \mid v(x) > \mu\},$$

where $\mu \neq \infty$ belongs to the chain of valuations I. They play the role of the fractional ideals in C-minimal fields: every definable subset of G is, modulo a finite set, a finite Boolean combinations of cosets of these subgroups. Macpherson and Steinhorn proved also that every proper definable subgroup is a finite union of cosets of one of these groups. We will recall their results in the next section.

In [8] we gave a partial characterization for abelian C-minimal valued groups and many new examples. We found, in that case, that for abelian valued C-groups $\mathbb{G}$ for which the C-relation satisfies some kind of compatibility with the multiplication by any prime number p, being C-minimal is equivalent to the o-minimality of the enriched chain $(I, \leqslant, (f_p)_{p\,\mathrm{prime}}, (R_n)_{n\in\mathbb{N}})$ where for each prime number p, f_p is the map induced on I by the multiplication by p in G, and for each natural number n, R_n is a unary relation defined on I such that $R_n(\mu)$ holds if and only if the quotient K_μ/H_μ has more than n elements.

In [9] we gave the first examples, as far as we know, of C-minimal groups that are not virtually abelian. These groups are nilpotent of class two. This proves that a large class of groups fits the setting of C-minimality. The problem remains of precisely how large is this class.

In the present paper we prove that every C-minimal valued group is virtually nilpotent. We do not have for the moment any example of a C-minimal group that is not virtually nil-2, and that would be the next question to study. There are some constraints: we prove here that the exponent of such a group is finite and that every definable subgroup has a connected component. Moreover there is a finite subset E of the associated chain I of valuations such that $I\backslash E$ is a finite union of dense intervals and the corresponding residual structures, i.e. the K_μ/H_μ's with $\mu\in I\backslash E$, are infinite and of the same cardinality and exponent.

1.2. Notation

Let $(G, ., ^{-1})$ be a group, A and B subsets of G, a and b elements of G and F a subgroup of G.

We denote by $\langle A\rangle$ and by $Z(A)$ respectively the subgroup generated by A and the centralizer of A in G. The conjugate a^b and the commutator of a and b are the elements:

$$a^b = b^{-1}ab \quad \text{and} \quad [a;b] := a^{-1}b^{-1}ab.$$

More generally we define the sets

$$[a;B] := \{[a;y] \mid y\in B\},$$

$$[A;B] := \{[x;y] \mid x\in A,\ \ y\in B\},$$
$$A^B := \{x^y \mid x\in A,\ \ y\in B\}.$$

If F is normal, $Z(a/F) := \{x\in G \mid [x;a]\in F\}$ and $Z(A/F)$ is the subgroup:

$$Z(A/F) := \{x\in G \mid [x;A]\subseteq F\}.$$

Let $A_1,\ldots,A_n$ be n subsets of G and $a_1,\ldots,a_n$ be elements of G. We define as usually the iterated commutators by

$$[a_1;a_2]_2 := [a_1;a_2]$$

and for $m<n$

$$[a_1;a_2;\dots;a_{m+1}]_{m+1} := [[a_1;a_2;\dots;a_m]_m;a_{m+1}].$$

Moreover,

$$[A_1;A_2;\dots;A_n]_n := \{[x_1;x_2;\dots;x_n]_n \mid x_i \in A_i\}.$$

For $a,b\in G$,

$$[a;_n b] := [a;b;b;\dots;b]_{n+1}.$$

If F is a subgroup of G, then $\{\gamma_n(F)\mid n\in\mathbb{N}^*\}$ is the lower central series of F, that is $\gamma_1(F):=F$ and $\gamma_n(F):=\langle[F;F;\dots;F]_n\rangle$.

A group F is said to be nilpotent of class n if $\gamma_{n+1}(F)=\{1\}$. We will say that F is *nil*-2 if it is nilpotent of class 2, that is $\gamma_3(F)=\{1\}$. The group F is said to be an *n-Engel* group if for all $x,y\in F$ we have $[x;_n y]=1$. If $\mathscr{P}$ is some property then we say that G is *virtually* $\mathscr{P}$ if G has a subgroup F of finite index such that F satisfies $\mathscr{P}$. For example *virtually nilpotent* means nilpotent-by-finite.

We will frequently use the following well-known identities:

1.2.1. $[a;xy]=[a;y][a;x]^y=[a;y][a;x][a;x;y]$.
1.2.2. $[xy;a]=[x;a]^y[y;a]=[x;a][x;a;y][y;a]$.

The following lemma is an easy consequence of 1.2.1 and 1.2.2:

Lemma 1.2.3. *Let G be a group, B a subgroup of G and Δ a normal subset of G (i.e. $\Delta^g=\Delta$ for every $g\in G$) containing* 1. *Then for every $a,c\in G$, $c\Delta\subseteq[a;B]$ if and only if $c\in[a;B]$ and $\Delta\subseteq[a;B]$.*

Proof. Suppose that $c\Delta\subseteq[a;B]$. Then clearly c belongs to $[a;B]$ and we can find $b_0\in B$ such that $c=[a;b_0]$. If $x\in\Delta$ then $x^{b_0}\in\Delta\subseteq c^{-1}[a;B]$ and there is $b\in B$ such that $x^{b_0}=[b_0;a][a;b]$, thus $x=[a;bb_0^{-1}]$ belongs to $[a;B]$. Conversely assume that $c=[a;b_0]$ with $b_0\in B$ and $\Delta\subseteq[a;B]$. Since for every $b\in B$, $[a;b_0][a;b]^{b_0}=[a;bb_0]$, we have $[a;b_0]\Delta=[a;b_0]\Delta^{b_0}\subseteq[a;B]$. □

1.3. Valued groups

In [8] we gave to the notion of *valued group* the following meaning: if $(G,\cdot,^{-1},1)$ is a group, and $(I,\leqslant,\infty)$ is a chain with a last element ∞ which has no immediate predecessor, then a *valuation* from G to I is a surjective map $v: G\to I$ satisfying:

(i) $v(x)=\infty$ iff $x=1$,
(ii) $v(x)<v(y)\to v(x^z)<v(y^z)$,
(iii) $v(xy^{-1})\geqslant\min\{v(x),v(y)\}$.

Note that by (iii) and (i), $v(x)=v(x^{-1})$ for every $x\in G$. Moreover, if $v(x)<v(y)$, then $v(x)=v(xyy^{-1})\geqslant\min\{v(xy),v(y)\}\geqslant v(x)$ therefore $v(xy)=v(x)$.

Definition 1.3.1. For each $v \in I \setminus \{\infty\}$, the sets

$$K_v := \{x \in G \mid v(x) \geqslant v\} \quad \text{and} \quad H_v := \{x \in G \mid v(x) > v\}$$

form a chain of subgroups of G such that for $v < v'$, $H_{v'} \subsetneq K_{v'} \subseteq H_v \subsetneq K_v$. Since $v(x) < v(y)$ implies $v(x) < v(y^x)$, H_v is normal in K_v. The quotients K_v/H_v will be called the *residual structures* of the valued group.

Note that G belongs to the chain defined above if and only if I has a first element v_0; then we have $G = K_{v_0}$. If this is not the case, and in order to simplify, we will set $H_{-\infty} := G$ and add $-\infty$ to the chain I as a first element (without successor). The valuation becomes a surjective map from G to $I \setminus \{-\infty\}$. For simplicity we write also $K_\infty = \{1\}$ and $H_\infty = \{1\}$.

Definition 1.3.2. We call Ω_G (or simply Ω) the chain of subgroups

$$\Omega_G := \{K_v \mid v \in I\} \cup \{H_v \mid v \in I\}.$$

Consider now the action of G on itself by conjugation. Clearly, this action induces, by (ii), an order preserving action on the chains Ω and I. For this reason each right coset of an element belonging to Ω is also a left coset of a, maybe different, element of Ω and vice versa. Moreover, the action of G on Ω is trivial if and only if each element of Ω is normal in G. Equivalently, for any x and y,

$$v(x^y) = v(x).$$

In that case, the action of G by conjugation induces an action on each residual structure K_v/H_v. If each of these actions is also trivial, we will say that the valued group is *plain*. It is easy to see that the following conditions are equivalent:

(i) G is plain,
(ii) for every x and y in $G \setminus \{1\}$, $v([x;y]) > \max\{v(x), v(y)\}$,
(iii) for every x in G, $Z(x/H_{v(x)}) = G$.

If G is plain then clearly the residual structures must be abelian. Moreover, for every $\eta \in I_G$ and every $a \in G$ the map

$$\begin{aligned} \sigma_{a,\eta} : Z(a/K_\eta) &\to K_\eta/H_\eta, \\ x &\mapsto [a;x]H_\eta \end{aligned}$$

is a morphism: using 1.2.1 and that G acts trivially on each residual structure we get that for $x, x' \in Z(a/K_\eta)$, $[a;xx']H_\eta = [a;x'][a;x]^{x'}H_\eta = [a;x][a;x']H_\eta$.

In the next section we will be interested in plain valued groups and in particular in plain valued groups of finite exponent. We prove below that such a group is locally finite and locally nilpotent. To prove this we will use the solution of the restricted Burnside Problem (see for example [10]): for every strictly positive integers k and q there is a bound on the orders of finite k-generator groups of exponent dividing q. It follows that there is a bound $c(k,q)$ on the nilpotency class of finite nilpotent

k-generator groups of exponent dividing q, and, if $N(q) := c(2,q)$, that any locally finite and locally nilpotent group H of exponent dividing q is an $N(q)$-Engel group: for all $x, y \in G$, $[x;_{N(q)} y] = 1$.

Lemma 1.3.3. *Let G be a plain valued group of finite exponent q. Then G is locally finite and locally nilpotent and hence an $N(q)$-Engel group.*

Proof. Let F be a subgroup of G generated by the finite subset $E_1 = \{a_0, a_1, \dots, a_{k-1}\}$, and F_n be the quotient $F/\gamma_n(F)$. These groups are nilpotent, generated by the classes of $a_0, a_1, \dots, a_{k-1}$ and of exponent dividing q. It follows that each F_n is finite (see for example Theorem 2.24 in [7]). By the solution of the restricted Burnside Problem their order is bounded. Therefore, the lower central series must be stationary.

