The existence of orthogonal diagonal Latin squares with subsquares

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Abstract

We prove that there exists a pair of orthogonal diagonal Latin squares of order \( v \) with missing subsquares of side \( n \) (ODLS\((v, n)\)) for all \( v \geq 3n + 2 \) and \( v - n \) even. Further, there exists a magic square of order \( v \) with missing subsquare of side \( n \) (MS\((v, n)\)) for all \( v \geq 3n + 2 \) and \( v - n \) even.

1. Introduction

A diagonal Latin square is a Latin square whose main diagonal and back diagonal are both transversals. Two diagonal Latin squares are orthogonal if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. We let ODLS\((v, n)\) denote a pair of orthogonal diagonal Latin squares of order \( v \) with missing subsquares of side \( n \) occupying the central positions in each. It is not necessary for such orthogonal diagonal subsquares of side \( n \) nearly to exist. We have the following more general definition.

A \((v, n)\)-IODLS (incomplete orthogonal diagonal Latin squares) is a pair of \( v \times v \) arrays which satisfy the following:

1. they are OLS\((v)\) with sub-OLS\((n)\) missing;
2. the first \( v - n \) elements in the main diagonal of each square are distinct and different from the missing elements;
3. the elements in the cells \((1, v - n), (2, v - n - 1), \ldots, (v - n, 1)\) of each square are also distinct and different from the missing elements.

When \( v - n \) is even, it is easy to see that the existence of \((v, n)\)-IODLS is equivalent to the existence of ODLS\((v, n)\).

The spectrum of orthogonal diagonal Latin squares was finally determined by Brown et al. [2].

Theorem 1.1. An orthogonal diagonal Latin square of order \( n \) exists if and only if \( n \neq 2, 3, \) or \( 6 \).
For \((v,n)\)-IODLS we have:

**Theorem 1.2** (Du [8]). *If there exists a \((v,n)\)-IODLS, then \(v \geq 3n + 2\).*

**Theorem 1.3** (Du [8]). *For any positive integer \(n \geq 100\), there exists a \((v,n)\)-IODLS if and only if \(v \geq 3n + 2\).*

We then have:

**Theorem 1.4.** *For any positive integer \(n \geq 100\), there exists an ODLS \((v,n)\) if and only if \(v \geq 3n + 2\) and \(v - n\) even.*

For ODLS\((v,n)\) we also have:

**Theorem 1.5** (Du [9]). *For any positive integer \(n \leq 6\), there exists an ODLS\((v,n)\) if and only if \(v \geq 3n + 2\) and \(v - n\) even.*

In this paper, we consider the remaining values of \(n\) and establish the following result.

**Theorem 1.6.** *For any positive integer \(n\), there exists an ODLS\((v,n)\) if and only if \(v \geq 3n + 2\) and \(v - n\) even.*

The terminology and notation used in this paper follow from [8, 9].

2. Some constructions

First we state a starter–adder type construction for \((v,n)\)-IODLS. The main idea is to generate each square under a cyclic group of order \(v - n\) from its first row and from the last \(n\) elements of the first column. Let \(X = \{0,1,\ldots,v-n-1\} \cap Y\), where \(Y = \{x_1, x_2, \ldots, x_n\}\). Suppose \(L\) is a square based on \(X\) with a hole indexed by \(Y\). We shall denote by \(e_L(i,j)\) the entry in the cell \((i,j)\) of the array \(L\). The first row is given by the vectors \(e = (e_L(0,0),\ldots,e_L(0,v-n-1))\) and \(f = (e_L(0,v-n),\ldots,e_L(0,v-1))\), and the last \(n\) elements of the first column are given by the vector \(g = (e_L(v-n,0),\ldots,e_L(v-1,0))\). The \(L\) is constructed modulo \(v - n\) in the range \(\{0,1,\ldots,v-n-1\}\), where the \(x_i\)'s act as 'infinity' elements as follows:

1. \(e_L(s+1,t+1) = e_L(s,t)\) if \(e_L(s,t) = x_i\), and \(e_L(s+1,t+1) \equiv e_L(s,t) + 1 \pmod{v-n}\) otherwise, where \(0 \leq s, t < v-n-1\);
2. \(e_L(s+1, v-n-1+t) \equiv e_L(s,v-n-1+t) + 1 \pmod{v-n}\), where \(1 \leq t \leq n\), \(0 \leq s < v-n-1\);
3. \(e_L(v-n-1+t,s+1) \equiv e_L(v-n-1+t,s) + 1 \pmod{v-n}\), where \(1 \leq t \leq n\), \(0 \leq s \leq v-n-1\).
We remark that there are obviously conditions which the vectors \( e, f, g \) must satisfy in order to produce the \((v, n)\)-IOILS, but we shall not concern ourselves with that, the reader may see [10]. For example, a \((15, 1)\)-IOILS can be constructed from

\[
\begin{align*}
e_A &= (0, 3, 12, 10, 2, x, 9, 8, 13, 4, 7, 1, 11, 5), \\
f_A &= (6), \\
g_A &= (8), \\
e_B &= (0, 6, 13, 4, 11, 9, 5, 1, 3, x, 12, 7, 10, 2), \\
f_B &= (8), \\
g_B &= (6).
\end{align*}
\]

**Lemma 2.1** (Du [8]). Suppose there exists a \((v, n)\)-IOILS constructed by the starter–adder method, \( v - n \) is even and the \( (1 + (v - n)/2) \)-th element in the starter set \( e \) is not an infinity element. Then there exists an ODLS\((v, n)\).

**Lemma 2.2.** For \((v, n) \in F\), there exists a \((v, n)\)-IOILS constructed by the starter–adder method such that \( (1 + (v - n)/2) \)-th element in the starter set is not an infinity element, where

\[
F = \{(38, 8), (3n + 6, n) : 8 \leq n \leq 32 \text{ and } n \text{ even}\}.
\]

**Proof.** For \((v, n) = (38, 8)\), see [16]. For the other cases \((v, n)\), see [18]. \(\square\)

Combining Lemmas 2.1 and 2.2 we have:

**Lemma 2.3.** There exists an ODLS\((v, n)\) for \((v, n) \in F\).

In the remainder of this paper we shall assume that the reader is familiar with the various methods of constructing \((v, n)\)-IOLS starting with an OLS\((n)\) (see e.g. [1, 3]), and starting with an \((n, k)\)-IOLS (see, e.g. [5, 20]). We shall also assume that the reader is familiar with the various techniques of constructing ODLS\((v)\) from OLS\((v)\) by permuting rows and columns (see, e.g. [4, 14, 15]).

For later use we need the following lemma, which is mainly the working corollary of Theorems 3.1 and 3.2 in [5]. So, we state it without proof. To state the construction we denote by ODLSt\((v, n)\) an ODLS\((v, n)\) with the property that there exists \( t \) disjoint common transversals including the main diagonal and the back diagonal.

**Lemma 2.4.** If ODLS\((u, n)\), OLS\((m)\), \((m + mj, mj)\)-IOILS (for \( j = 1, 2 \)) and \((m + mj, mj) - IOLS \) (for \( j = 3, 4, \ldots, t \)) all exist, then there exists an ODLS \((mn + \sum_{1 \leq j \leq t} mj, mu + \sum_{1 \leq j \leq t} mj)\). Moreover, there exists an ODLS\((mn + \sum_{1 \leq j \leq t} mj, n)\) if an ODLS\((mu + \sum_{1 \leq j \leq t} mj, u)\) exists and \( m > 2mj \).

