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On the genus of joins and compositions of graphs

David L. Craft*

Department of Mathematics and Computer Science, Muskingum College, New Concord, OH 43762, USA

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Abstract

We will use a surgical technique to imbed the composition graph $G[H]$ when G has minimum degree 2 and H has even order. This imbedding is shown to be minimal when G is triangle-free. Along the way, we construct genus imbeddings of the join$G + H$ when both factors have even order, G is empty or 1-regular, and G has order at least twice that of H. Variants and applications to specific graphs, such as complete tripartite graphs, are given.

1. Introduction

The history of the genus parameter for graphs is a rich one which dates to 1890 when Heawood [5] first published his Map Coloring Conjecture. Ringel began working on this problem in the early 1950s and, with numerous contributions by others, most notably Youngs and Gustin, finally established the genus of the complete graphs in 1968. (See [8]). As this work was taking place, interest grew in the genus parameter as applied to other graphs. (See [9] or [13] for lists of genus results as of 1978 and 1984, respectively.) Although work has been done on various complete multipartite graphs (which can be expressed as joins) as well as other graph products such as cartesian products, amalgamations, and tensor products, to name a few, very little has been written on the genus of graphical joins and compositions in general. The sole published exception is a paper by White [11] in which he established the genus of compositions $G[H]$ in which H has even order and maximum degree less than two. The current paper is a strengthened version of a portion of [4] which focused on the problem of determining the genus of joins and compositions in general. (It is hoped that the remainder of this previous work will appear in the near future.)

Before setting about the work at hand, a word or two about basic terminology and notation is in order. A graph G is here understood to be devoid of multiple edges and

* E-mail: craft@muskingum.edu.

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loops. A (p, q)-graph having order p and size q. The *complement* of a graph G is denoted $\bar{G}$. A $(p, 0)$-graph is said to be *empty* and is denoted by $\bar{K}_p$. The disjoint union of m copies of G is denoted by mG. A *triangle-free* graph contains no 3-cycles. Finally, the *genus* of G is denoted by $\gamma(G)$. We will define *join* and *composition* in Section 3 as well as various terms of our own invention throughout the paper. Other terms and notation used, but not explicitly defined here, can be found in [3] or [13].

The goal in this paper is to use a surgical technique based on graphical surfaces to find minimal or near-minimal imbeddings of joins and compositions of graphs. The concept of a *graphical surface*, which is simply a fattened graph, will be developed in Section 2. We review the definitions of these graph products in Section 3 and state the main theorems to be proved. At this point, an overview of the proof of Theorem 1 is given which motivates the invention of a combinatorial structure called a *proposition chain*. In Section 4, we explore proposition chains and manufacture the machinery necessary to prove our main theorems. The remainder of the paper consists of these proofs as well as variations and applications of these theorems. The reader will observe that the proof of Theorem 2 depends very heavily on Theorem 1 and its proof, as composition is seen to be a generalization of join.

2. Graphical surfaces

Any connected graph G can be used as a template for building an orientable surface $S(G)$ which is called the *graphical surface based on* G. This surface is formed by associating a sphere with each vertex of G and joining two such spheres with a tube if and only if the vertices associated with these spheres are adjacent in G. Recalling that the betti number $\beta(G)$ of a connected (p, q)-graph G is calculated as $q - p + 1$, which is simply the number of edges of G not included in any given spanning tree, we have the following.

Lemma 1. $\gamma(S(G)) = \beta(G)$.

Proof. First observe that for any tree T, $\gamma(T) = 0$ and that a straightforward induction on the order of T yields $\gamma(S(T)) = 0$ Now, let T be any spanning tree for G. The surface $S(G)$ can be constructed from $S(T)$ by simply adding a tube for each edge in $E(G) - E(T)$. Thus, $S(G)$ is formed by adding $\beta(G)$ handles to a surface which is homeomorphic to a sphere, creating an orientable surface of genus $\beta(G)$. $\square$

For purposes of clearly describing imbeddings of graphs in these graphical surfaces, we will impose a bit more structure on $S(G)$ and develop an associated vocabulary and notation. Begin with the connected graph G imbedded in some orientable surface S_k. (This imbedding need not be minimal.) If this surface S_k is situated in $\boldsymbol{R}_3$, it partitions $\boldsymbol{R}_3$ into the surface itself and two half-spaces — the unbounded outside and the bounded inside. For each vertex $u \in V(G)$, place a sphere $S(u)$ in $\boldsymbol{R}_3$ centered at u. Now, for each edge $e = uv \in E(G)$, place a tube $T(e) = T(uv)$ in $\boldsymbol{R}_3$ centered along e. By choosing a small enough common radius for these spheres and a much smaller common radius for the tubes, we can guarantee that (1) no two spheres intersect, (2) two tubes intersect only if the corresponding edges are adjacent and then they only intersect inside the sphere which corresponds to the common endpoint. (3) no tube is self-intersecting, and (4) a sphere intersects a tube if and only if the corresponding vertex and edge are incident and, in this case, the intersection consists of a single circle. We will also assume that the surface S_k is locally flat at each vertex of G to a distance at least equal to the common radius of the spheres and that the portions of the edges of G imbedded in these flat subregions lie along straight lines. Finally, at each sphere $S(u)$, delete the portion inside $S(u)$ of every tube $T(e)$ corresponding to an edge e incident with u, and delete the portion of $S(u)$ which is inside each such tube. The resultant surface is clearly the surface $S(G)$. Despite these final deletions, we will continue to use the notations $S(u)$ and $T(e)$ for the remaining portion of the original structure so named.