Consider, for $n > 0$, the finite set:

$$E_{n+1} := \{[a_{i_0}; a_{i_1}; \dots; a_{i_n}]_{n+1} \mid i_0, \dots, i_n \in k\} = [E_n, E_1].$$

Then, for every $n > 0$, $\gamma_n(F) = \langle E_n^F \rangle$, the group generated by the normal closure of E_n in F (see for example Lemma 2.1.10 in [10]). Let v_n be the first element of $\{v(x) \mid x \in E_n\}$. Since any element $x \in \gamma_n(F)$ can be written $x = \prod_{i \in l} (m_i^{\varepsilon_i})^{g_i}$ with $g_0, \dots, g_{l-1} \in F$, $\varepsilon_0, \dots, \varepsilon_{l-1} \in \{-1, 1\}$ and $m_0, \dots, m_{l-1} \in E_n$, it follows, by the properties of the valuation, that $v(x) \geqslant \min\{v(m_i) \mid i \in l\} \geqslant v_n$. But $E_n \subseteq \gamma_n(F)$, thus $v_n = \min\{v(x) \mid x \in \gamma_n(F)\}$. Moreover, $E_{n+1} = [E_n; E_1]$ and, since G is plain, it follows that if $v_n \neq \infty$ then $v_n < v_{n+1}$. Thus $\gamma_n(F) \neq \{1\}$ implies $\gamma_{n+1}(F) \subsetneq \gamma_n(F)$. The lower central series being stationary we conclude that $\gamma_N(F) = \{1\}$ for some integer N and then F is finite and nilpotent. This proves that G is locally finite and locally nilpotent, and by the remark above, that G is an $N(q)$-Engel group. □

1.4. *C-structures*

We recall now some relevant facts and results about C-structures, C-minimal structures and C-minimal groups. Most of them can be found in [2,5,8].

A *C-structure* is a structure (M, C) where $C(x; y, z)$, the *C-relation*, is a ternary relation satisfying the following axioms:

- $\mathscr{C}_1 : \forall xyz\ (C(x; y, z) \to C(x; z, y))$,
- $\mathscr{C}_2 : \forall xyz\ (C(x; y, z) \to \neg C(y; x, z))$,
- $\mathscr{C}_3 : \forall xyzw\ [C(x; y, z) \to (C(w; y, z) \vee C(x; w, z))]$,
- $\mathscr{C}_4 : \forall xy\ \exists z\ [x \neq y \to (y \neq z \wedge C(x; y, z))]$.

We will also call C-structure any expansion $\mathbb{M} = (M, C, \dots)$ of a structure like above, for instance groups or fields with a C-relation. Such a structure is *C-minimal* if for every elementary extension $\mathbb{M}'$ of $\mathbb{M}$, any parameter-definable subset of M' is quantifier-free definable in (M', C). That is definable with quantifier-free formulae using only the C-relation and the equality, allowing also parameters from M'.

When studying groups or fields endowed with a C-relation one asks for some kind of compatibility with the operations: for instance a *C-group* is a C-structure

$\mathbb{G}=(G,C,\cdot,^{-1},1)$, where $(G,\cdot,^{-1},1)$ is a group, C is a C-relation and $\mathbb{G}$ satisfies furthermore the following axiom:

$$\mathscr{C}_g: \forall xyzuv\ (C(x;y,z) \leftrightarrow C(uxv;uyv,uzv)).$$

Any valued field, with valuation v, can be endowed with a C-relation defined by

$$C(x;y,z) \quad \text{iff} \quad v(z-y) > v(z-x)$$

for which it is a *C-field*: the C relation is compatible in the sense above with the addition and the multiplication by non-zero elements. In [8] we saw that any valued group can be endowed with a structure of a C-group: let $(G,\cdot,^{-1},1)$ be a group, and v a valuation from G to a chain I; then

$$C(x;y,z) \quad \text{iff} \quad v(zy^{-1})>v(zx^{-1}) \quad (\text{iff } v(y^{-1}z)>v(x^{-1}z))$$

defines a C-relation on G, and $\mathbb{G}=(G,C,\cdot,^{-1},1)$ is a C-group. Moreover, $\mathbb{G}$ satisfies the following sentence:

$$\mathscr{C}_v: \forall x\ \neg C(x;x^{-1},1).$$

Conversely, if $\mathbb{G}=(G,C,\cdot,^{-1},1)$ is a non-trivial C-group, that is with at least 2 elements, then the relation $\preccurlyeq$ given by

$$x \preccurlyeq y \quad \text{iff } \neg C(y;x,1)$$

defines a total preorder on G. The quotient of G by the associated equivalence relation is a totally ordered set I, with a last element, the class of 1. This class contains a single element and has no immediate predecessor. The canonical surjection satisfies the axioms (i) and (ii) of valuations. The third property is satisfied iff and only if $\mathbb{G} \models \mathscr{C}_v$. We will call *valued C-group* any C-group whose C-relation can be defined as above from a valuation. The class of valued C-groups is hence axiomatized by the axioms of groups together with the set $\{\mathscr{C}_1,\mathscr{C}_2,\mathscr{C}_3,\mathscr{C}_4,\mathscr{C}_g,\mathscr{C}_v\}$.

Let $\mathscr{F}$ be a C-field. In [2], it was proved that in the C-field $\mathscr{F}$ the C-relation always comes from a non-trivial field valuation v. The additive group of the field together with the C-relation is obviously a valued C-group. The multiplicative group with the same C-relation is a C-group but not a valued C-group: if x has a strictly negative valuation then we have $C(x;x^{-1},1)$. On the other hand, the multiplicative subgroup of elements of valuation 0, that is the invertible elements of the valuation ring, with the induced C-relation, is a valued C-group; the associated valuation is no longer v but may be identified with the map $x \mapsto v(x-1)$.

If $\mathbb{G}=(G,C,\cdot,^{-1},1)$ is a valued C-group, we write I_G and v_G (or simply I and v) for the associated chain and valuation. The elements of the chain of subgroups Ω_G can easily be defined by means of the C-relation: if a is any element of valuation v then

$$K_v = \{x \in G \mid \neg C(x;a,1)\} \quad \text{and} \quad H_v = \{x \in G \mid C(a;x,1)\}.$$

2. The structure of non-abelian C-minimal valued groups

In this section we will prove, among other things, the following results:

Theorem I. *Let $\mathbb{G}$ be a C-minimal valued group. Then G is virtually plain and virtually nilpotent.*

The following theorem gives more information in the case where G is not nil-2-by-finite.

Theorem II. *Let $\mathbb{G}$ be a C-minimal valued group. If G is not nil-2-by-finite then*:

- *G has finite exponent*,
- *I_G is a finite union of points and dense intervals*,
- *every definable subgroup of G is virtually connected. The exponent of the connected component of G is a prime power.*

The only known examples of non-virtually abelian C-minimal valued groups are virtually nil-2 (see below). It is relevant that these examples satisfy also the conclusion of Theorem II. It seems not difficult to construct an example whose exponent is infinite but we do not know any example of a non-virtually abelian C-minimal valued group that is not virtually connected or whose chain of valuations is discrete.

We will keep to the following framework: in (2.1) we summarize the already known results and some direct consequences. We will make extensive use of these results and for most of them we give a proof here in order to be self-contained. We will also recall an example of a C-minimal valued group that is not virtually abelian. This will help us to understand the remainder. In (2.2) we prove that a C-minimal valued group is virtually plain, the first part of Theorem I. This is fundamental: to prove the remaining results we can in fact work under the assumption that $\mathbb{G}$ is plain. In (2.3) we study the behavior of the commutator function on definable subgroups of a plain C-minimal valued group $\mathbb{G}$, especially on definably connected subgroups. This subsection contains essentially all techniques we will use to prove the remaining part of Theorems I and II. In (2.4) we prove that a C-minimal valued group that is not virtually nil-2 has finite exponent and that any C-minimal valued group is virtually an n-Engel group for some integer n. In (2.5) we work under the assumption that G is connected. We prove that any connected C-minimal valued group is nilpotent. If G has finite exponent we prove moreover that this exponent is a prime power. In (2.6) we prove that a C-minimal valued group that is not virtually nil-2 has a connected component. As this component is nilpotent this will finish the proof of Theorem I. We prove also the remaining statements of Theorem II. In the last Section (2.7) we work under the assumption that G is connected, not nil-2, and the chain I_G is dense. The aim is to understand better the structure of C-minimal valued groups that are not nil-2. We prove among other things that all the residual structures are then infinite groups of the same finite exponent p (a prime number) and same cardinality.

2.1. We begin this section by summarizing some known results on non-abelian valued C-minimal groups and direct consequences. Most of them can be found in [5].

In [5] a description of definable subsets of a C-minimal structure was given. They may be described as finite Boolean combinations of sets of the form $\{x \in M \mid C(a; b, x)\}$ or $\{x \in M \mid C(x; a, b)\}$ where $a, b \in M$. Clearly, in a valued C-group $\mathbb{G} = (G, C, \cdot, ^{-1}, 1)$, either these sets or their complements are left cosets (and right cosets) of elements of the chain Ω_G. Then in our setting we get the following description: if $\mathbb{G}$ is C-minimal then any definable subset E of G is a Boolean combination of cosets of elements of Ω_G. More precisely, E can be written as a finite disjoint union of sets of the form

$$(a_0.\Delta_0)\backslash((a_1.\Delta_1) \cup \cdots \cup (a_n.\Delta_n))$$

(or simply $(a_0.\Delta_0)$ if $n = 0$) where $a_0, \dots, a_n$ are elements of G and $\Delta_0, \dots, \Delta_n$ are elements of Ω_G.

As a consequence it was shown in [5].

Fact 2.1.1. *Let $\mathbb{G}$ be a C-minimal valued C-group*:

(i) *The chain $(I, \leqslant)$ is o-minimal in the following sense*: *any $\mathbb{G}$-definable subset of I is a finite union of intervals with endpoints in I.*
(ii) *For $v \in I$, the residual structure K_v/H_v is either finite or strongly minimal*: *for every elementary extension $\mathbb{G}'$ of $\mathbb{G}$, any $\mathbb{G}'$-definable subset of $K_v^{\mathbb{M}'}/H_v^{\mathbb{M}'}$ is finite or cofinite.*

It comes that any infinite residual structure K_v/H_v is elementary abelian or divisible abelian. We also easily deduce:

Lemma 2.1.2. *Let $\mathbb{G}$ be a C-minimal valued C-group and Λ be a definable subchain of Ω.*

(i) *The union of all the elements of Λ is in Ω. If Λ has no last element* (*for the inclusion*) *then this union is an H_v where v has no successor in I.*
(ii) *The intersection of all the elements of Λ is in Ω. If Λ has no first element* (*for the inclusion*) *then this intersection is a K_v where v has no predecessor in I.*

Notation. Let E be a subset of G defined by a formula $\phi(x, \bar{a})$. If E contains 1 then the union of all K_v such that $K_v \subseteq E$ is a non-empty definable subgroup Δ_E that belongs to Ω. Note that Δ_E may be defined by the formula:

$$\forall y\ (\neg C(y; x, 1) \rightarrow \phi(y, \bar{a})).$$

If F is a definable subgroup of G, we will denote by $DFI(F)$ the family of definable subgroups of F of finite index.

The following results are explicitly or implicitly in [5]. Since we will use them very often, we give a proof here.