To apply the above lemma we need some input designs, which we state below. Its proof is familiar.
Lemma 2.5. (1) If \( v \in E_1 \), then there exists an ODLS\(^4\)(v, 2), where
\[
E_1 = \{46, 54, 58, 62, 66, 70\}.
\]
(2) If \( v \in E_3 \), then there exist ODLS\(^6\)(v, 6) and ODLS\(^8\)(v + 2, 8), where
\[
E_3 = \{34, 38, 42, 46, 50, 54\}.
\]
(3) If \( v \in E_4 \), then there exists an ODLS\(^4\)(v, 1), where
\[
E_4 = \{15, 21, 33, 39, 55, 69\}.
\]

Proof. (1) From the following decomposition the result is obvious.
\[
46 = 4 \times 11 + 2, \quad 58 = 8 \times 7 + 2, \quad 66 = 8 \times 8 + 2,
54 = 4 \times 13 + 2, \quad 62 = 4 \times 15 + 2, \quad 70 = 4 \times 17 + 2.
\]
(2) Write \( v = 4t + 6, v + 2 = 4t + 8 \) (7 \( \leq t \leq 12 \)), the result is obvious.
(3) From the following decomposition the result is obvious.
\[
21 = 5 \times 4 + 1, \quad 39 = 5 \times 7 + (2 + 2), \quad 69 = 5 \times 13 + (2 + 2),
33 = 8 \times 4 + 1, \quad 55 = 5 \times 11.
\]
The case \( v = 15 \) follows from the above (15, 1)-IOILS.

Finally, we present the following construction, the proof of which is easy.

Lemma 2.6. (1) If \((n + m_j, m_j)\)-IOLS (for \( j = 1, 2 \)) and \((n + m_j, m_j)\)-IOLS (for \( j = 3, 4 \)) all exist, then there exists an ODLS\((4n + \sum_{1 \leq j \leq 4} m_j, \sum_{1 \leq j \leq 4} m_j)\). Moreover, there exists an ODLS\((4n + \sum_{1 \leq j \leq 4} m_j, n)\) if an ODLS\((\sum_{1 \leq j \leq 4} m_j)\) exists and \( n > 2m_j \).

(2) If ODLS\((n + m_1, m_1)\) and \((n + m_j, m_j)\)-IOLS (for \( j = 2, 3, 4 \)) all exist, then there exists an ODLS\((4n + \sum_{1 \leq j \leq 4} m_j, n + m_1)\) if an ODLS\((\sum_{1 \leq j \leq 4} m_j, m_1)\) exists.

(3) If \( n \) is even, ODLS\((n)\) and \((n + m_j, m_j)\)-IOLS (for \( j = 1, 2, 3, 4 \)) all exist, then there exists an ODLS\((5n + \sum_{1 \leq j \leq 4} m_j, n)\) if an ODLS\((\sum_{1 \leq j \leq 4} m_j)\) exists.

3. A general bound

A Latin square is self-orthogonal if it is orthogonal to its transpose. A self-orthogonal Latin square (SOLS) of order \( v \) will be denoted by SOLS\((v)\). We also denote by \((v, n)\)-ISOLS an incomplete self-orthogonal Latin square.

A Latin square is symmetric if it is equal to its transpose. We denote by SOLSSOM\((v)\) a self-orthogonal Latin square of order \( v \) with a symmetric orthogonal mate, USOLSSOM\((v)\) a self-orthogonal Latin square of order \( v \) with a constant main diagonal symmetric orthogonal mate. It is easy to see that the existence of an USOLSSOM\((v)\) requires that \( v \) is even.
Let $P = \{S_1, S_2, \ldots, S_n\}$ be a partition of a set $S$, where $n \geq 2$. A partitioned incomplete Latin square (PILS) having partition $P$ is an $|s| \times |s|$ array $L$, indexed by $S$, which satisfies the following properties:

1. a cell of $L$ either contains a symbol from $S$ or is empty;
2. the subarray indexed by $S_i \times S_i$ are empty for $1 \leq i \leq n$ (these subarrays are called holes);
3. the elements occurring in row (or column) $s$ of $L$ are precisely these in $S \setminus S_i$, where $s \in S_i$.

The type of $L$ is the multiset $\{|S_1|, |S_2|, \ldots, |S_n|\}$. We use the notation $1^{u_1}2^{u_2} \ldots$ to describe a type, where there are precisely $u_i$ occurrences of $i$ for $i = 1, 2, \ldots$.