Let us now consider any particular sphere $S(u)$ in the surface $S(G)$. Because of the local flattening of the surface holding G, S_k intersects $S(u)$ in a great circle which we will call the *equator* of $S(u)$. The antipodal points in $S(u)$ which are the greatest distance from the equator will be called the poles of $S(u)$. In particular, the pole outside S_k will be called the *north pole* of $S(u)$ and the one inside S_k will be called the *south pole* of S_k. Furthermore, *meridian* refers to any great arc in $S(u)$ which has the poles as its endpoints (or, at least that portion of one which remains after the aforementioned deletions).

Lemma 2. *If G is any connected graph then the composition graph $G[\bar{K}_2]$ has a quadrilateral imbedding in the surface $S(G)$. Furthermore, if G is triangle-free then this imbedding is minimal. (That is, $\gamma(G[\bar{K}_2]) \leqslant \beta(G)$ with equality if G is triangle-free.)*

Proof. Begin by placing a vertex at each pole of each sphere in the graphical surface $S(G)$. Now, for each pair of adjacent vertices u and v in G, the four edges joining the poles of $S(u)$ to the poles of $S(v)$ must be imbedded. First, imbed the edge joining the north poles of these spheres so that the portions of this edge which lie in the spheres follow meridians of the spheres and so that the portion of this edge which lies in the tube $T(uv)$ consists of the locus of points in the northern half of $T(uv)$ which are farthest from S_k. Likewise, imbed the south-pole-to-south-pole edge to follow meridians in $S(u)$ and $S(v)$ and to be the locus of points in the southern half of $T(uv)$ which are farthest from S_k. These two edges separate the remainder of $T(uv)$ into two tubal regions. If, in imbedding the remaining two edges, we continue to insist on following meridians in the spheres, we are forced to choose between the two possible assignments of the remaining two edges to these two tubal regions. Fig. 1 shows these two possible imbeddings. A tube containing these four imbedded edges will be said to have either a positive or negative *orientation*, depending on the way in which the north-to-south edges are imbedded. The (arbitrary) assignment of these terms is also indicated in the figure, where a bar above the notation for a tube indicates a 'negative' orientation and the absence of one indicates a positive one.

That this mapping of $G[\bar{K}_2]$ to the surface $S(G)$ is an imbedding, or can easily be made one (that is, that no edges cross) is guaranteed by the requirement that portions of edges lying in a sphere follow meridians, since a meridian can intersect at most one tube. That this imbedding is quadrilateral is illustrated in Fig. 2, which shows the four possible region shapes. These shapes are determined by two factors: (1) the original imbedding of G in S_k (i.e., the order in which the edges incident with a vertex u radiate out from u, which is the same as the order in which the tubes joined to $S(u)$ are arranged around the equator of $S(u)$; and (2) the orientations of the tubes. Thus, the imbedding is quadrilateral.

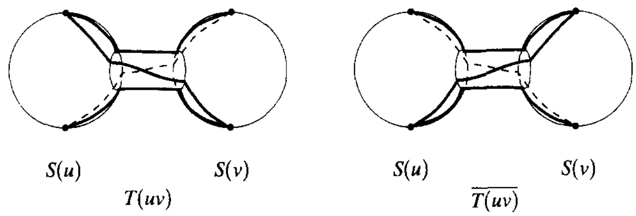

Fig. 1.

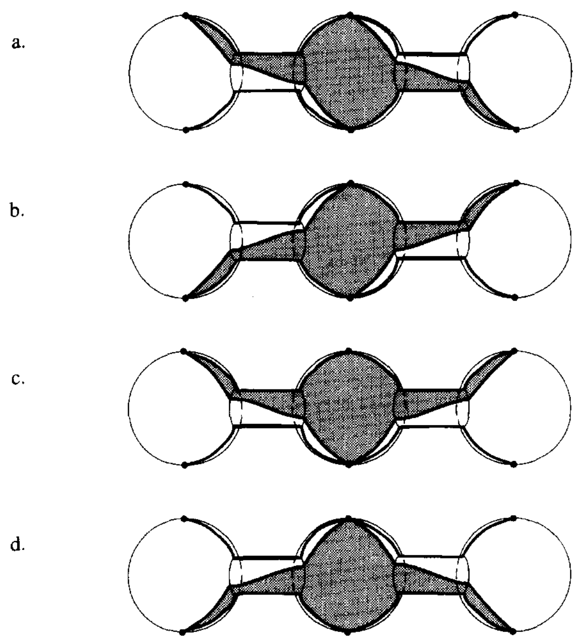

Fig. 2.

Finally, if G is a triangle-free graph then so is $G[\bar{K}_2]$. The standard argument from the Euler–Poincaré formula shows that a quadrilateral imbedding of a triangle-free graph must be minimal. □

The imbeddings to be constructed will begin with the quadrilateral imbedding described in Lemma 2, and will be completed by imbedding diagonals in some of these quadrilateral regions. Consider Fig. 2 again. Notice that, in each case, one pair of opposite vertices in the region boundary consists of the two poles of a single sphere. Such a region is said to be *centered on this sphere* and the diagonal of the region which joins the poles of this sphere will be called the *short diagonal*. The other diagonal will be called the *long diagonal* of the region. All four regions shown in Fig. 2 are viewed from outside of the surface, with a vertical short diagonal — north pole on top. Using this perspective, the four types of regions shown can be named according to the poles which would be joined by the long diagonal — reading from left to right. Using N for north and S for south, the *region types* shown in the figure, from the top down, are NS, SN, NN, and SS.