Lemma 2.1.3. *Let $\mathbb{G}$ be a C-minimal valued C-group, E a definable subset of G and F a definable subgroup of G.*

(i) *E is infinite if and only if it contains a coset of some H_v with $v \neq \infty$.*
(ii) *If E intersects infinitely many cosets of some non-trivial $\Delta \in \Omega$ then it contains infinitely many cosets of Δ.*
(iii) *If for some $\mu \in I \setminus \{\infty\}$, E intersects infinitely many cosets of K_μ then it contains a coset of H_v for some $v < \mu$.*
(iv) *If F intersects infinitely many cosets of H_μ then it contains K_μ.*
(v) *Δ_F is a subgroup of F of finite index. Hence every definable subgroup of G is a finite union of cosets of some element of Ω and every definably connected definable subgroup of G belongs to Ω.*
(vi) *Let Λ be a definable family of subgroups of F, and assume that in every elementary extension of $\mathbb{G}$, $\Lambda \subseteq DFI(F)$. Then the intersection of the elements of Λ belongs to $DFI(F)$ (hence Λ must be finite).*

Proof. These results come from the description of definable subsets. We may assume that $E = D_0 \setminus (D_1 \cup D_2 \cup \cdots \cup D_n)$ where the D_i are cosets of elements of Ω. We allow of course the case where $n = 0$ and $E = D_0$. If $\Delta \in \Omega$, at most one coset of Δ may intersect D_i without being contained in it. This proves (ii). Suppose now that E intersects (hence contains) infinitely many cosets of some K_μ. To prove (i) and (iii) we may assume that $\mu \in I$ has no predecessor (this includes the case $\mu = \infty$) for if v is the predecessor of μ in I then $H_v = K_\mu$. Using a translation we may also assume that E contains K_μ, so D_0 is itself in Ω while $D_1, \ldots, D_n$ are not in Ω. By the assumptions the set $\Sigma = \{v \in I \mid K_\mu \subsetneq H_v \subsetneq D_0\}$ is an infinite interval with no last element and K_μ is the intersection all the H_v with $v \in \Sigma$. For each $i \in \{1, \ldots, n\}$ there is exactly one $v_i \in \Sigma$ such that $D_i \subseteq K_{v_i} \setminus H_{v_i}$. If $v \in \Sigma$ is greater than $v_1, \ldots, v_n$ then H_v must be disjoint from $D_1, \ldots, D_n$ and $H_v \subseteq E$.

(iv) By (ii) we may assume that F intersects finitely many cosets of K_μ. Then $F \cap K_\mu$ contains infinitely many cosets of H_μ, and the result follows from the strong minimality of K_μ / H_μ.

(v) The group Δ_F is the greatest element of Ω contained in F. By (iii) and (iv) F cannot contain infinitely many cosets of Δ_F.

(vi) We may assume that G is ω-saturated. If $\Lambda = \{F_l \mid l \in L\}$ is a definable family of subgroups of finite index of F, then, by (v), the family $\Lambda' = \{\Delta_{F_l} \mid l \in L\}$ is a definable chain of subgroups of finite index of F. By compactness and ω-saturation this chain must be finite. Its smallest element belongs to $DFI(F)$ and is contained in the intersection of the elements of Λ. Alternatively, we can derive this result from the fact that the theory of a C-minimal structure does not have the independence property (see [5,6, Lemma 1.3]). □

If $\phi(x, \bar{y})$ is a formula such that for every tuple $\bar{a}$ the set of realizations of $\phi(x, \bar{a})$ is a subgroup $F_{\bar{a}}$ of G, we can easily deduce from 2.1.3 (v) and by compactness that there is a bound N_ϕ on the indexes of the groups $\Delta_{F_{\bar{a}}}$ in $F_{\bar{a}}$, and we can take for N_ϕ the least common multiple of these indexes. Applying this for instance to the double centralizer:

$$Z(Z(a)) := \{x \in G \mid \forall y \in G\ ([y;a] = 1 \to [y;x] = 1)\}$$

and writing e_G the least common multiple of the indexes of $\Delta_{Z(Z(a))}$ in $Z(Z(a))$, we get that $a^{e_G} \in \Delta_{Z(Z(a))}$ for every $a \in G$. Moreover, since Ω is a totally ordered by inclusion, for every $b \in G$ we have either $\Delta_{Z(Z(a))} \subseteq \Delta_{Z(Z(b))}$ or $\Delta_{Z(Z(b))} \subseteq \Delta_{Z(Z(a))}$. Hence

Corollary 2.1.4. *Let $\mathbb{G}$ be a C-minimal valued C-group. For any $a, b \in G$, either $[a^{e_G}; b] = 1$ or $[a; b^{e_G}] = 1$.*

The subgroup generated by an element $a \in G$ acts by conjugation on the chain Ω. As this action preserves the inclusion, the orbit of $\Delta \in \Omega$ under this action must be trivial or infinite. But if $\Delta \nsubseteq Z(a)$ then $\Delta_{Z(Z(a))} \subseteq \Delta$ and $\Delta^{a^{e_G}} = \Delta$, thus the orbit must be trivial. Therefore

Corollary 2.1.5 (Macpherson and Steinhorn [5]). *Let $\mathbb{G}$ be a C-minimal valued C-group. The elements of the chain Ω are normal subgroups of G.*

The family $\{K_\mu \mid K_\mu$ is abelian$\}$ is definable. The union of the elements of this chain is the definable abelian normal subgroup of G

$$A_G := \{x \in G \mid \forall y \in G\ (\neg C(y;x,1) \to [y;x] = 1)\}.$$

By 2.1.2 A_G belongs to Ω. If F is any definable abelian subgroup of G then $\Delta_F \subseteq A_G$. In particular, for every $a \in G$, $\Delta_{Z(Z(a))} \subseteq A_G$. Hence, the group A_G has the following properties (see also [5]):

Lemma 2.1.6. *For every definable abelian subgroup F of G, $A_G \cap F \in DFI(F)$. The quotient G/A_G has finite exponent which is a divisor of e_G.*

Remark 2.1.7. By Lemma 2.1.3 (v), if Δ is a *connected* definable subgroup of G, i.e. Δ has no proper definable subgroup of finite index, then Δ is in Ω. If F and Δ are two definable subgroups of G, we will say that Δ is a *connected component* of F if $\Delta \in DFI(F)$ and Δ is connected. The group F may not have a connected component, but if such a component exists it is unique and belongs to Ω.

We may characterize the connected subgroups of G: consider the following subchains of I and Ω:

Definition 2.1.8.

$$I_M := \{\eta \in I \mid K_\eta/H_\eta \text{ is infinite}\},$$

$$I_S := \{\eta \in I \backslash \{\infty\} \mid \eta \text{ has no successor}\},$$

$$I_{S'} := \{\eta \in I \mid \exists \eta' > \eta\ (\eta, \eta') \text{ is dense}\},$$

$$I_P := \{\eta \in I \mid \eta \text{ is not the first element of } \Omega \text{ and } \eta \text{ has no predecessor}\},$$

$$\Omega_M := \{K_\eta \mid \eta \in I_M\}, \quad \Omega_S := \{H_\eta \mid \eta \in I_S\},$$

$$\Omega_{S'} := \{H_\eta \mid \eta \in I_{S'}\}, \quad \Omega_P := \{K_\eta \mid \eta \in I_P\}.$$

The sets I_S, $I_{S'}$ and I_P are definable in the chain $(I, \leqslant)$. If $(I, \leqslant)$ is o-minimal then $I_S \backslash I_{S'}$ and $I_P \backslash I_{S'}$ are finite. Moreover, if $I_{S'}$ is non-empty then it is a finite disjoint union of intervals $[\mu_i; \nu_i)$, where $i \in n$ and $-\infty \leqslant \mu_0 < \nu_0 < \cdots < \mu_i < \nu_i < \cdots < \mu_{n-1} < \nu_{n-1} \leqslant \infty$. A subgroup F belongs to Ω_S if and only if it is an H_η that is not a $K_{\eta'}$. A subgroup F belongs to Ω_P if and only if it is a K_η that is not an $H_{\eta'}$. Moreover $\Omega_M \cup \Omega_S$ is the set of definable connected subgroups of G.

We will use also the following characterization of the sets Ω_M, Ω_S, $\Omega_{S'}$ and Ω_P:

Lemma 2.1.9. *Let $\mathbb{G}$ be a C-minimal valued group. Let F be a definable subgroup of G.*

(i) *$F \in \Omega_M$ iff F is connected and there is a definable normal subgroup H of F such that F/H is infinite and strongly minimal.*
(ii) *$F \in \Omega_S$ iff F is connected and is the union of an increasing definable family of proper subgroups.*
(iii) *$F \in \Omega_{S'}$ iff $F = \bigcup_{H \subsetneq F, H \in \Omega_S} H$.*
(iv) *$F \in \Omega_P$ iff F is the intersection of a definable subchain $\Lambda \subseteq \Omega$ such that $F \notin \Lambda$.*

Proof. (iv) follows immediately from 2.1.2. (iii) follows from 2.1.2 and the o-minimality of $(I, \leqslant)$.

(i) and (ii): If $K_\eta \in \Omega_M$ then clearly H_η is a proper definable subgroup of K_η and K_η/H_η is infinite and strongly minimal. If $H_\eta \in \Omega_S$ then H_η is the union of all the K_ν for $\nu > \eta$ and all these groups are proper subgroups of H_η. Assume now for contradiction that F is connected, F is the union of an increasing definable family $(F_i)_{i \in L}$ of proper subgroups and that there is a proper definable normal subgroup H of F such that F/H is infinite and strongly minimal. If for all $i \in L$, $\Delta_{F_i} \subseteq H$, then H has finite index in each group E_i generated by F_i and H. This is not possible since the strongly minimal group F/H cannot be the union of an increasing definable family of finite subgroups. Therefore there is $i \in L$ such that $\Delta_H \subseteq \Delta_{F_i}$. Without loss of generality we can then assume that for all $i \in L$, $H \subseteq F_i$. By strong minimality of F/H the group H has finite index in each F_i. Again we get a contradiction because F/H cannot be the union of an increasing definable family of finite subgroups. □

In [9] we gave the following example of a non-abelian-by-finite C-minimal group. To my knowledge so far such an example is unique:

Example 2.1.10. Given an algebraically closed valued field (F, v) of characteristic $p > 0$, with valued group Γ, valuation ring $A_v = \{x \in F \mid v(x) \geqslant 0\}$ and maximal ideal $M_v = \{x \in F \mid v(x) > 0\}$, and an element $\varepsilon \in M_v$ of strictly positive valuation $\gamma = v(\varepsilon)$, we consider the quotient of A_v by its ideal $\varepsilon^2.A_v = \{x \in F \mid v(x) \geqslant 2\gamma\}$ and we denote its underlying set by G. We define a new operation on G by setting, for any $a, b \in A_v$,

$$(a + \varepsilon^2.A_v) * (b + \varepsilon^2.A_v) = a + b + \varepsilon a^p b + \varepsilon^2.A_v.$$

It is easy to verify that this operation is a group law on G, with unit $0 + \varepsilon^2.A_v$. The inverse of any element $a + \varepsilon^2.A_v$ being $-a + \varepsilon a^{p+1} + \varepsilon^2.A_v$. The commutator of $a + \varepsilon^2.A_v$ and $b + \varepsilon^2.A_v$ is $\varepsilon(a^p b - ab^p) + \varepsilon^2.A_v$. This group becomes a plain valued group for the valuation v' from G to the dense chain $I = [0, 2\gamma) \cup \{\infty\}$ defined by $v'(a + \varepsilon^2.A_v) = v(a)$ if $v(a) < 2\gamma$, and $v'(\varepsilon^2.A_v) = \infty$. For the corresponding C-relation C' we obtain a C-group $\mathbb{G} = (G, C', *, ^{-1}, 0)$ which is interpretable in the C-minimal field $(F, C, +, ., -, 0, 1)$ where C is the natural C-relation on F defined by $C(x; y, z)$ if and only if $v(z - x) < v(z - y)$. It is easy to prove that the group $\mathbb{G}$ is also C-minimal. Moreover $\mathbb{G}$ is a nil-2 and of exponent p if p is odd and 4 if $p = 2$. The group G is not abelian-by-finite. Note also that the chain Ω is exactly the set of connected definable subgroups of $\mathbb{G}$. In particular, all the residual structures are infinite groups of exponent p.