Suppose $L$ and $M$ are PILS having the same partition $P$. We say that $L$ and $M$ are orthogonal if their superposition yields every ordered pair in $S^2 \setminus (\bigcup S_i^2)$. The term ‘orthogonal PILS’ is abbreviated to OPILS.

We shall assume that the reader is familiar with the standard terminology of group-divisible designs (GDDs) and Wilson’s ‘Fundamental Construction’ (see, e.g. [17]). Of course, a GD$[k, 1, n; kn]$ is equivalent to $k - 2$ POLS$(n)$.

**Lemma 3.1** (Heinrich et al. [11], Heinrich and Zhu [12]). If $v \geq 3n + 1$ and $v \neq 6$, then there exists a $(v, n)$-ISOLS except possible for $(v, n) = (6m + 2, 2m)$.

We need the following recursive construction for OPILS.

**Lemma 3.2** (Dinitz and Stinson [3]). Suppose that $(X, G, A)$ is a GDD, $w$ is a weighting, and let $k \geq 1$. Further, suppose that, for every block $A \in A$, there are $k$ OPILS of type $w(A)$. Then there are $k$ OPILS of type $\{\sum_{x \in G} w(x) : G \in G\}$.

We now state the main constructions.

**Lemma 3.3** (Du [8]). Suppose $n, m$ and $k$ are positive integers, $m$ odd, $2 \leq n \leq 3m - 3$, $1 \leq k \leq 2m$ and $k \neq 3, 4$, such that there exists a GD$[10, 1, m; 10m]$, then there exists a $(7m + n + k, n)$-IODLS. Further, for $7m + n + 5 \leq v \leq 9m + n$, there exists a $(v, n)$-IODLS.

**Lemma 3.4** (Du [9]). Suppose $n, m$ and $k$ are positive integers, $m$ even, $2 \leq n \leq 3m$, $1 \leq k \leq 2m$ and $k \neq 2, 3, 6$, such that there exists a GD$[10, 1, m; 10m]$, then there exists a $(7m + n + k, n)$-IODLS. Further, for $7m + n + 7 \leq v \leq 9m + n$, there exists a $(v, n)$-IODLS.

**Lemma 3.5.** Suppose $n, m$ are positive integers, $m$ even and $2 \leq n \leq 3m + 1$ such that there exists a GD$[9, 1, m; 9m]$, then there exists a $(7m + n + 6, n)$-IODLS.

**Proof.** In all groups but two of the GD$[9, 1, m; 9m]$, we give the points weight 1. In the second last group, we give 6 points weight 1 and give the remaining points weight 0. In
the last group, we give weight 0, 2 or 3, such that total weight be $n - 1$. We can apply Lemma 3.2 with the necessary input designs from Lemma 3.1 to obtain an OPILS$(m^76^4(n - 1)^1)$. We then fill size $m$ holes with ODLS$(m + 1, 1)$, size 6 hole with ODLS$(7, 1)$ and to obtain the required design by permuting rows and columns as Wallis did in [15].

**Lemma 3.6** (Du [8]). For any positive integer $n > 48$, if $v \geq 10n/3 + 66$, then there exists a $(v, n)$-IODLS.

Put $M' = \{9, 11, 13\} \cup M$ where $M = \{17, 19, 23, 25, 27, 29, 31, 37, 41, \ldots\}$ as in [8]. Apply Lemma 3.3 with $m \in M'$ and Lemmas 3.4 and 3.5 with $m = 16$ we have:

**Lemma 3.7.** For any positive integer $17 \leq n \leq 48$, if $v \geq 10n/3 + 29$, then there exists a $(v, n)$-IODLS.

**Lemma 3.8.** For any positive integer $7 \leq n \leq 16$, if $v \geq n + 64$, $v \neq n + 66$ or $v \neq n + 67$, then there exists a $(v, n)$-IODLS.