Our work hinges on being able to construct regions with specific long diagonals. (Imbedding short diagonals will, in general, not be a problem. To imbed an edge joining the poles of, say, $S(u)$, we need only find a region centered on $S(u)$ in which the long diagonal need not be imbedded. Notice that there are $\deg_G u$ such regions.) To construct a region whose long diagonal joins a pole of $S(v)$ to a pole of $S(w)$, find a vertex $u \in V(G)$ which is a common neighbor of v and w and arrange the tubes around the equator of $S(u)$ so that $T(uv)$ and $T(uw)$ are next to each other; i.e., one of the two portions of the equator of $S(u)$ connecting $T(uv)$ and $T(uw)$ is unbroken by other tubes. Part (a) of Fig. 2 shows that when both tubes are positively oriented, the long diagonal would join the north pole of the westmost sphere to the south pole of the eastmost sphere. Part (b) of Fig. 2 shows that when both tubes are negatively oriented, the long diagonal joins the south pole of the westmost sphere to the north pole of the eastmost sphere. So, to construct a region through which the edge joining the north pole of $S(v)$ to the south pole of $S(w)$ can be imbedded, either attach $T(uv)$ immediately to the west of $T(uw)$ along the equator of $S(u)$ and orient both tubes positively, or attach $T(uv)$ immediately to the east of $T(uw)$ along the equator of $S(u)$ and orient both tubes negatively. That is, either of the following orderings of the oriented tubes to $S(v)$ and $S(w)$ from west to east around the equator of $S(u)$ will serve the purpose: $\ldots\ T(uv)\ T(uw)\ \ldots$ or $\ldots\ \overline{T(uw)}\ \overline{T(uv)}\ \ldots$

Similarly, either $\ldots\ T(uw)\ T(uv)\ \ldots$ or $\ldots\ \overline{T(uv)}\ \overline{T(uw)}\ \ldots$ will form a region whose long diagonal joins the south pole of $S(v)$ to the north pole of $S(w)$. To form a region whose long diagonal joins the north poles of $S(v)$ and $S(w)$ requires, in addition to the adjacency of attachment sites of the tubes along the equator of $S(u)$, only that the western tube be negatively oriented and the eastern tube be positively oriented as in $\ldots\ \overline{T(uv)}\ T(uw)\ \ldots$ or $\ldots\ \overline{T(uw)}\ T(uv)\ \ldots$ Finally, according to part (d) of Fig. 2, the south to south diagonal can be found in the region formed by $\ldots\ T(uv)\ \overline{T(uw)}\ \ldots$

or ... $T(uw)$ $\overline{T(uv)}$ ... Thus, our task will be accomplished by judiciously orienting tubes and ordering them around the equators of the spheres to which they are attached. These observations motivate the development of proposition chains below, but let us first describe the graphs with which we will be working and state the theorems to be proved.

3. Joins, compositions, and theorems

Given two graphs G and H, the *join* $G + H$ is formed from the disjoint union of these two graphs by adding all edges of the type uv where $u \in V(G)$ and $v \in V(H)$, as illustrated in Fig. 3.

Observe that if G and H are empty graphs then $G + H$ is a complete bipartite graph. That is, $\overline{K_m} + \overline{K_n} = K_{m,n}$. Thus, Lemma 2 gives us our first genus formula for joins. (It should be mentioned that the following is a special case of the general genus formula for complete bipartite graphs proved by Ringel in 1965 [7].)

Lemma 3. *The complete bipartite graph with even order partite sets $K_{2m,2n}$ has a quadrilateral imbedding in the surface $S(K_{m,n})$. Thus, $\gamma(K_{2m,2n}) = (m-1)(n-1)$.*

Proof. This is an immediate corollary of Lemma 2 which follows from the following three observations: (i) $K_{2m,2n} = K_{m,n}[\overline{K_2}]$; (ii) $K_{m,n}$ is a connected, triangle-free graph; and (iii) $\beta(K_{m,n}) = mn - (m+n) + 1 = (m-1)(n-1)$. $\square$

The second type of graph product of interest is the *composition* (or lexicographic product), denoted $G[H]$. This graph is constructed from G by replacing each vertex of G by a copy of H and then adding edges as follows. If H_u and H_v are the copies of H which replace vertices u and v respectively, add all edges of the type $u'\,v'$ where $u' \in V(H_u)$ and $v' \in V(H_v)$ if and only if $uv \in E(G)$. For example, $K_2[H] = H + H$. In this way, each edge of G may be said to induce a subgraph of $G[H]$ which is isomorphic with $H + H$. This connection between joins and compositions will be

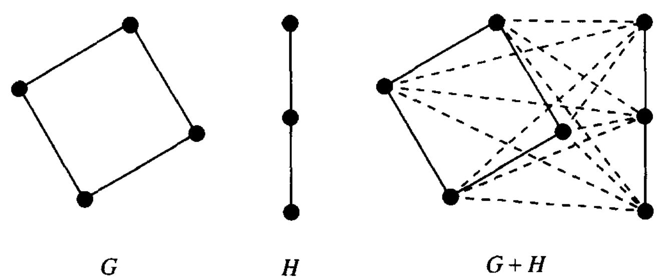

Fig. 3.

exploited later. Just as the complete bipartite graphs are the simplest joins, it could be argued that graphs of the form $G[\overline{K_n}]$ are the simplest compositions. The analogy leads us to the following.