The commutator has a particular behavior in this group. Let Φ be the function defined from $I \times I$ to I by $\Phi(\mu, v) = \gamma + \min\{p\mu + v, pv + \mu\}$ if $\min\{p\mu + v, pv + \mu\} < \gamma$ and $\Phi(\mu, v) = \infty$ otherwise. This function is continuous and increasing in each of its variables (strictly increasing as long as ∞ is not reached). We have the following relations:

$$[K_\mu; K_v] := \{[x; y] \mid x \in K_\mu,\ \ y \in K_v\} = K_{\Phi(\mu,v)},$$

$$[H_\mu; K_v] = [K_\mu; H_v] = [H_\mu; H_v] = H_{\Phi(\mu,v)}.$$

In particular, $[G; G] = K_\gamma = Z(G)$.

What is relevant here is that the set of commutators of two definable connected subgroups of G is a definable connected subgroup. In the next section we will prove that this situation can be generalized in some way to any C-minimal plain valued group.

2.2. Remember that $\mathbb{G}$ is plain if and only if for every $x, y \in G \setminus \{1\}$, $v([x; y]) > \max\{v(x), v(y)\}$. Equivalently the action of G on itself by conjugation induces trivial action on each residual structure, i.e. $Z(a/H_{v(a)}) = G$ for any $a \in G$. We prove the first assertion of Theorem I: every C-minimal valued group is virtually plain.

Theorem 2.2.1. *Let $\mathbb{G}$ be a C-minimal valued group. There is a definable subgroup $F \in \Omega$ which is plain and of finite index in G.*

Proof. We may assume that $\mathbb{G}$ is ω-saturated. The groups $Z(a/H_{v(a)})$ form a uniformly definable family of subgroups of G. Their intersection is a definable subgroup $F = \{x \in G \mid \forall y \in G \setminus \{1\}\ v([x; y]) > v(y)\}$ and Δ_F is plain. If $F \in DFI(G)$ then we are done. By 2.1.3 (vi) we just need to prove the following statement:

$$\text{for every } a \in G, \quad Z(a/H_{v(a)}) \in DFI(G).$$

So we study the action of G by conjugation on some residual structure K_μ/H_μ. Let us introduce some notation: for $a\in K_\mu\backslash H_\mu$ and $\Delta\in\Omega$, a_μ^Δ is the orbit of the coset $a.H_\mu$ under the action of Δ. The stabilizer of $a.H_\mu$ in G is the subgroup $Z(a/H_\mu)$. Let $D(a)\in DFI(Z(a/H_\mu))$ be the normal subgroup $D(a):=\Delta_{Z(a/H_\mu)}$. Clearly a_μ^Δ is finite if and only if $Z(a/H_\mu)\cap\Delta$, and thus $D(a)\cap\Delta$, belong to $DFI(\Delta)$. If a_μ^Δ is not trivial then $D(a)\subsetneq\Delta$. We want to prove that a_μ^G is finite. Assume for contradiction that a_μ^G is infinite. As K_μ/H_μ is finite or strongly minimal, a_μ^Δ is finite or cofinite for every $\Delta\in\Omega$.

Claim 1. If a_μ^Δ is not trivial then either $D(a)\in DFI(\Delta)$ or $\Delta\in DFI(G)$ and $a_\mu^\Delta=a_\mu^G$.

Proof of Claim 1. If $D(a)\notin DFI(\Delta)$, then a_μ^Δ is infinite. Since a_μ^G can be written as a union of Δ-orbits of the same cardinality we must have $a_\mu^\Delta=a_\mu^G$ by strong minimality of K_μ/H_μ. But then each coset of Δ in G contains an element of $Z(a/H_\mu)$. Since $D(a)\in DFI(Z(a/H_\mu))$, the index of Δ in G must be finite. □

By ω-saturation it follows easily that the set $\{\Delta\in\Omega\mid D(a)\subseteq\Delta\}$ must be finite. Moreover, there is $v\in I$ such that $D(a)\in DFI(H_v)$ and $K_v\in DFI(G)$. As $D(a)$ is not of finite index in G we have $v\in I_M$, i.e. K_v/H_v is infinite and strongly minimal. Moreover, $a_\mu^{K_v}=a_\mu^G$ and w.l.o.g. we may assume that $G=K_v$. The group G is then connected.

Claim 2. $G/D(a)$ is abelian. If $b\in K_\mu\backslash H_\mu$ then either $b_\mu^G=a_\mu^G$ or $b_\mu^G=\{b.H_\mu\}$.

Proof of Claim 2. As G/H_v is abelian, the conjugates of any element $x\in G$ belong to the coset $x.H_v$. But $H_v/D(a)$ is finite, and modulo $D(a)$, x has a finite number of conjugates so $Z(x/D(a))\in DFI(G)$. As G is connected, it follows that $Z(x/D(a))=G$ and $G/D(a)$ is abelian. As the orbit a_μ^G is cofinite in K_μ/H_μ, either $b\in a_\mu^G$ or b_μ^G is finite and $Z(b/H_\mu)\in DFI(G)$. As G is connected, b_μ^G is trivial in the second case. □

Let $B:=Z(G/H_\mu)\cap K_\mu$ and $C:=Z(a/B)=\{x\in G\mid [a;x]\in B\}$.

Claim 3. B and C are definable normal subgroups of G. Moreover G/C and K_μ/B are infinite strongly minimal groups and G/C acts regularly on $(K_\mu/B)\backslash\{1\}$ by conjugation.

Proof of Claim 3. B is obviously a normal subgroup of G and we have $H_\mu\subseteq B\subsetneq K_\mu$. In particular, $H_\mu\in DFI(B)$ and K_μ/B is infinite and strongly minimal. As B is normal, C is also a subgroup of G. The group C contains $D(a)$ and is thus normal in G by the second claim. From that claim we also deduce that B/H_μ is the complement of a_μ^G in K_μ/H_μ. It follows easily from all this that G/C acts transitively on $(K_\mu/B)\backslash\{1\}$ and that the stabilizers are trivial. The action of G/C on $(K_\mu/B)\backslash\{1\}$ is then regular and these two sets are in definable bijection which implies that G/C is also strongly minimal. □

The action of G/C on (K_μ/B) allows us to endow K_μ/B with a new operation: for $x,x'\in G$ and $y\in K_\mu/B$ we set $a^xB\times a^{x'}B:=a^{xx'}B$ and $y\times 1=1\times y:=1$. Then it is easy to see that the structure $(K_\mu/B,\cdot,\times,1,aB)$ is an infinite (strongly minimal) field. The group G/C is isomorphic to the multiplicative group of this field. Therefore G/C, is of unbounded exponent. But this contradicts 2.1.6: since the group A_G is abelian and belongs to Ω, we have either $A_G\subseteq H_\mu$ or $K_\mu\subseteq A_G$, and in both cases $A_G\subseteq C$. $\square$

Theorem 2.2.1 implies that every connected C-minimal group is plain. In [5] it was proved that only a finite number of quotients K_η/H_η may not be abelian. Theorem 2.2.1 proves, furthermore, that if K_η/H_η is not abelian then $H_\eta\in DFI(G)$.

2.3. In this subsection we assume that the C-minimal valued group $\mathbb{G}$ is plain. We study the behavior of the commutator function on definable subgroups of G.

Lemma 2.3.1. *Let a be an element of G and B be a definable subgroup of G.*

(i) *The set $[a;B]$ is a finite union of cosets of $\Delta_{[a;B]}$.*
(ii) *Assume that $B\nsubseteq Z(a)$ and let L be the set $L:=\{\Delta\in\Omega\mid\Delta\subseteq B$ and $[a;B]=[a;\Delta]\}$. Then $L\subseteq DFI(B)$ and is finite. If $\Delta\in L$, then $Z(a)$ intersects every coset of Δ in B.*

Proof. As G is plain the residual structures are abelian and for every $\eta\in I$ and every $a\in G$ the map

$$\sigma_{a,\eta}: Z(a/K_\eta)\to K_\eta/H_\eta,$$
$$x\mapsto [a;x]H_\eta$$

is a morphism. Moreover any subset of K_η which is a union of cosets of H_η is normal in G.

(i) By 2.1.2 (i) the union of all the H_v contained in $[a;B]$ is an H_η for some $\eta\geqslant v(a)$. Lemmas 1.2.3 and 2.1.3 (iii) imply that $[a;B]$ is a union of cosets of H_η and intersects a finite number of cosets of K_η. Let A be the set $A:=[a;B]\cap K_\eta$. Then A is a union of cosets of H_η and as $A=\{x\in G\mid xH_\eta\in\sigma_{a,\eta}(B\cap Z(a/K_\eta))\}$, it is a normal subgroup of G. Now, again by 1.2.3, $[a;B]$ is a union of cosets of A and the intersection of $[a;B]$ with any coset of K_η contains at most one coset of A: for if c and d belong to the same coset of K_η and are such that $cA\cup dA\subseteq[a;B]$ then, as $A'=A\cup c^{-1}dA$ is a normal subset of G, we must have $A'\subseteq[a;B]$ thus $cA=dA$. Hence $[a;B]$ is a finite union of cosets of the subgroup A. Note that, since K_η/H_η is finite or strongly minimal, A is either a finite union of cosets of H_η or equal to K_η. Clearly, $\Delta_{[a;B]}=\Delta_A$ and $[a;B]$ is a finite union of cosets of $\Delta_{[a;B]}$.