4. The case $n$ even

**Lemma 4.1** (Du [7], Wang [16]). If $n$ is even and $n \notin E_2$, then there exists an USOLSSOM$(n)$, where

$$E_2 = \{2, 6, 10, 14\} \cup E_1.$$

**Lemma 4.2** (Du [8]). If $n$ is even and $n \notin E_2$, then there exists a $(3n + k, n)$-IODLS, for $2 \leq k \leq n$ and $k \neq 2, 3$ or 6.

**Lemma 4.3.** If $n$ is even and $n \in E_1$, then there exists an ODLS$(3n + k, n)$ for $2 \leq k \leq n$.

**Proof.** Apply Lemma 2.4 with $u = 2$, $m = 3$ and $n \in E_1$. The result is true for $k < n - 2$. The conditions ODLS$^u - 4(v, 2)$ and ODLS$(k + 6, 2)$ exist by Lemma 2.5(1) and Theorem 1.5, respectively. For $k = n - 2$, ODLS$(3 \times 54 + 52, 54)$ comes from $3 \times 54 + 52 = 4 \times 53 + 2$ and apply Lemma 2.4 with $u = 13$, $m = 4$ and $n = 53$, the condition ODLS$(53, 13)$ comes from the decomposition $53 = 13 \times 4 + 1$. For the other cases come from Lemma 3.3 with the following decomposition:

$$3 \times 46 + 44 = 7 \times 17 + 46 + 17, \\ 3 \times 58 + 56 = 7 \times 23 + 58 + 11, \\ 3 \times 62 + 60 = 7 \times 25 + 62 + 9.$$

This completes the proof. □
Lemma 4.4. If even \( n \geq 8 \), then there exists an ODLS\((3n + 2, n)\).

Proof. Apply Lemma 2.4 with \( u = 2, m = 3 \) and \( n \geq 8 \), the condition ODLS\(^2(n, 2)\) comes from Theorem 1.5. □

Lemma 4.5 (Du [8]). If \( n \) is even and \( n \geq 56 \), then there exists an ODLS\((3n + 6, n)\).

Lemma 4.6. If \( n \) is even and \( n \geq 8 \), then there exists an ODLS\((3n + 6, n)\).

Proof. For the cases \( 8 \leq n < 34 \) see Lemma 2.3. For the other cases, apply Lemma 2.4 with \( u = 6 \) or \( 8 \), \( m = 3 \) and \( 34 \leq n < 56 \). The conditions ODLS\(^6(n, 6)\), ODLS\(^6(n, 8)\) and ODLS\((24, 6)\), ODLS\((30, 8)\) come from Lemma 2.5(2) and Theorem 1.5, respectively. □

Lemma 4.7. If \( n \) is even and \( n \not\in E_2 \), then there exists a \((4n + k, n) - IODLS\), for \( 0 \leq k \leq n \) and \( k \not= 2, 3 \) or \( 6 \).

Proof. If \( n \) is even and \( n \not\in E_2 \), then there exists an ODLS\((n)\) which possesses \( n \) disjoint common transversals including the main diagonal and the back diagonal (see [8]). We fill the \( k \) disjoint common transversals with \((5, 1)\)-IODLS and the others with ODLS\((4)\), but the back diagonal with modified \((5, 1)\)-IODLS or ODLS\((4)\), that is, by permuting the first \( 4 \) columns so that the main diagonal of the upper left part in the \((5, 1)\)-IODLS becomes its back diagonal. Note that there exists ODLS\((k)\) from Theorem 1.1. We obtain the required design by permuting rows and columns. □

Lemma 4.8. (1) If \( n \in E_1 \), then there exists an ODLS\((4n + k, n)\) for \( 0 \leq k \leq n \) and \( k \not= 2 \) or \( 6 \).

(2) If \( n \in \{10, 14\} \), then there exists an ODLS\((4n + k, n)\) for \( 0 \leq k < n \).