Lemma 4. *If G is a nontrivial, connected (p, q)-graph and n is a positive integer, then $G[\overline{K_{2n}}]$ has a quadrilateral imbedding in the surface $S(G[\overline{K_n}])$. Thus $\gamma(G[\overline{K_{2n}}]) \leqslant n^2q - np + 1$, with equality if G is triangle-free.*

Proof. As with Lemma 3, this also follows immediately from Lemma 2. It suffices to observe that: (i) $G[\overline{K_{2n}}] = (G[\overline{K_n}])[\overline{K_2}]$; (ii) $\beta(G[\overline{K_n}]) = n^2q - np + 1$; and (iii) if G is triangle-free then so is $G[\overline{K_n}]$. □

We are now in a position to state our main theorems, which are generalizations of Lemmas 3 and 4.

Theorem 1. *If m and n are positive integers and H is any graph of order $2n$, then*

(i) $\gamma(\overline{K_{2m}} + H) = (m - 1)(n - 1)$, *provided* $m \geqslant 2(n - 1)$, *and*

(ii) $\gamma(mK_2 + H) = (m - 1)(n - 1)$, *provided* $m \geqslant 2n$.

Theorem 2. *If G is a nontrivial, connected (p, q)-graph with minimum degree at least 2, and H is a graph of positive even order $2n$, then $\gamma(G[H]) \leqslant n^2q - np + 1$, with equality if G is triangle-free.*

The proofs of both theorems begin in the same fashion. Start with the quadrilateral imbeddings mentioned in Lemma 3 or 4 respectively, and add edges as diagonals of some of the quadrilateral regions. This final step is preceded by the work of configuring the imbedding so as to contain regions with the required diagonals. Although the proofs of these theorems must be postponed until after the next section, let us take a closer look at the work ahead in proving Theorem 1.

We begin with part (i). Observe that it suffices to imbed the graph $\overline{K_{2m}} + K_{2n}$ in a minimal surface for $K_{2m,2n}$. The surface in which $\overline{K_{2m}} + K_{2n}$ will be imbedded is $S(K_{m,n})$, a fattened version of the complete bipartite graph $K_{m,n}$. Think of this surface as being composed of two clusters of spheres joined by all possible tubes running from spheres in the first cluster to spheres in the second cluster. The poles of the m spheres in the first cluster hold the vertices of $\overline{K_{2m}}$ and the n spheres in the second cluster hold the vertices of K_{2n}. Each of the tubes carries the four edges which join the poles of the two end spheres as has been described. The way in which these edges are imbedded in a tube imparts an orientation to the tube and the orientations of the tubes together with the order in which they are attached to the spheres in the first cluster determine the long diagonals of the regions centered on the spheres in the first cluster. Our objective is to order and orient the tubes so as to form a collection of regions with the property that, for every pair of poles of different spheres in the second cluster, there is a region centered on a sphere in the first cluster whose long diagonal joins these poles.

Each sphere in the first cluster is attached to n tubes (joining it to the n spheres in the second cluster). Thus there are n regions centered on each sphere in the first cluster. Since there are $\binom{2n}{2} - n = 2n(n-1)$ long diagonals to imbed (n of the edges of K_{2n} will be imbedded as short diagonals in regions centered on the n spheres in the second cluster), there must be $2n(n-1)$ regions centered on spheres within the first cluster and, hence, there must be at least $2(n-1)$ spheres in the first cluster. We will show that this is sufficient to prove part (i). The only difference between the parts of Theorem 1 is that in part (ii), one of the n regions centered on each sphere in the first cluster will be used to imbed a short diagonal and will, therefore, be unavailable to use for the purpose of imbedding a long one. Thus, there must be at least $2n$ spheres in the first cluster. We will show that this is sufficient to prove part (ii).

Thus, the proposed procedure can be summarized as follows. Order and orient the n tubes attached to the first sphere in the first cluster to form n regions, centered on this sphere, which have long diagonals joining n different pairs of poles of spheres in the second cluster. This order is cyclic in the proof of part (i) and linear in part (ii). This is done for each sphere in the first cluster, in turn, until all pairs of poles of second-cluster spheres can be joined by long diagonals through the regions formed. For our purposes, the n tubes being ordered and oriented around each sphere in the first cluster can be treated as the same n objects — identified with the second-cluster spheres to which they are attached. In order to study these collections of ordered and oriented objects, we introduce a combinatorial structure called a proposition chain.

4. Proposition chains

A *linear* (or *cyclic*) *n-chain* is a linear (respectively, cyclic) ordering of the elements of a set $L = \{\ell_0, \ell_1, \dots, \ell_{n-1}\}$. Call the elements of L the *links* of the n-chain. When denoting a cyclic n-chain, the 'first' link will be repeated as the 'last' to indicate the wrapping around of the chain. Thus, the absence of this repeated element will indicate a linear n-chain. For example, $\ell_3\ell_1\ell_0\ell_4\ell_2\ell_3$ is a cyclic 5-chain whereas $\ell_3\ell_1\ell_0\ell_4\ell_2$ is a linear 5-chain. When n is small and there is little chance of confusion, chains may be represented as a chain of subscripts. That is, the examples just given may be rendered as 310423 and 31042.

It is immediate that $n!$ is the number of linear n-chains on a given set of n elements and $(n-1)!$ is the number of cyclic n-chains. Our interest here lies in slightly more challenging counting problems involving couples and propositions chains, both of which are defined below.

Define a *couple* to be any (unordered) pair of adjacent links in a chain. Our first problem is to find the minimum number of n-chains necessary to form all possible couples, specifying that these chains are either all linear or all cyclic. Such a collection will be called a *complete collection*. The solution to this problem is the following.