(ii) If $\Delta\in\Omega$ is a subgroup of B and if $[a;B]=[a;\Delta]$ then B is equal to the group generated by Δ and the centralizer $Z(a)\cap B$ of a in B. Assume that B/Δ is infinite. As $\Delta_{Z(a)}\in DFI(Z(a))$ and Ω is totally ordered by inclusion, $\Delta\subseteq\Delta_{Z(a)}\subseteq Z(a)$ and $[a;B]=\{1\}$, a contradiction. Moreover, the cardinality of $L:=\{\Delta\in\Omega\mid\Delta\subseteq B$ and $[a;B]=[a;\Delta]\}$ is bounded by the index of $\Delta_{Z(a)}$ in $Z(a)$. $\square$

This has interesting consequences when we suppose that B belongs to the chains Ω_M, Ω_S, $\Omega_{S'}$ or Ω_P defined in 2.1.8:

Corollary 2.3.2. *Let a be an element of G and B be a definable connected subgroup of G.*

(i) *$[a;B]$ is a connected subgroup of G. If $B \nsubseteq Z(a)$ then $[a;B] \in \Omega_M$ (resp. $[a;B] \in \Omega_S$, $[a;B] \in \Omega_{S'}$) if and only if $B \in \Omega_M$ (resp. $B \in \Omega_S$, $B \in \Omega_{S'}$).*

(ii) *If $\{1\} \neq B \subseteq [a;G]$ then there is a unique definable connected subgroup B' of G such that $[a;B'] = B$. Moreover B' is the connected component of $Z(a/B)$ and $B' \in \Omega_M$ (resp. $B' \in \Omega_S$, $B' \in \Omega_{S'}$) if and only if $B \in \Omega_M$ (resp. $B \in \Omega_S$, $B \in \Omega_{S'}$).*

Proof. We prove the first parts of (i) and (ii):

(i) If $\Delta \in DFI(\Delta_{[a;B]}) \cap \Omega$ then $Z(a/\Delta) \cap B \in DFI(B)$. If B is connected we conclude that $B \subseteq Z(a/\Delta)$, $[a;B] = \Delta_{[a;B]}$ and $\Delta_{[a;B]}$ is connected.

(ii) Clearly, $[a, Z(a/B)] = B$. If $\Delta \in DFI(Z(a/B)) \cap \Omega$ then it is easy to see that $\Delta_{[a;\Delta]}$ must be of finite index in B, thus equal to B. Therefore $[a;\Delta] = [a, Z(a/B)] = [a; \Delta_{Z(a/B)}]$. By the proof of 2.3.1 (ii) the set $L := \{\Delta \in \Omega \mid [a;\Delta] = B\}$ is finite and has a smallest element B'. Moreover, this subgroup has finite index in $Z(a/B)$ and must be connected. Clearly B' is the unique connected subgroup belonging to L.

To finish the proof it is now sufficient to verify the following:

Claim. Let B and C be two non-trivial definable connected subgroups of G. Assume that $[a,B] = C$. Then

- $B \in \Omega_M$ if and only if $C \in \Omega_M$.
- If $B, C \in \Omega_S$ then $B \in \Omega_{S'}$ if and only if $C \in \Omega_{S'}$.

Proof of the Claim. We use the characterization of 2.1.9.

If $C = K_\nu$ where $\nu \in I_M$, the group $F = B \cap Z(a/H_\nu)$ is a proper normal subgroup of B and $\sigma_{a,\nu}$ induces a definable isomorphism between B/F and K_ν/H_ν which is strongly minimal. This implies that $B \in \Omega_M$.

If $C = H_\nu$ where $\nu \in I_S$ then $H_\nu = \bigcup_{\eta > \nu} K_\eta$ and B is the union of the increasing definable family of proper subgroups $F_\eta = B \cap Z(a/K_\eta)$ for $\eta > \nu$, thus $B \in \Omega_S$.

This proves $B \in \Omega_M$ (resp. Ω_S) if and only if $C \in \Omega_M$ (resp. Ω_S). The second assertion comes easily from 2.1.9 (iii). $\square$

Lemma 2.3.3. *Let a be an element of G and $B \in \Omega_P$. Then*

(i) *$[a;B] \in \Omega_P$.*

(ii) *If $Z(a/B) \notin DFI(G)$ then there is $B' \in \Omega_P$ such that $[a;B'] = B$. If $B \neq \{1\}$ then B' is unique and $B' = \Delta_{Z(a/B)}$.*

Proof. (i) We may assume $[a;B] \neq 1$. Suppose $B = K_\mu$ with $\mu \in I_P$. Then, for each $\eta < \mu$, H_η/K_μ is infinite and $K_\mu = \bigcap_{\eta<\mu} H_\eta$. Clearly, $[a;K_\mu] \subseteq \bigcap_{\eta<\mu} [a;H_\eta]$.

Suppose that $y \in \bigcap_{\eta<\mu} [a;H_\eta] \setminus [a;K_\mu]$. Then, for each $\eta < \mu$, the set $\{x \in H_\eta \setminus K_\mu \mid [a;x] = y\}$ is not empty and we can construct a sequence $(x_n)_{n\in\omega}$ such that for each $n \in \omega$, $v(x_n) < v(x_{n+1}) < \mu$ and $[a;x_n] = y$. For $n \neq n'$, we have $x_n x_{n'}^{-1} \in Z(a) \setminus K_\mu$. Together with

2.1.3 (v) this would imply that $K_\mu \subseteq Z(a)$, contradicting our hypothesis. Therefore $[a;K_\mu] = \bigcap_{\eta<\mu}[a;H_\eta]$. Note that $[a;K_\mu]$ is a proper subset of each $[a;H_\eta]$ by 2.3.1 (ii). The group $\Delta := \bigcap_{\eta<\mu} \Delta_{[a;H_\eta]}$ is a subset of $[a;K_\mu]$, and a proper subset of each $\Delta_{[a;H_\eta]}$: otherwise, there would be some $\eta<\mu$, such that $[a;H_\eta]=[a;K_\mu]$. By 2.1.9, $\Delta \in \Omega_P$. For each $\eta<\mu$, $Z(a/\Delta_{[a;H_\eta]}) \cap H_\eta \in DFI(H_\eta)$, hence contains K_μ. This proves that $[a;K_\mu]=\Delta$ and $[a;K_\mu]\in\Omega_P$.

(ii) Assume now that $Z(a/K_\mu)\notin DFI(G)$. Then $K_\mu \subsetneq \Delta_{[a;G]}$ and, as μ has no predecessor, there is $v<\mu$ such that $H_v \subseteq [a;G]$, and $K_\mu = \bigcap_{v<\eta<\mu} H_\eta$. It is easy to see that $\Delta_{Z(a/K_\mu)} = \bigcap_{v<\eta<\mu} \Delta_{Z(a/H_\eta)}$ and that $\Delta_{Z(a/K_\mu)}$ is a proper subgroup of each $\Delta_{Z(a/H_\eta)}$. By (i) and 2.1.9, $\Delta_{Z(a/K_\mu)}$ and $[a;\Delta_{Z(a/K_\mu)}]$ belong to Ω_P. Now, this last subgroup has finite index in $[a;Z(a/K_\mu)]=K_\mu$ and it is thus equal to K_μ. To prove the unicity, it is sufficient to remark that if $B_1 \subsetneq B_2$ are two elements of Ω_P then the index of B_1 in B_2 is infinite. Thus $[a;B_1]=[a;B_2]$ implies $B_1,B_2\subseteq Z(a)$ by 2.3.1 (ii). □

Remark 2.3.4. We have proved that for every $\mu\in I_P$ and every $a\in G$, $\Delta_{Z(a/K_\mu)}$ belongs either to $DFI(G)$ or to Ω_P. If E is a definable subset of G then $Z(E/K_\mu)=\bigcap_{a\in E} Z(a/K_\mu)$. If for every $a\in E$, $\Delta_{Z(a/K_\mu)}\notin\Omega_P$, this will be true in every elementary extension of $\mathbb{G}$. By 2.1.3 (vi) $Z(E/K_\mu)\in DFI(G)$. If for some $a\in E$, $Z(a/K_\mu)\notin DFI(G)$ then $\Delta_{Z(E/K_\mu)}\in\Omega_P$: clearly $\Delta_{Z(E/K_\mu)}=\bigcap_{a\in E}\Delta_{Z(a/K_\mu)}\in\Omega_P$ by 2.1.2. If for example $\mu=\infty$ then $Z(E/K_\infty)=Z(E)$ and either $Z(E)\in DFI(G)$ or $\Delta_{Z(E)}\in\Omega_P$.

For $\mu\in I_P$ let E_μ be the set $\{a\in G \mid Z(a/K_\mu)\in DFI(G)\}$. This set is definable since $a\in E_\mu$ if and only if $\Delta_{[a;G]}\subseteq K_\mu$. It follows that $Z(E_\mu/K_\mu)\in DFI(G)$ for every $\mu\in I_P$ and hence the group $G_P := \bigcap_{\mu\in I_P} Z(E_\mu/K_\mu)$ belongs to $DFI(G)$ by 2.1.3. We have: for every definable subset E of G and every $\mu\in I_P$, either $G_P\subseteq Z(E/K_\mu)$ or $\Delta_{Z(E/K_\mu)}\in\Omega_P$. For every interval $(v_1,v_2]\subseteq(I_{G_P}\setminus I_P)$, $H_{v_2}\subseteq Z(E/K_\mu)$ implies $K_{v_1}\subseteq Z(E/K_\mu)$.

Proposition 2.3.5. *Let B and C be two definable subgroups of G such that $[B;C]\neq 1$. Then $[B;C]$ intersects finitely many cosets of $\Delta_{[B;C]}$.*

If $C\in\Omega_S\cup\Omega_M\cup\Omega_P$ then $[B;C]\in\Omega$. Moreover:

(1) *If $C\in\Omega_S$ then $[B;C]\in\Omega_S$.*
(2) *If $C\in\Omega_M$ then*
  (i) *either $[B;C]\in\Omega_M$ and there is $b\in B$ such that $[B;C]=[b;C]$*
  (ii) *or $[B;C]\in\Omega_S$ and there is $\Delta\in DFI(B)\cap\Omega_S$ and $c\in C$ such that $[B;C]=[B;c]=[\Delta;c]$.*
(3) *If $C\in\Omega_P$ then*
  (i) *either $[B;C]\in\Omega_P$ and there is $b\in B$ such that $[B;C]=[b;C]$*
  (ii) *or $[B;C]\in\Omega_{S'}$ and there is $\Delta\in DFI(B)\cap\Omega_{S'}$ and $c\in C$ such that $[B;C]=[B;c]=[\Delta;c]$. In this case C has a connected component.*

Proof. For $b\in B$, we have $\Delta_{[b;C]}\subseteq\Delta_{[B;C]}$ hence $C\cap Z(b/\Delta_{[B;C]})$ is of finite index in C. By 2.1.3 (vi) the group $C'=C\cap\bigcap_{b\in B} Z(b/\Delta_{[B;C]})\in DFI(C)$. For the same reason, $B'=B\cap\bigcap_{c\in C} Z(c/\Delta_{[B;C]})\in DFI(B)$. Then $[B',C]\subseteq\Delta_{[B;C]}$ and $[B,C']\subseteq\Delta_{[B;C]}$. Using the identities on commutators we find that $[B;C]$ intersects finitely many cosets of $\Delta_{[B;C]}$.

Assume now that $C\in\Omega_S\cup\Omega_M\cup\Omega_P$. As $[B;C]=\bigcup_{b\in B}[b;C]$, the set $[B;C]$ is, by Lemmas 2.3.2 and 2.3.3, the union of a definable family of elements of Ω and by 2.1.2 (i) belongs to Ω. Note that $[B;C]\notin\Omega_S$ implies $[B;C]=[b;C]$ for some $b\in B$. By 2.3.2 and 2.3.3 we just have to consider the cases where $C\in\Omega_M\cup\Omega_P$ and $[B;C]=H_\eta\in\Omega_S$. Clearly $[B;C]=\bigcup_{\eta'>\eta}K_{\eta'}$.