Proof. (1) For \( k \neq 0 \), write \( 4n + k = 4(n - 2) + (2 + 8 + 2 \times (k - 2)/2) \) and apply Lemma 2.6(2) with \( m_1 = 2, m_2 = 8 \) and \( m_3 = m_4 = (k - 2)/2 \). For \( k = 0 \), write \( 4n = 4(n - 2) + 4 \times 2 \) and apply Lemma 2.6(2) with \( m_j = 2 \) (\( 1 \leq j \leq 4 \)).

(2) The proof is similar to (1), but ODLS\((4 \times 10 + 8, 10)\) comes from the decomposition \( 4 \times 10 + 8 = 4 \times 10 + 4 \times 2 \) and apply Lemma 2.6(1) with \( m_j = 2 \) (\( 1 \leq j \leq 4 \)). □

Lemma 4.9. If \( n \) is even and \( n \geq 8 \), then there exists an ODLS\((4n + 2, n)\).

Proof. Apply Lemma 2.4 with \( u = 2, m = 4 \) and \( n \geq 8 \), the condition ODLS\(^2(n, 2)\) comes from Theorem 1.5. □

Lemma 4.10. If \( n \) is even and \( n \geq 8 \), then there exists an ODLS\((4n + 6, n)\).
Proof. The case \( n = 8 \) comes from Lemma 2.3. For the other cases with \( n \geq 10 \), write \( 4n + 6 = 4(n-2) + (2 + 3 \times 4) \) and apply Lemma 2.6(2) with \( m_1 = 2 \) and \( m_2 = m_3 = m_4 = 4 \).

Up to now we have:

**Lemma 4.11.** If \( n \) is even and \( n > 14 \), then there exists an \( ODLS(3n + k, n) \) for \( 2 \leq k \leq 2n \).

Combining Lemmas 3.6 and 3.7 we have:

**Theorem A.** If \( n \) is even and \( n > 16 \), then there exists an \( ODLS(v, n) \) if and only if \( v \geq 3n + 2 \) and \( v - n \) even.

5. The case \( n \) odd

We need the following results about SOLSSOM(\( n \)).

**Lemma 5.1** (Lindner et al. [13], Wang [16], Zhu [19]). If \( n \) is odd and \( n > 3 \), then there exists a SOLSSOM(\( n \)).

**Lemma 5.2.** If there exists a SOLSSOM(\( n \)), then there exists an \( ODLS(n) \) which possesses \( n \) disjoint common transversals including the back diagonal, each of which meets the main diagonal precisely once.

**Proof.** Say the SOLSSOM consists of a self-orthogonal Latin square \( A \) with a symmetric orthogonal mate \( C \). By applying a permutation simultaneously to the rows and columns as Wallis did in [14], we can produce a Latin square with constant back diagonal. We carry out the same permutation on self-orthogonal Latin squares \( A, A' \) and obtain required \( ODLS(n) \).

**Lemma 5.3.** If \( n \) is odd and \( n > 3 \), then there exists an \( ODLS(3n + k, n) \) for \( 2 \leq k \leq n + 1 \) and \( k \neq 4 \).

**Proof.** We begin with the \( ODLS(n) \) in Lemma 5.2, and fill the main diagonal and \( k - 1 \) disjoint common transversals including the back diagonal with \( (4, 1)\)-IOILS or \( (5, 1, 1)\)-IOILS; fill the back diagonal with modified \( (4, 1)\)-IOILS, and leave the central cell empty. For all other cells with OLS(3). Finally fill the size 5 hole with \( ODLS(5) \) and the size \( k - 1 \) hole with \( ODLS(k - 1, 1) \), and permute rows and columns.

**Lemma 5.4** (Du [6], Wallis and Zhu [15]). If \( n \) is odd and \( n > 5 \) and \( n \notin E_4 \), then there exist 4 PODLS(\( n \)).
Lemma 5.5 (Du [8]). If \( n \) is odd and \( n > 5 \) and \( n \notin E_4 \), then there exists an ODLS(\( 3n + 4, n \)).

Lemma 5.6 (Du[8]). Suppose there exists an ODLS\(^4(n, 1)\), then there exists an ODLS(\( 3n + 4, n \)).

