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25 Pretty graph colouring problems

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Even if there is nothing more to say about the 4-colour-problem, there are very many easily formulated unsolved graph colouring problems left. We have selected a list of 25 pretty problems.

Problem 1 (Hadwiger, 1943). If a class of graphs is closed under minors (deletions and contractions), is the maximum chromatic number of graphs in the class equal to the largest order of a complete graph in the class?

Problem 2 (Berge, 1961). Assume that all vertex-critical induced proper subgraphs of a vertex-critical graph G are complete. Is G complete, an odd cycle or an odd cycle complement?

Problem 3 (Hadwiger; Nelson, 1961). What is the chromatic number k of the graph whose vertices are all points in the plane, with two vertices joined by an edge when they are of distance 1? It is known that $4 \leqslant k \leqslant 7$.

Problem 4 (Erdős, Faber and Lovász, 1972). If a simple graph G is the edge-disjoint union of k complete k-graphs, is G then k-colourable?

Problem 5 (Erdős and Lovász, 1966). If G is k-chromatic and $G - x - y$ is $(k-2)$-colourable for all edges (x, y) in G, does G then contain the complete k-graph?

More generally, If G is $(a+b-1)$-chromatic without a complete $(a+b-1)$-graph as a subgraph ($a \geqslant 2$, $b \geqslant 2$), does G contain vertex disjoint subgraphs of chromatic numbers a and b?

Problem 6 (Gupta; Albertson and Collins; Erdős, 1979; Bollobás and Harris, 1985). Is the list-edge-chromatic number of every multigraph equal to its edge-chromatic number?

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Problem 7 (Borodin and Kostochka, 1977; Reed, 1998). For $\omega(G)+1=\Delta(G)\geqslant 9$ is G colourable with $\Delta(G)-1$ colours?

Is a graph G always $\lceil(\Delta(G)+1+\omega(G))/2\rceil$-colourable?

Problem 8 (Vizing, 1964; Behzad, 1965). Can the vertices and edges of a graph G be coloured in $\Delta(G)+2$ colours so that no two adjacent or incident elements are coloured the same? If G is a multigraph of multiplicity μ can the vertices and edges then be such coloured in $\Delta+\mu+1$ colours?

Problem 9 (Goldberg, 1973). Let G be a multigraph with edge-chromatic number $k\geqslant\Delta(G)+2$. Does G contain a subgraph H with $2n+1$ vertices and m edges such that $(k-1)n<m$? By a theorem of J. Edmonds this is equivalent to asking whether k is equal to the upper integer part of the fractional edge-chromatic number of G.

Problem 10 (Hajós). Does every 5-chromatic graph contain a subdivision of the complete 5-graph as a subgraph?

Problem 11 (Ringel, 1959). How few colours suffice to colour any map on two spheres (earth and moon) where each country consists of a connected region on each sphere? This problem asks for the maximum chromatic number χ of graphs of thickness 2. It is known that $9\leqslant\chi\leqslant 12$.

Problem 12 (Dirac, 1957; Gallai, 1963; Ore, 1967). What is the minimum number of edges of a k-critical graph on n vertices? Is it $\lfloor 5n/3\rfloor$ for $k=4$?

Problem 13 (Nešetřil and Rödl, 1972). Does every large k-critical graph contain a large $(k-1)$-critical subgraph?

Problem 14 (Hedetniemi, 1966). Let $G\times G'$ denote the graph with vertex set $V(G)\times V(G')$ and edges $((a,a'),(b,b'))$ for edges (a,b) and (a',b') of G and G', respectively. Is $\chi(G\times G')=\min\{\chi(G),\chi(G')\}$?

Problem 15 (Johnson, 1974). Are there positive constants c and ε and a polynomial graph colouring algorithm which uses at most

$$c|V(G)|^{1-\varepsilon}\chi(G)$$

colours to colour every graph G? U. Feige and J Kilian have shown that this would imply efficient randomized algorithms for all problems in NP.

Problem 16 (Erdős, Rubin and Taylor, 1979). Can the list-chromatic number of the union of two graphs G and H exceed the product of the list-chromatic numbers of G and H?

Problem 17 (Ore and Plummer, 1968). Are $3\Delta(G)/2$ colours sufficient to colour the countries of a map G (i.e. a plane 2-connected multigraph) such that any two countries with a common boundary curve or a common boundary point get different colours?

Problem 18 (Albertson, 1981). For a surface S, does there exist a constant k such that for all graphs G embeddable on S, all but k of the vertices of G can be 4-coloured? In particular, can all but three vertices of a toroidal graph be 4-coloured?

Problem 19 (Berge; Fulkerson, 1971). If G is a 3-regular simple graph without bridges and H is obtained from G by duplicating each edge, is H then 6-edge-colourable?

Problem 20 (Sachs, 1968). Let M be a set of unit-spheres in $\mathbb{R}^n$ such that no two have interior points in common. Let G be a graph with vertex set M and edges xy for spheres x and y that touch. What is the maximum chromatic number χ_n for these graphs. It is easy to see that $\chi_2 = 4$, but even χ_3 is unknown ($5 \leqslant \chi_3 \leqslant 9$).

Problem 21 (Tarsi; Klein and Schönheim, 1991). If G is the union of a forest and a graph each non-empty subgraph of which has a vertex of degree at most 2, can G then be 5-coloured?

Problem 22 (Alon, 1993). Does there exist a function g and a polynomial algorithm which for every input graph G gives as output a number $s \leqslant \chi(G)$ and a $g(s)$-colouring of G? Even restricted to 3-colourable input graphs the answer is unknown.

Problem 23 (Gyárfás, 1997). If the vertices of every path in G span a 3-colourable subgraph, is G then 4-colourable? Is there at least a constant c such that G is c-colourable?

Problem 24 (Grötzsch; Seymour, 1981). If G is planar with maximum degree 3, is G 3-edge-colourable if and only if G has no subgraph in which one vertex has degree 2 and all others have degree 3?

Problem 25 (Erdős, 1985). Given two graphs of uncountable chromatic numbers, is there a 4-chromatic graph which is isomorphic to a subgraph of both?

Unlinked Bibliography: [1]

Acknowledgements

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References

[1] T.R. Jensen, B. Toft, Graph Coloring Problems, Wiley-Interscience, New York, 1995; http://www.imada.sdu.dk/Research/Graphcol/.