Assume first that $C\in\Omega_M$ and suppose for contradiction that for every $c\in C$, $[B;c]\subsetneq[B;C]$. Then, for every $c\in C$, $\Delta_{[B;c]}$ is a proper subgroup of H_η and there is $\eta'>\eta$ such that $[B;c]\subseteq K_{\eta'}$. It follows that $C=\bigcup_{\eta'>\eta}\{x\in C\mid[B;x]\subseteq K_{\eta'}\}$: this means that C is the union of an increasing definable family of proper subgroups, which is not possible by 2.1.9. Hence $[B;c]=H_\eta$ for some $c\in C$ and B has a connected component $\Delta\in\Omega_S$ such that $[\Delta;c]=H_\eta$ by 2.3.2.

Assume now that $C\in\Omega_P$. Since $H_\eta=\bigcup_{b\in B}[b;C]$ is a union of elements of Ω_P it follows, by o-minimality, that $\eta\in I_{S'}$. Moreover, for some $b\in B$ and some $\eta'\in S'$, $[b;C]=K_{\eta'}$. This implies, from 2.3.2, that C has a connected component C_0 that belongs either to Ω_M or to $\Omega_{S'}$ according to whether $K_{\eta'}$ belongs to Ω_M or not. Write C as a finite disjoint union $C_0\cup C_0.c_1\cup C_0.c_1\cup\cdots\cup C_0.c_{n-1}$ of cosets of C_0. Using the identities on commutators and that $[B;C_0]\in\Omega$ is normal, we get that for $i\in n$, $[B;C_0.c_i]\subseteq[B;c_i].[B;C_0]$. Suppose for contradiction that for every $c\in C$, $[B;c]\subsetneq[B;C]$. Then for every $i\in n$ there is $\eta_i>\eta$ such that $[B;c_i]\subseteq K_{\eta_i}$. As $[B;C_0]\in\Omega$ we must have $[B;C]=[B;C_0]$. Moreover, by the case above, we have $C_0\in\Omega_S$. Since $H_\eta=\bigcup_{b\in B}[b;C_0]$, we can find some $b\in B$ such that $[B;c_i]\subseteq K_{\eta_i}\subseteq[b;C_0]$ for each $i\in n$. But then $[b;C]=[b;C_0]\cup[b;c_1].[b;C_0]\cup[b;c_2].[b;C_0]\cup\cdots\cup[b;c_n].[b;C_0]=[b;C_0]\in\Omega_S$, which is impossible since $[b;C]\in\Omega_P$. Hence $[B;c]=H_\eta$ for some $c\in C$ and B has a connected component $\Delta\in\Omega_{S'}$ such that $[\Delta;c]=H_\eta$ by 2.3.2. □

We easily deduce from 2.3.5 the following corollary. The proof is left to the reader.

Corollary 2.3.6. *Let B and C be two elements of $\Omega_M\cup\Omega_P$ such that $[B;C]\neq 1$. Then there is $b\in B$ and $c\in C$ such that $[B;C]=[B;c]=[b;C]$. It follows*:

(1) *$B,C\in\Omega_M$ implies $[B;C]\in\Omega_M$.*
(2) *$B,C\in\Omega_P$ implies $[B;C]\in\Omega_P$.*
(3) *$B\in\Omega_M$ and $C\in\Omega_P$ implies $[B;C]\in\Omega_P\cap\Omega_M$ and C has a connected component that belongs to Ω_M.*

2.4. We prove here that a C-minimal valued group $\mathbb{G}$ that is not virtually nil-2 has finite exponent (this is the first assertion of Theorem II). From 1.3.3 it will follow that $\mathbb{G}$ is virtually an N-Engel group for some integer N.

Proposition 2.4.1. *Let $\mathbb{G}$ be a C-minimal valued group. There is a subgroup $F\in DFI(G)$ such that $\gamma_2(F)$ is of finite exponent. If G is not of finite exponent then F is nil-2.*

Proof. We may assume that $\mathbb{G}$ is plain.

Remember that $e = e_G$ is the least common multiple of the indexes of $\Delta_{Z(Z(a))}$ in $Z(Z(a))$ for $a \in G$. Set $G_1 := \bigcap_{a\in G} Z(a/\Delta_{[a;G]})$. By 2.3.1 (i) each $Z(a/\Delta_{[a;G]})$ has finite index in G (and the same will be true in every elementary extension of $\mathbb{G}$). By 2.1.3 (vi), $G_1 \in DFI(G)$. For any $a \in G$, $[a; G_1] \subseteq \Delta_{[a;G]}$. Let $c = [a; b]$ with $a, b \in G_1$. Thanks to 2.1.4 we have either $[a; b^e] = 1$ or $[a^e; b] = 1$. Using the identities on commutators and by induction, $c^e = [a; b]^e$ belongs to $\Delta_{[c;G]}$. Then $u^{-1}cu = c^{e+1}$ for some $u \in G$. Again $[c; u^e] = 1$ or $[c^e; u] = 1$. It follows that either $c = c^{(e+1)e}$ or $c^e = c^{e(e+1)}$. Hence the exponent of the group $\Delta_{[G_1;G_1]}$ is finite. By 2.3.5, for $a \in G_1$, the group $G_1 \cap Z(a/\Delta_{[G_1;G_1]})$ belongs to $DFI(G_1)$ and 2.1.3 (vi) implies that $F := G_1 \cap \bigcap_{a\in G_1} Z(a/\Delta_{[G_1;G_1]})$ also belongs to $DFI(G_1)$. Then $[F; F] \subseteq \Delta_{[G_1;G_1]}$ and $\gamma_2(F)$ has finite exponent. Moreover $F/\Delta_{[G_1;G_1]}$ is abelian. Let L be the normal subgroup $L := \{x \in F \mid x^e \in \Delta_{[G_1;G_1]}\}$. If $a \in F\backslash L$ then, as $a^e \in \Delta_{Z(Z(a))}$, we have $\Delta_{[G_1;G_1]} \subseteq \Delta_{Z(Z(a))}$ and $a \in Z(\Delta_{[G_1;G_1]})$. It follows that $Z(\Delta_{[G_1;G_1]})$ contains every non-trivial coset of L. We have two cases: either $F = L$ and hence G has finite exponent, or $Z(\Delta_{[G_1;G_1]}) = F$ and F is nil-2. □

In 1.3.3 we proved that any plain valued group of finite exponent is an N-Engel group for some integer N. As any nilpotent group of class n is obviously n-Engel we easily deduce:

Corollary 2.4.2. *Let $\mathbb{G}$ be a C-minimal valued group. There is $F \in DFI(G)$ which is an N-Engel group for some integer N.*

2.5. From the last subsection we deduce that any connected subgroup of a C-minimal valued group $\mathbb{G}$ is an N-Engel group for some integer N. We will use this to prove that any connected C-minimal valued group $\mathbb{G}$ is nilpotent. If $\mathbb{G}$ has finite exponent we will also prove that this exponent is a prime power. Proposition 2.3.5 shows that if B belongs to $\Omega_S \cup \Omega_M$ then we can define a map $C \mapsto [B; C]$ from $\Omega_S \cup \Omega_M$ to $\Omega_S \cup \Omega_M \cup \{1\}$. The following Lemma proves that this map has no fixed points.

Lemma 2.5.1. *Let B and C be two non-trivial definable connected subgroups of G. Then $[B; C]$ is a proper* (*connected*) *subgroup of $B \cap C$.*

Proof. The groups B and C belong to Ω and we may assume that $C \subseteq B$. As B is a connected C-minimal valued group, B is plain and an N-Engel group for some integer N. By 2.3.5 $[B; C]$ is a connected subgroup of C and we want to prove that $[B; C] \neq C$. This is clear if $C = K_\mu \in \Omega_M$: as B is plain, for any $b \in B$ and $c \in C$ we have $v([b; c]) > \mu$, hence $[B; C] \subseteq H_\mu$. Assume now that $C = H_\mu \in \Omega_S$ and, for contradiction, that $[B; C] = C$. Then for each $x \in B$ such that $C \nsubseteq Z(x)$, $[x; C] \in \Omega_S$. The N-Engel condition implies that $[C;_N x] = 1$ therefore $[x; C] \subsetneq C$. As $C = \bigcup_{x\in B} [x; C]$, we have $C \in \Omega_{S'}$ by o-minimality of I: there is some $\lambda \in I$ such that $\lambda > \mu$ and the interval (μ, λ) is densely ordered. Since $[C;_N B; B; \ldots; B] = C$ there are $z \in C$ and $y_1, \ldots, y_N \in B$ such that $\mu < v([z, y_1, \ldots, y_N]_{N+1}) < \lambda$. Write $z_0 = z$, and, for $0 < n \leqslant N$, $z_n = [z; y_1; \ldots; y_n]_{n+1}$. We have $\mu < v(z_0) < v(z_1) < \cdots < v(z_N) < \lambda$, and $K_{v(z_n)} \in \Omega_P$ for $n \in N + 1$. The group $F_n = \{x \in B \mid [K_{v(z_{n-1})}; x] \subseteq H_{v(z_n)}\}$ is a proper definable subgroup of the connected group B. By 2.3.3, we see that if $x \in B \setminus F_n$ then $K_{v(z_n)} \subseteq [K_{v(z_{n-1})}, x]$. The set $\bigcup_{0<n\leqslant N} F_n$ intersects finitely many cosets of the group $\bigcup_{0<n\leqslant N} \Delta_{F_n}$ hence the set $B \setminus \bigcup_{0<n\leqslant N} F_n$ is not empty. If a belongs to this set then $z_N \in [K_{v(z_0)};_N a] = \{1\}$: a contradiction. □

Proposition 2.5.2. *If $\mathbb{G}$ is connected then it is nilpotent.*

Proof. We know that $\mathbb{G}$ is an N-Engel group for some integer N. By 2.5.1, and induction, for every integer $n>0$, $\gamma_{n+1}(G) = [\gamma_n(G); G]$ is a connected definable proper subgroup of G and belongs to Ω. If $\gamma_n(G)$ is not trivial then $\gamma_{n+1}(G) \subsetneq \gamma_n(G)$. Assume that $\gamma_{2N}(G) \neq \{1\}$. Then, for $0<n\leqslant 2N$, $\gamma_{n+1}(G) \subsetneq \gamma_n(G)$. For $0<n\leqslant N$ the set $F_n = \{x \in G \mid [\gamma_{2n-1}(G), x] \subseteq \gamma_{2n+1}(G)\}$ is a proper definable subgroup of G. If $x \in G \setminus F_n$ then $\gamma_{2n+1}(G) \subseteq [\gamma_{2n-1}(G), x]$ since these two subgroups are in Ω. The set $\bigcup_{0<n\leqslant N} F_n$ intersects finitely many cosets of the group $\bigcup_{0<n\leqslant N} \Delta_{F_n}$ hence the set $G \setminus \bigcup_{0<n\leqslant N} F_n$ is not empty. If a belongs to this set then $\gamma_{2N+1}(G) \subseteq [G;_N a] = \{1\}$ thus G is nilpotent of class at most $2N$. □