Lemma 5.7. If \( n \) is odd and \( n \notin E_4 \), then there exists an ODLS(\( n + 4, n \)).

Proof. Apply Lemma 5.6 with \( n \notin E_4 \), the condition ODLS\(^4(n, 1)\) comes from Lemma 2.5(3). \[\]

Lemma 5.8. If \( n \) is odd and \( n > 3 \), then there exists a \((4n + k, n)\)-IODLS for \( 0 \leq k \leq n \), and \( k \neq 2, 3 \) or 6.

Proof. If \( n \) is odd and \( n > 3 \), then by Lemmas 5.1 and 5.2 there exists an ODLS\((n)\) which possesses \( n \) disjoint common transversals. The remaining proof is similar to Lemma 4.7. \[\]

Lemma 5.9. If \( n \) is odd and \( n \geq 7 \), then there exists an ODLS(\( 4n + 3, n \)).

Proof. Write \( 4n + 3 = 4(n - 1) + (1 + 3 \times 2) \) and apply Lemma 2.6(2) with \( m_1 = 1 \) and \( m_2 = m_3 = m_4 = 2 \). \[\]

Up to now we have obtained:

Lemma 5.10. If \( n \) is odd and \( n > 5 \), then there exists an ODLS(\( 3n + k, n \)) for \( 2 \leq k \leq 2n \).

Combining Lemmas 3.6 and 3.7 we have:

Theorem B. If \( n \) is odd and \( n > 15 \), then there exists an ODLS(v, n) if and only if \( v \geq 3n + 2 \) and \( v - n \) even.

6. The main result

We need the following construction.

Lemma 6.1 (Du [9]). Suppose \( n, m \) are positive integers, \( m \) even and there exists an \((m + 1, 1)-IODLS, 2 \leq n \leq 3m - 3, 1 \leq k \leq m, k \) even and \( k \neq 2 \), such that there exists a GD[9, 1, m; 9m]. Then there exists a \((7m + n + k, n)\)-IODLS.

Applying Lemma 6.1 with \( m = 8 \) and combining Lemma 3.8 we have:
Lemma 6.2. For any positive integer $7 \leq n \leq 16$, if $v \geq n + 60$ and $v \neq n + 66$, then there exists an ODLS$(v, n)$.

Put $F_1 = \{10, 14\}$, $F_2 = \{8, 12, 16\}$, $F_3 = \{7, 9, 11, 13, 15\}$.

Lemma 6.3. If $n \in F_1$, then there exists an ODLS$(5n + k, n)$ for $0 \leq k \leq 2n$.

Proof. The proof of the cases $k \leq n - 4$ is similar to Lemma 4.8. For the cases $k > n - 4$, write $5n + k = 5n + (m_1 + m_2 + m_3 + m_4)$ and apply Lemma 2.6(3). □

Lemma 6.4. If $n \in F_1$, then there exists an ODLS$(v, n)$ if and only if $v \geq 3n + 2$ and $v - n$ even.

Proof. (1) The case $n = 10$: From Lemmas 4.4, 4.6, 4.8 and 6.3 we only need to consider the cases $v = 34, 38$ and $76$. For the case $v = 34$, write $34 = 4 \times 8 + 2$ and apply Lemma 2.6(2). For the other cases apply Lemma 2.4 with the following decomposition

$$38 = 5 \times 7 + (2 + 1), \quad 76 = 23 \times 3 + (7 \times 1).$$

The conditions ODLS$^2(5, 1)$ and ODLS$^7(23, 1)$ is obvious.

(2) The case $n = 14$: From Lemmas 4.4, 4.6, 4.8 and 6.3 we only need consider the cases $v = 46, 50, 52$ and $54$. For the case $v = 50$, write $50 = 4 \times 12 + 2$ and apply Lemma 2.6(2). For the case $v = 54$, write $54 = 4 \times 10 + (4 + 4 + 4 + 2)$ and apply Lemma 2.6(1). For the other cases apply Lemma 2.4 with the following decomposition

$$46 = 7 \times 6 + (1 + 1 + 1 + 1), \quad 52 = 7 \times 7 + (1 + 1 + 1).$$

The condition ODLS$^4(7, 1)$ is obvious, but we fill using (6, 2)-IOLS, (7, 2, 1)-IOLS and (7, 2)-IOLS, (8, 2, 1)-IOLS, respectively. □

Lemma 6.5. If $n \in F_2$, then there exists an ODLS$(5n + k, n)$ for $0 \leq k \leq n$.