Lemma 5. *If n is a positive integer then the minimum cardinality of a complete collection of linear (or cyclic) n-chains is $\lceil n/2 \rceil$ (respectively, $\lceil (n-1)/2 \rceil$).*

Proof. First observe that there is a total of

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

couples to be formed in the collection. Since each linear n-chain forms $n - 1$ couples, a complete collection of linear n-chains must contain at least $n/2$ chains. Similarly, since each cyclic n-chain forms n couples, a complete collection of cyclic n-chains must contain at least $(n-1)/2$ chains. Thus, the indicated cardinalities are seen to be lower bounds. In order to establish them as upper bounds as well, we need only exhibit complete collections having these cardinalities.

The problem is easily solved by associating each of the n links with a vertex in the complete graph K_n and translating the problem into one of factoring the complete graph into spanning paths or cycles, which are then associated with linear or cyclic chains, respectively. It is well known that K_n is factorable into spanning cycles if and only if n is odd and into spanning paths if and only if n is even. (cf. [3, p. 237]). Thus, if n is odd, the $(n-1)/2$ spanning cycles into which K_n is factorable correspond to $(n-1)/2$ cyclic n-chains in a complete collection. Similarly, if n is even, the $n/2$ spanning paths into which K_n is factorable correspond to $n/2$ linear n-chains in a complete collection.

Two cases remain. If n is odd, a complete collection of $\lceil n/2 \rceil$ linear n-chains can be constructed from the complete collection of linear $(n+1)$-chains mentioned above, by deleting the same vertex from each of the chains. If n is even, a complete collection of $\lceil (n-1)/2 \rceil$ cyclic n-chains can be constructed from the complete collection of linear n-chains mentioned above, by joining the first and last links in each chain. □

Understanding how these complete collections are constructed is vital to the work ahead. Consequently, we will briefly review the idea behind the proof of the factorization theorem referred to in our proof of Lemma 5. We begin by showing how to factor K_n into spanning paths when n is even. Arrange the n vertices as vertices of a regular n-gon and label these vertices with the elements of the cyclic group Z_n as in Fig. 4.

The first spanning path zigzags across the n-gon starting with one edge of the n-gon and ending with the opposite edge (shown as the solid path). All subsequent paths are obtained by rotation. (The second path is shown as a dotted line.). The $n/2$ spanning paths are the rows in the table below. Note that each entry, beyond those in the first row, is obtained from the one directly above it by adding 1

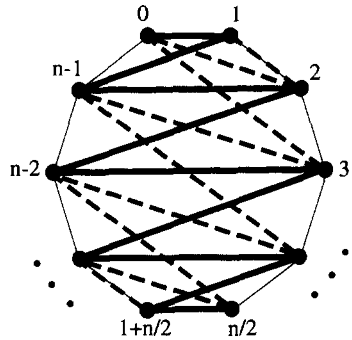

Fig. 4.

(mod n).

$$\begin{array}{ccccccc} 0 & 1 & n-1 & 2 & \dots & \frac{n}{2}+1 & \frac{n}{2} \\ 1 & 2 & 0 & 3 & \dots & \frac{n}{2}+2 & \frac{n}{2}+1 \\ 2 & 3 & 1 & 4 & \dots & \frac{n}{2}+3 & \frac{n}{2}+2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{n}{2}-1 & \frac{n}{2} & \frac{n}{2}-2 & \frac{n}{2}+1 & \dots & 0 & n-1 \end{array}$$

In order to obtain the factorization into spanning cycles of a complete graph K_n of odd order, begin with the factorization of K_{n-1} into spanning paths. Then insert vertex $n-1$ between the first and last vertices in each path. For instance, if the table above is modified by adding n to the beginning and end of each row, the factorization of K_{n+1} into spanning cycles would result.

Example. The upper left-hand table below represents the decomposition of K_6 into spanning paths and so, under the correspondence between vertices of complete graphs and link sets, it also represents a complete collection of linear 6-chains. The upper right-hand table represents the decomposition of K_7 into spanning cycles as well as a complete collection of cyclic 7-chains. The table at the lower left represents a complete collection of cyclic 6-chains. Note that each of the last three of these collections is derived from the first.

0	1	5	2	4	3	6	0	1	5	2	4	3	6
1	2	0	3	5	4	6	1	2	0	3	5	4	6
2	3	1	4	0	5	6	2	3	1	4	0	5	6

0	1	2	4	3	0	1	5	2	4	3	0
1	2	0	3	4	1	2	0	3	5	4	1
2	3	1	4	0	2	3	1	4	0	5	2

Let us now move on to proposition chains. Suppose each of the links of a chain is an ordered pair; i.e., $\ell_i = (a_i, b_i)$ for $i = 0, 1, 2, \dots, n-1$. Define a second set of links based on the first set by $\bar{\ell}_i = (b_i, a_i)$ for $i = 0,1,2, \dots, n-1$ and say that ℓ_i and $\bar{\ell}_i$ are the positive and negative orientations of the ith link, respectively. A *linear* (or *cyclic*) *n-proposition chain* is a linear (respectively, cyclic) ordering of n oriented links. Observe that an n-proposition chain on the link set $\{\ell_0, \ell_1, \dots, \ell_{n-1}\}$ *induces* a chain (of the same type—linear or cyclic) on the set $A = \{a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1}\}$. A *complete collection of n-proposition chains* is a collection of n-proposition chains on the same link set L, either all linear or all cyclic, such that the set of induced chains on the set A is a complete collection of $(2n)$-chains. Since all couples with the same subscript (i.e., $a_i b_i$ or equivalently $b_i a_i$) are formed in each n-proposition chain, define a proper couple or *Couple* (with a capital C) to mean a couple whose elements have different subscripts; i.e., couples comprised of coordinates from different links. So, in constructing complete collections of proposition chains, our concern will focus on forming all Couples. Our next problem is analogous to our first: find the minimum cardinalities among all complete collections of linear (or cyclic) n-proposition chains. However, before we set about solving this latest problem, an explanation of the name is in order. (The following paragraph serves only to explain our choice of vocabulary and can be omitted without loss of continuity.)