Assume that G is connected. Then the derived subgroup of G is equal to $[G;G]$ and is definable. For $p \in \mathbb{P}$, the set of prime numbers, define

$$\phi_p(G) := G^p.[G;G] = \{x^p.[G;G] \mid x \in G\}.$$

$\phi_p(G)$ is a definable normal subgroup of G: the p-Frattini subgroup of G (see [11]). Moreover, $\phi_p(G)$ is connected: if $F \in DFI(\phi_p(G))$ then, as $[G;G]$ is connected, we have $[G;G] \subseteq F$ and the set $\{x \in G \mid x^p \in F\}$ is a definable subgroup of G of finite index. Thus $\phi_p(G) \in \Omega$. If $\phi_p(G)$ is a proper subgroup of G then for any $n \in I \setminus \{\infty\}$ such that $\phi_p(G) \subseteq H_\eta$ we have that K_η/H_η is a group of exponent p. It follows that, for $x \in G \setminus \phi_p(G)$, $v(px) > v(x)$, and if $q \in \mathbb{P} \setminus \{p\}$ then $v(qx) = v(x)$ hence $\phi_q(G) = G$. Since G is nilpotent, if G has finite exponent then the exponent of $G/[G;G]$ and consequently that of G (see for example [11] Lemma 3.13) is a power of p. If the exponent of G is infinite, then G is nil-2 and for $e = e_G$ and $a, b \in G$ we have $[a^e; b] = [a; b^e] = [a;b]^e = 1$. It follows that e is a power of p like the exponents of $G/Z(G)$ and of $[G;G]$. We have proved the following:

Theorem 2.5.3. *Let $\mathbb{G}$ be a connected C-minimal valued group. Then $G/Z(G)$ and $[G;G]$ have finite exponent, and this exponent is a prime power. If G is of finite exponent, then this exponent is a prime power.*

2.6. The main result of this subsection is that a C-minimal valued group that is not virtually nil-2 has a connected component. Together with 2.5.2 this will prove that every C-minimal valued group is virtually nilpotent and hence conclude the proof of Theorem I.

By Remark 2.3.4 there is $G_P \in DFI(G) \cap \Omega_G$ such that G_P is plain, and for every $a \in G_P$ and $\mu \in I_P$, either $G_P \subseteq Z(a/K_\mu)$ or $\Delta_{Z(a/K_\mu)} \in \Omega_P$ and $[a; \Delta_{Z(a/K_\mu)}] = K_\mu$. Note that, with the induced C-relation, G_P is a C-minimal valued group whose chain I_{G_P} is a cofinite final segment of I_G.

It follows that, if $I_P=\{\infty\}$, then the centralizer of every element is either finite or contains G_P. But 2.4.2 implies that the centralizer of every element is infinite, thus G_P is abelian and G is abelian-by-finite.

We assume that $I_P\neq\{\infty\}$ and let α and β be respectively the greatest lower bound of I_P and the least upper bound of $I_P\backslash\{\infty\}$. We consider the subgroups F of G_P having the following property:

(A) There is η_1, η_2 in I such that $\eta_1\in I_{S'}$, $\eta_1\geqslant\eta_2$, and either $H_{\eta_1}\in DFI(F)$, or $H_{\eta_1}\in DFI(H_{\eta_2})$ and $K_{\eta_2}\in DFI(F)$.

We have:

- if $\eta\in I_{S'}$ then K_η and H_η satisfy (A).
- If F satisfies (A), then obviously F has a connected component, which is either H_{η_1} or K_{η_2} if this group belongs to Ω_M and $K_{\eta_2}\in DFI(F)$.
- If $F'\in DFI(F)$ then F satisfies (A) if and only if F' satisfies (A).
- If F and F' are two infinite definable subgroups of G and if for some $a\in G$, $[a;F]=F'$ then F satisfies (A) if and only if F' satisfies (A) (this follows easily from Corollary 2.3.2).

Lemma 2.6.1. *Suppose $F\in\Omega_{G_P}$ and $a\in G_P$.*

(i) *If F does not satisfy (A), then $K_\alpha\subseteq Z(F)$.*
(ii) *If $a\in Z(K_\alpha)$ then $[a;G_P]\subseteq K_\beta$.*
(iii) *If $a\in Z(Z(K_\alpha))$ and $F\neq\{1\}$ does not satisfy (A) then $\Delta_{[a;G_P]}\subsetneq F$.*

Proof. (i) By 2.3.4 $K_\alpha\subseteq Z(F)$ if and only if $H_\alpha\subseteq Z(F)$ if and only if $[F;K_\mu]=1$ for every $\mu\in I_P$. Assume for contradiction that $[F;K_\mu]\neq 1$ for some $\mu\in I_P$. By 2.3.5, $[F;K_\mu]=K_\nu$ with $\nu\in I_P\backslash\{\infty\}$. If for every $a\in K_\mu$, we have $Z(a/H_\nu)\cap F\in DFI(F)$ then, by 2.1.3 (vi), $F':=\Delta_{Z(K_\mu/H_\nu)\cap F}\in DFI(F)$. But then, $[F';K_\mu]=K_{\nu'}$ with $\nu'\in I_P$ and $\nu'>\nu$. By 2.3.4 $F\subseteq Z(K_\mu/K_{\nu'})$, a contradiction. It follows that, for some $a\in K_\mu$, $Z(a/H_\nu)\cap F\notin DFI(F)$. Then $\nu\in I_M$ and $[a;F]=K_\nu$. By 2.3.2 F has a connected component K_η with $\eta\in I_M$. By 2.3.1 (ii) we have $Z(b/H_\nu)\cap H_\eta\in DFI(H_\eta)$ for every $b\in K_\mu$ and by 2.1.3 (vi), $F':=\Delta_{Z(K_\mu/H_\nu)\cap H_\eta}\in DFI(H_\eta)$. Using that F does not satisfy (A) and 2.3.5 we get as before $[F';K_\mu]=K_{\nu'}$ with $\nu'\in I_P$ and $\nu'\neq\nu$ and this leads again to a contradiction, because it follows easily, by 2.3.4, that H_η, K_η and finally F are subgroups of $Z(K_\mu/K_{\nu'})$.

(ii) We just need to prove that $G_P\subseteq Z(a/K_\mu)$ for every $\mu\in I_P\backslash\{\infty\}$. But this follows from 2.3.3 and 2.3.4 since we cannot have $\Delta_{Z(a/K_\mu)}\in\Omega_P$ and $[a;\Delta_{Z(a/K_\mu)}]\neq 1$.

(iii) Assume for contradiction that $F\subseteq[a;G_P]$. Then $[a;Z(a/F)]=F$. Now $Z(a/F)$ does not satisfy (A) and $\Delta_{Z(a/F)}\subseteq Z(K_\alpha)$ by (i). This implies $F=\{1\}$. □

We distinguish two cases according to whether G_P satisfies (A) or not:

First case: Assume that G_P does not satisfy (A). Then Lemma 2.6.1 (i) and (ii) imply that $K_\alpha\subseteq Z(G_P)$ and $[G_P;G_P]\subseteq K_\beta$. Clearly G_P is nil-2 and G is virtually nil-2. Note that, if G_P is not abelian, then $K_\alpha\in DFI(Z(G_P))$ and $\alpha\in I_P$ by 2.3.4.

Second case: Assume that G_P satisfies (A). Then G_P (and a fortiori G) has a connected component G°. In this case, we have $\alpha \in I_{S'} \backslash I_P$. Moreover, if $H_\alpha \in DFI(G_P)$ then $G^\circ = H_\alpha$, and if $H_\alpha \notin DFI(G_P)$ then for some $\eta \in I_M$, $G^\circ = K_\eta$ and $H_\alpha \in DFI(H_\eta)$. In both cases $Z(G_P) = Z(K_\alpha)$ by 2.3.4. If G_P is not abelian then $\Delta_{Z(G_P)} \in \Omega_P$. Assume that some $F \in \Omega$ does not satisfy (A). Then Lemma 2.6.1 implies $F \subseteq \Delta_{Z(G_P)}$ and for every $a \in G_P$, $\Delta_{[a;G_P]} \subseteq F \subseteq \Delta_{Z(G_P)}$. It follows that $G_P \subseteq Z(a/\Delta_{Z(G_P)})$ by 2.3.4 and $[G_P; G_P] \subseteq Z(G_P)$. The group G_P is nil-2.

We can finish now the proofs of Theorem I and II:

Proof of Theorem I. The first part is 2.2.1. By what we have just seen, if G is not virtually nil-2 then it has a connected component which is nilpotent by 2.5.2 $\square$

Proof of Theorem II. If G is not nil-2-by-finite then G_P is not nil-2. It follows that every definable subgroup of G satisfies property (A) and has a connected component. Assume for contradiction that $I \backslash I_{S'}$ is infinite. Then it contains by o-minimality an infinite interval $[v, \mu)$. Obviously K_v does not satisfy (A): a contradiction. The other assertions follow directly from 2.4.1 and 2.5.3. $\square$

The following problems arise naturally:

- construct a C-minimal group that is not virtually nil-2;
- is there a C-minimal valued group that is not virtually abelian and does not satisfy property (A)?

2.7. In this last subsection we want to give more information about the structure of a C-minimal valued group $\mathbb{G}$ that is not virtually nil-2, if such a group exists. From Theorem II we know that the chain I is not far from being dense. To avoid being too tedious we will assume that G is connected (hence plain) and that the chain I is dense. This is not far from the general case, and from what follows it is indeed not difficult to obtain a general result, which we will state without proof among some final remarks.

If I is dense then every element $\mu \in I$ that is not the first element of I belongs to I_P. In the case where I has a first element $\alpha \neq -\infty$, that is $G = K_\alpha$, then the connectedness of G implies $\alpha \in I_M$. Hence $I = I_M \cup I_P$. By 2.3.6 the commutator of two elements of $\Omega_M \cup \Omega_P$ also belongs to $\Omega_M \cup \Omega_P$. It follows that we have a definable symmetric function Φ from $I \times I$ to I such that for every $\mu, v, \eta, \in I$,

$$\Phi(\mu, v) = \eta \quad \text{if and only if} \quad [K_\mu; K_v] = K_\eta.$$

This function has the following properties:

Lemma 2.7.1. *Fix μ and v in $I \backslash \{\infty\}$.*

(i) $[K_\mu; H_v] = [H_\mu; K_v] = H_{\Phi(\mu,v)}$,
(ii) $\Phi(\mu, v) > \max\{\mu, v\}$.