Proof. If $n \in F_2$, then there exists an ODLS$(n)$ which possesses $n$ disjoint common transversals including the main diagonal and the back diagonal (see [8]). We fill the $k$ disjoint common transversals with (7, 2)-IOLS and the others with ODLS$(5)$ and fill the back diagonal with modified (7, 2)-IOLS or ODLS$(5)$. For $k \neq 2, 6$, note that there exist ODLS$(k)$ from Theorem 1.1, so the result is true for $k \leq 2n$ and $k \neq 2, 6$. The proof of the result for $2n < k \leq 5n$ and $k \neq 2, 6$ is similar to the preceding, but fill using (9, 2)-IOLS, (10, 3)-IOLS or ODLS$(7)$.

For the cases $5n + k, k = 2, 6$, we begin with the above ODLS$(n)$ and fill the $(n + k)/2$ disjoint common transversals not including the main diagonal and the back
diagonal with (6,2)-IOILS and the others with ODLS(4). Note that there exist ODLS(n + k) from Theorem 1.1, the result is true for 5n + k, k = 2, 6. The proof of the cases 7n + k, k = 2, 6, is similar to the preceding, but fill using (7,1)-IODLS or (8,2)-IODLS.

Combining Lemmas 3.8, 4.2, 4.4, 4.6, 4.7, 4.9, 4.10 and 6.5 we have:

Lemma 6.6. If \( n \in F_2 \), then there exists an ODLS(\( v, n \)) if and only if \( v \geq 3n + 2 \) and \( v - n \) even.

Lemma 6.7. If \( n \in F_3 \), then there exists an ODLS(\( v, n \)) for \( 0 \leq k \leq 5n - 1 \).

Proof. The proof of the cases \( k \leq 2n \) and \( k \neq 2, 6 \); and \( 2n \leq k \leq 5n \) and \( k \neq 2, 6 \) is similar to Lemma 6.5. For the cases \( 5n + k, k = 2, 6 \), apply Lemma 2.4 with \( u = 1 \), \( t = 3 \), \( m_j = 0 \) or \( 2 \), the condition ODLS\(^3(n, 1)\) is obvious. The proof of the cases \( 7n + k, k = 2, 6 \), is similar to the preceding.

Combining Lemmas 3.8, 5.10 and 6.7 we have:

Lemma 6.8. If \( n \in F_3 \), then there exists an ODLS(\( v, n \)) if and only if \( v \geq 3n + 2 \) and \( v - n \) even.

Combining Lemmas 6.4, 6.6 and 6.7 we then have:

Theorem C. If \( 7 \leq n \leq 14 \), then there exists an ODLS(\( v, n \)) if and only if \( v \geq 3n + 2 \) and \( v - n \) even.

Combining Theorems 1.1 and 1.5 and Theorems A–C we have Theorem 1.6.

7. Remark

A magic square of order \( n \), denoted by MS(\( n \)), is an \( n \times n \) array whose entries are \( n^2 \) distinct positive integers such that the sum in each row, column, main diagonal and back diagonal is a constant. If a magic square has a subsquare occupying the central positions, the subsquare must itself be magic square. We refer to it as a magic subsquare. We denote by MS(\( v, n \)) a magic square of order \( v \) with missing subsquare of side \( n \).

It is easy to see that the existence of an ODLS(\( v, n \)) implies the existence of a MS(\( v, n \)). From Theorem 1.6 we have:

Theorem 7.1. For any positive integer \( n \), there exists a MS(\( v, n \)) if \( v \geq 3n + 2 \) and \( v - n \) even.
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References