Suppose p and q are propositions as in the study of elementary logic. Suppose also, that some liberties are taken with the usual logical notation. Let a bar over a proposition denote negation and let juxtaposition of two propositions denote a conditional statement; i.e., use pq to replace the more customary $p \to q$. Furthermore, let a chain of propositions, either linear or cyclic, be understood as being merely a formal shorthand for the collection of conditional statements denoted by the several adjacent pairs of propositions in the chain. For example, the cyclic chain $pqrsp$ represents the collection of conditional statements pq, qr, rs, and st. (The analogy with logic breaks down a bit in that transitivity is not implied. We cannot conclude, for instance, that pr follows from the compound statement pqr.) The *contrapositive* of such a chain of propositions is a second chain which is constructed from the first by writing the propositions in reverse order and negating all of them. Thus, the contrapositive of a chain of propositions represents the collection of contrapositives of the component conditional statements comprising the original chain. For this reason, we say that the contrapositive of a chain of propositions is (logically) equivalent to the original. It is precisely this equivalence which motivates the name *proposition chain.*

Define the *contrapositive of an n-proposition chain* to be a second n-proposition chain which is constructed from the first by writing the links in reverse order and changing the orientation of each. Also, we say that two n-proposition chains are *equivalent* if and only if they form precisely the same couples.

Lemma 6. *An n-proposition chain is equivalent to its contrapositive.*

Proof. The couple $a_i b_j$ is formed by $\overline{\ell_i}\,\overline{\ell_j} = (b_i, a_i)(b_j, a_j)$ or by $\ell_j \ell_i = (a_j, b_j)(a_i, b_i)$. Similarly, the couple $a_i a_j$ is formed by $\overline{\ell_i}\ell_j$ or by $\overline{\ell_j}\ell_i$ and the couple $b_i b_j$ is formed by $\ell_i\overline{\ell_j}$ or by $\ell_j\overline{\ell_i}$. Thus, any couple formed by an adjacent pair of oriented links in an n-proposition chain is also formed by the contrapositive. □

Thus for each pair of links ℓ_i and ℓ_j, $0 \leqslant i \leqslant j \leqslant n$, a complete collection of n-proposition chains must contain at least four chains containing ℓ_i and ℓ_j, as adjacent links — one for each of the following: $\ldots \ell_i \ell_j \ldots$ or $\ldots \overline{\ell_j}\,\overline{\ell_i}, \ldots$, $\ldots \overline{\ell_i}\ell_j \ldots$ or $\ldots \overline{\ell_j}\ell_i, \ldots$, $\ldots \ell_i\overline{\ell_j} \ldots$ or $\ldots \ell_j\overline{\ell_i}, \ldots$, and $\ldots \overline{\ell_i}\,\overline{\ell_j} \ldots$ or $\ldots \ell_j \ell_i \ldots$ Since the number of couplings has quadrupled from simple chains to proposition chains, one might expect that the minimum cardinality of complete collections might also have quadrupled. Lemma 7 shows that this is very nearly the case.

Lemma 7. *If n is a positive integer then the minimum cardinality of a complete collection of linear (or cyclic) n-proposition chains is $2n$ (respectively, $2(n-1)$).*

Proof. In each of the four cases below, a collection which contains all $2n(n-1)$ Couples will be constructed. Since each linear n-proposition chain forms $n-1$ Couples, a complete collection of linear n-proposition chains must contain at least $2n$ proposition chains. Similarly, since each cyclic n-proposition chain forms n Couples, a complete collection of cyclic n-proposition chains must contain at least $2(n-1)$ proposition chains. Thus, the indicated cardinalities are lower bounds. In order the establish them as upper bounds as well, we need only exhibit complete collections having these cardinalities. (In each case below, an example is provided to help clarify the construction. These examples are not part of the proof per se.)

Case 1 (*linear, n even*): In the proof of Lemma 5, in the case of n even, we constructed a complete collection of linear n-chains consisting of $n/2$ chains with the property that each couple appeared exactly once. A complete collection of n-proposition chains can be formed from four copies of this previous collection of chains by orienting links as follows: in the first copy all links have positive orientation, in the second copy all links have negative orientation, in the third copy orientations of links in each proposition chain alternate with the first link being positive, and the orientations in the last copy are precisely opposite to those in the third. The result of this procedure is a collection of $4(n/2) = 2n$ proposition chains. This collection is clearly complete. Choose integers i and j such that $0 \leqslant i < j \leqslant n$. Then, by definition, the complete collection with which we started contains a chain in which ℓ_i and ℓ_j are adjacent, say ℓ_i is first. Then this chain corresponds to four proposition chains in the collection just constructed in which the first contains $\ldots \ell_i \ell_j \ldots$, the second contains $\ldots \overline{\ell_i}\,\overline{\ell_j} \ldots$, and the third and fourth contain $\ldots \overline{\ell_i}\ell_j \ldots$ and $\ldots \overline{\ell_i}\ell_j \ldots$ in some order (depending on whether ℓ_i is in an even or odd position in the original chain).