(iii) *The partial map $\Phi(\mu,.)$ is increasing, strictly increasing on $\{\xi \in I \mid \Phi(\mu,\xi) \neq 1\}$ and satisfies the intermediate value property.*

Proof. The proof of (i) and (ii) is trivial if $\Phi(\mu,v)=\infty$ so we may assume $\Phi(\mu,v)\neq\infty$. (i) By the hypothesis made on I we have $v\in I_S$ and thus $[K_\mu;H_v]=H_\eta$ with $\eta\in I_S\cup\{\infty\}$ by 2.3.5. Clearly η is not the first element of I, hence $\eta\in I_P$ and 2.3.4 implies that either $\Delta_{Z(K_\mu/K_\eta)}$ is equal to G or $\Delta_{Z(K_\mu/K_\eta)}$ belongs to Ω_P. It follows that $H_\eta \subsetneq [K_\mu;K_v]\subseteq K_\eta$ and $\Phi(\mu,v)=\eta$.

(ii) This is a consequence of the plainness of G: for every $x,y\in G\backslash\{1\}$, $v([x;y])>\max\{v(x),v(y)\}$.

(iii) The first part is a direct consequence of (i). To prove that $\Phi(\mu,.)$ satisfies the intermediate value property we take $\eta>\Phi(\mu,v)$ and we show that there exists $\xi\in I$ such that $[K_\mu;K_\xi]=K_\eta$. We may assume $\eta\neq\infty$. We distinguish two cases:

First case: $K_\mu\notin\Omega_M$. As in the proof of 2.3.5 we take a set $\{a_0,a_1,\dots,a_{n-1}\}$ of representatives of the cosets of H_μ in K_μ. Fix any $K\in\Omega_M\cup\Omega_P$. Since $[H_\mu;K]$ and each $[a_i;K]$ belong to Ω we get, using the identities on commutators, that $[K_\mu;K]=[H_\mu;K]\cup[a_0;K]\cup[a_1;K]\cup\cdots\cup[a_{n-1};K]$. As $[H_\mu;K]\in\Omega_S\cup\{1\}$ and $[K_\mu;K]\notin\Omega_S$ we deduce that $[K_\mu;K]=[a_0;K]\cup[a_1;K]\cup\cdots\cup[a_{n-1};K]=[a_i;K]$ for some $i\in n$. Since $[K_\mu;G]\nsubseteq K_\eta$, for at least one $i\in n$ we have $Z(a_i/K_\eta)\neq G$ and in this case $\Delta_{Z(a_i/K_\eta)}\in\Omega_P$ and $[a_i;\Delta_{Z(a_i/K_\eta)}]=K_\eta$ by 2.3.3 and 2.3.4. It follows that $\Delta_{Z(a_1/K_\eta)}\cap\cdots\cap\Delta_{Z(a_{n-1}/K_\eta)}=K_\xi\in\Omega_P$ and $[K_\mu;K_\xi]=K_\eta$.

Second case: $K_\mu\in\Omega_M$. Consider the following definable subset of I:

$$L:=\{\xi\in I\mid \exists x\in K_\mu,[x;K_\xi]=K_\eta\}.$$

The set L is not empty: for every $x\in K_\mu$, $[x;K_v]$ belongs to Ω and for at least one of them $K_\eta\subseteq[x;K_v]$; apply then 2.3.4. For each $\xi\in L$, $F_\xi:=\{x\in K_\mu\mid[x;K_\xi]\subseteq K_\eta\}$ is a definable subgroup of K_μ. Clearly, if $\xi<\xi'$ then $F_\xi\subseteq F_{\xi'}$. Moreover, $K_\mu=\bigcup_{\xi\in L}F_\xi$: if $x\in K_\mu$ then either $[x;K_\xi]\subseteq K_\eta$ for every $\xi\in L$, or there is $\xi\in L$ such that $[x;K_\xi]=K_\eta$ by 2.3.3. By 2.1.9, for some $\xi\in L$, $K_\mu=F_\xi$. Therefore $[K_\mu;K_\xi]\subseteq K_\eta$. Since there is $x\in K_\mu$ such that $[x;K_\xi]=K_\eta$, we have $[K_\mu;K_\xi]=K_\eta$. □

From 2.7.1 (i) and 2.3.1 (i) it follows that, for every $\mu,v\in I$ such that $\Phi(\mu,v)\neq\infty$, we can define a function $\delta_{\mu,v}$ from $K_\mu/H_\mu\times K_v/H_v$ to $K_{\Phi(\mu,v)}/H_{\Phi(\mu,v)}$ with

$$\delta_{\mu,v}(xH_\mu,yH_v)=[x;y]H_{\Phi(\mu,v)}.$$

This function is bilinear: For $a,a'\in K_\mu/H_\mu$ and $b,b'\in K_v/H_v$,

$$\delta_{\mu,v}(a\cdot a',b)=\delta_{\mu,v}(a,b)\cdot\delta_{\mu,v}(a',b)$$

and

$$\delta_{\mu,v}(a,b\cdot b')=\delta_{\mu,v}(a,b)\cdot\delta_{\mu,v}(a,b').$$

We show now Proposition 2.7.2 under the assumptions made above.

Proposition 2.7.2. *All the residual structures of* $\mathbb{G}$ *are infinite and of the same cardinality.*

Proof. To prove the first part we may assume that $\mathbb{G}$ is ω-saturated. We begin by proving that I_M is not empty. Assume the contrary. Then the cardinality of the residual structures is bounded. We claim that we can find two infinite intervals $[\mu,\nu]$ and $[\mu',\nu']$ of I such that:

- $\Phi(\mu,\mu)=\mu'$ and $\Phi(\nu,\nu)=\nu'$,
- there are integers $n,m>0$ such that $|K_\eta/H_\eta|=n$ for every $\eta\in[\mu,\nu]$ and $|K_\eta/H_\eta|=m$ for $\eta\in[\mu',\nu']$,
- the restriction of Φ to $[\mu,\nu]\times[\mu,\nu]$ strictly increasing and continuous in each variable.

This easily follows from o-minimality of I and Proposition 2.7.1, so we leave it to the reader (to prove the two first items consider for example the map $\eta\mapsto\Phi(\eta,\eta)$, the last one follows then immediately from 2.7.1 (iii)). Fix $\alpha,\beta\in[\mu,\nu]$, and a set $a_0,\dots,a_{n-1}\in K_\alpha$ of representatives of the cosets of H_α in K_α. If $\eta=\Phi(\alpha,\beta)$ then, by the proof of 2.7.1 (iii), there is $i\in n$ such that $K_\eta=[a_i;K_\beta]$. The map $x\mapsto\delta_{\alpha,\beta}(a_i,x)$ induces a morphism from K_β/H_β onto K_η/H_η. As H_β, $[a_i;H_\beta]$ and H_η are connected we see that $[a_i;H_\beta]=H_\eta$ and the kernel of this morphism is the projection of $Z(a_i)\cap K_\beta$ by 2.3.1. Taking $\alpha=\beta$ we get $m=|K_\eta/H_\eta|<|K_\alpha/H_\alpha|=n$. But this means that for every $\beta\in[\mu,\nu]$, and every $i\in n$, $Z(a_i)\cap(K_\beta\backslash H_\beta)$ is not empty. This implies $K_\beta\subseteq Z(a_i)$ for every $\beta\in(\mu,\nu]$ and $i\in n$ by 2.1.3 (v). It follows that $[K_\alpha;K_\beta]=1$ for every $\beta\in(\mu,\nu]$, a contradiction.

Thus I_M is not empty. Take $K_1\in\Omega_M$ and assume, for contradiction, that there is some non-trivial $K_2\in\Omega_P\backslash\Omega_M$. If $K_1\subseteq Z(G)$ then $K_1\subsetneq[G;G]$. It follows that $K_1\subseteq[a;G]$ for some $a\in G$, and $K_1=[a;K]$ for some $K\in\Omega_M$ applying 2.3.2. The same is true if we replace K_1 by K_2 and Ω_M by $\Omega_P\backslash\Omega_M$ (this time we use 2.3.3 and 2.3.2). It follows that we may assume that $[K_1;G]\neq 1$ and $[K_2;G]\neq 1$. Consequently $[K_1;K]\neq 1$ and $[K_2;K]\neq 1$ for some $K\in\Omega_P\cup\Omega_M$. By 2.3.6 K and K_2 belong to Ω_M, a contradiction.

We can drop now the hypothesis of ω-saturation. We have proved that all the residual structures are infinite, thus strongly minimal. We find that, if $\Phi(\mu,\nu)\neq\infty$, then every partial function $\delta_{\mu,\nu}(a,.)$ (or $\delta_{\mu,\nu}(.,b)$) is a morphism that is either trivial or onto with finite kernel. It follows that K_μ/H_μ, K_ν/H_ν and $K_{\Phi(\mu,\nu)}/H_{\Phi(\mu,\nu)}$ all have the same (infinite) cardinality (and the same exponent). Consider now the set $I_1=\{\mu\in I\mid\exists\xi\ \ \Phi(\mu,\xi)\neq\infty\}$. If $\mu,\nu\in I_1$ then we can always find $\xi\in I$ such that $\Phi(\mu,\xi)\neq\infty$ and $\Phi(\nu,\xi)\neq\infty$. Hence the cardinality of K_μ/H_μ is the same for every $\mu\in I_1$. If $\eta\notin I_1$ then there are $\mu,\nu\in I_1$ such that $\Phi(\mu,\nu)=\eta$ because G is not nil-2. We conclude then that all the residual structures have the same infinite cardinality. □

Final remarks. (1) In the general case we obtain that if $\mathbb{G}$ is a C-minimal valued group that is not virtually nil-2, then all, but a finite number, of residual structures are infinite (and then have the same infinite cardinality).

(2) We could have considered more generally any C-minimal valued group satisfying property (A). We have proved that in this case G has a connected component G°. Let

J be the following subinterval of I:

$$J := \{\eta \in I \mid [G^\circ; G^\circ] \subseteq H_\eta \subseteq Z(G^\circ)\}.$$

Note that J is not empty if and only if G° is nil-2. We can prove that there is a finite subset E of I such that $I \backslash (J \cup E) \subseteq (I_{S'} \cap I_M)$.

(3) Let $\mathbb{G}$ be a non-abelian C-minimal valued group whose residual structures are infinite. This is equivalent to say that G and all the elements of the chain Ω are connected. It is easy to prove that in this situation we can define on I a function Φ from $I \times I$ to I such that for every $\mu, v, \eta, \in I$,

$$\Phi(\mu, v) = \eta \text{ if and only if } [K_\mu; K_v] = K_\eta$$

and this function has the properties of Lemma 2.7.1. It will follow that, if

$$J := \{\eta \in I \mid [G; G] \subseteq H_\eta \subseteq Z(G)\},$$

then $I \backslash J$ is a union of two (or only one if J is empty) dense intervals. In particular $\mathbb{G}$ satisfies property (A). To prove this it is enough to see that if we suppose that there is μ such that $\mu \in I \backslash I_S$ and $\Phi(\mu, \mu) \neq \infty$ and if we write μ^+ for the successor of μ, then $\mu^+ \in I \backslash I_S$, $\Phi(\mu^+, \mu^+) \neq \infty$ and $\Phi(\mu^+, \mu^+) = \Phi(\mu, \mu)^{++}$: it follows that the range of the definable map $\eta \mapsto \Phi(\eta, \eta)$ is not a finite union of intervals, contradicting the o-minimality of I.

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