For example, with $n = 6$ we have the following. (Notice that in this, and all subsequent examples, only the subscripts are given.)

$$
\begin{array}{cccccc@{\qquad}cccccc}
0 & 1 & 5 & 2 & 4 & 3 & \bar{0} & \bar{1} & \bar{5} & \bar{2} & \bar{4} & \bar{3} \\
1 & 2 & 0 & 3 & 5 & 4 & \bar{1} & \bar{2} & \bar{0} & \bar{3} & \bar{5} & \bar{4} \\
2 & 3 & 1 & 4 & 0 & 5 & \bar{2} & \bar{3} & \bar{1} & \bar{4} & \bar{0} & \bar{5}
\end{array}
$$

$$
\begin{array}{cccccc@{\qquad}cccccc}
0 & \bar{1} & 5 & \bar{2} & 4 & \bar{3} & \bar{0} & 1 & \bar{5} & 2 & \bar{4} & 3 \\
1 & \bar{2} & 0 & \bar{3} & 5 & \bar{4} & \bar{1} & 2 & \bar{0} & 3 & \bar{5} & 4 \\
2 & \bar{3} & 1 & \bar{4} & 0 & \bar{5} & \bar{2} & 3 & \bar{1} & 4 & \bar{0} & 5
\end{array}
$$

Case 2 (*cyclic, n odd*): We follow a procedure quite similar to that used in the proof of Lemma 5. Suppose n is odd and begin with the complete collection of linear $(n-1)$ proposition chains constructed in case 1. Insert the positively oriented ℓ_{n-1} between the first and last links in the first n proposition chains and the negatively oriented $\overline{\ell_{n-1}}$ between the first and last links in the last n proposition chains. Consider the following depiction of the collection just constructed to verify that all Couples involving the $(n-1)$st link are formed. (Only the subscripts are given in the four tables below.)

$$
\begin{array}{cccccccccc}
(n-1) & 0 & \ldots & (\frac{n-1}{2}) & (n-1) & (n-1) & \bar{0} & \ldots & \overline{(\frac{n-1}{2})} & (n-1) \\
(n-1) & 1 & \ldots & (\frac{n-1}{2}+1) & (n-1) & (n-1) & \bar{1} & \ldots & \overline{(\frac{n-1}{2}+1)} & (n-1) \\
(n-1) & 2 & \ldots & (\frac{n-1}{2}+2) & (n-1) & (n-1) & \bar{2} & \ldots & \overline{(\frac{n-1}{2}+2)} & (n-1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n-1) & (\frac{n-1}{2}-1) & \ldots & (n-2) & (n-1) & (n-1) & \overline{\frac{n-1}{2}-1} & \ldots & \overline{(n-2)} & (n-1)
\end{array}
$$

$$
\begin{array}{cccccccccc}
\overline{(n-1)} & 0 & \ldots & (\overline{\tfrac{n-1}{2}}) & \overline{(n-1)} & \overline{(n-1)} & \bar{0} & \ldots & (\frac{n-1}{2}) & \overline{(n-1)} \\
\overline{(n-1)} & 1 & \ldots & \overline{(\frac{n-1}{2}+1)} & \overline{(n-1)} & \overline{(n-1)} & \bar{1} & \ldots & (\frac{n-1}{2}+1) & \overline{(n-1)} \\
\overline{(n-1)} & 2 & \ldots & \overline{(\frac{n-1}{2}+2)} & \overline{(n-1)} & \overline{(n-1)} & \bar{2} & \ldots & (\frac{n-1}{2}+2) & \overline{(n-1)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\overline{(n-1)} & (\frac{n-1}{2}-1) & \ldots & \overline{(n-2)} & \overline{(n-1)} & \overline{(n-1)} & \overline{(\frac{n-1}{2}-1)} & \ldots & \overline{(n-2)} & \overline{(n-1)}
\end{array}
$$

For example, with $n = 7$ we have the following;

$$
\begin{array}{cccccccc@{\qquad}cccccccc}
6 & 0 & 1 & 5 & 2 & 4 & 3 & 6 & 6 & \bar{0} & \bar{1} & \bar{5} & \bar{2} & \bar{4} & \bar{3} & 6 \\
6 & 1 & 2 & 0 & 3 & 5 & 4 & 6 & 6 & \bar{1} & \bar{2} & \bar{0} & \bar{3} & \bar{5} & \bar{4} & 6 \\
6 & 2 & 3 & 1 & 4 & 0 & 5 & 6 & 6 & \bar{2} & \bar{3} & \bar{1} & \bar{4} & \bar{0} & \bar{5} & 6
\end{array}
$$

$$
\begin{array}{cccccccc@{\qquad}cccccccc}
\bar{6} & 0 & \bar{1} & 5 & \bar{2} & 4 & \bar{3} & \bar{6} & \bar{6} & \bar{0} & 1 & \bar{5} & 2 & \bar{4} & 3 & \bar{6} \\
\bar{6} & 1 & \bar{2} & 0 & \bar{3} & 5 & \bar{4} & \bar{6} & \bar{6} & \bar{1} & 2 & \bar{0} & 3 & \bar{5} & 4 & \bar{6} \\
\bar{6} & 2 & \bar{3} & 1 & \bar{4} & 0 & \bar{5} & \bar{6} & \bar{6} & \bar{2} & 3 & \bar{1} & 4 & \bar{0} & 5 & \bar{6}
\end{array}
$$

Case 3 (*linear, n odd*): This case requires a tactic different than beginning with copies of the corresponding complete collection of n-chains and orienting links in a particular way. Here we begin with the following double cover of K_n.

$$\begin{array}{cccccccccc}
0 & 1 & (n-1) & 2 & (n-2) & 3 & \ldots & \frac{n-1}{2} & \frac{n+2}{2} \\
1 & 2 & 0 & 3 & (n-1) & 4 & \ldots & \frac{n-1}{2}+1 & \frac{n+1}{2}+1 \\
2 & 3 & 1 & 4 & 0 & 5 & \ldots & \frac{n-1}{2}+2 & \frac{n+1}{2}+2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
(n-1) & 0 & (n-2) & 1 & (n-3) & 2 & \ldots & \frac{n-1}{2}-1 & \frac{n+1}{2}-1
\end{array}$$

The pattern is quite similar to the complete collection of linear n-chains when n is even, with some significant exceptions. Most notable among these is that the complete collection for n even consists of $n/2$ chains and each edge of K_n appears once, whereas this latest collection of linear n-chains for n odd consists of n chains and each edge of K_n appears twice — with its vertices listed in the same order. To verify this fact, observe that each row is generated from its first entry by adding 1 (mod n) to obtain the second entry, then subtracting 2 (mod n) to obtain the third entry, then adding 3 (mod n), then subtracting 4 (mod n), and so forth until the last entry is obtained from the previous one by subtracting $n-1$ (mod n) which is, of course, the same as adding 1 (mod n). Note also that the columns are generated from the top down by adding 1 (mod n) as before. Since there are n rows, each column contains all elements of Z_n. Thus, since the first two entries in the first row are 0 and 1, the first pair of columns contains all edges of K_n which join vertices labeled with consecutive elements of Z_n and all such edges appear with the smaller vertex listed first. But this is also true of the last pair of columns. Since two is subtracted from the second entry of each row to obtain the third entry, this pair of columns contains all edges joining vertices whose labels differ by two and all these edges appear with the larger vertex listed first. This is also true of the $(n-3)$rd and $(n-2)$nd columns where the transition rule requires adding $n-2$ or equivalently subtracting 2 (mod n). The pattern continues from both sides until we see that each edge of length $(n-1)/2$ appears twice — once immediately to the left of the $((n+1)/2)$th or middle column and once immediately to the right. This symmetry about the middle column will be used in assigning orientations to the links.

To construct the complete collection of linear n-proposition chains in this case, begin with two copies of the chain collections given in the table above (i.e., the collection representing the double covering of the edges of K_n) and orient the entries as follows: In the first copy, all links to the left of the middle column are positively oriented and the ones to the right of the middle alternate in orientation, beginning with the middle column entries being positively oriented. The orientation of each entry in the second copy is the opposite of the corresponding entry in the first copy.

Since this collection contains the prescribed number of n-proposition chains, $2n$, it remains to verify that this collection is complete. Given two links ℓ_i and ℓ_j, these appear as adjacent links twice in each half of the collection — in the same order, say ℓ_i ℓ_j. Left of the middle column in the first half of the collection both links are positively oriented and in the corresponding position in the second half they are both negatively oriented. In their other two occurrences as adjacent links they are alternately oriented in both ways (positive — negative once and negative — positive once).

For example, with $n = 7$ we have the following;

0	1	6	2	$\bar{5}$	3	$\bar{4}$		$\bar{0}$	$\bar{1}$	$\bar{6}$	$\bar{2}$	5	$\bar{3}$	4
1	2	0	3	$\bar{6}$	4	$\bar{5}$		$\bar{1}$	$\bar{2}$	$\bar{0}$	$\bar{3}$	6	$\bar{4}$	5
2	3	1	4	$\bar{0}$	5	$\bar{6}$		$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	0	$\bar{5}$	6
3	4	2	5	$\bar{1}$	6	$\bar{0}$		$\bar{3}$	$\bar{4}$	$\bar{2}$	$\bar{5}$	1	$\bar{6}$	0
4	5	3	6	$\bar{2}$	0	$\bar{1}$		$\bar{4}$	$\bar{5}$	$\bar{3}$	$\bar{6}$	2	$\bar{0}$	1
5	6	4	0	$\bar{3}$	1	$\bar{2}$		$\bar{5}$	$\bar{6}$	$\bar{4}$	$\bar{0}$	3	$\bar{1}$	2
6	0	5	1	$\bar{4}$	2	$\bar{3}$		$\bar{6}$	$\bar{0}$	$\bar{5}$	$\bar{1}$	4	$\bar{2}$	3

Case 4 (*cyclic*, *n even*): This complete collection is constructed from the one in case 3 (for $n - 1$ odd, since n is even in this case) by inserting the positively oriented ℓ_{n-1} between the first and last links in all of the proposition chains in the collection. As in case 2, we need only verify that all Couples involving link ℓ_{n-1} are formed (since the previous case demonstrates the formation of all others). For $i = 0, 1, \dots, n - 1$, row $i + 1 \pmod n$ begins $(n - 1)\, i \dots$, row $n + i + 1 \pmod n$ begins $(n - 1)\, \bar{i} \dots$, and rows $i + 1 + ((n - 1)/2) \pmod n$ and $n + i + 1 + ((n - 1)/2) \pmod n$ end in $\dots\, i\,(n - 1)$ and $\dots\, \bar{i}\,(n - 1)$ in an order determined by whether $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$. Thus all Couples involving ℓ_{n-1} are formed in the collection.

For example, with $n = 8$ we have the following;

7	0	1	6	2	$\bar{5}$	3	$\bar{4}$	7		7	$\bar{0}$	$\bar{1}$	$\bar{6}$	$\bar{2}$	5	$\bar{3}$	4	7
7	1	2	0	3	$\bar{6}$	4	$\bar{5}$	7		7	$\bar{1}$	$\bar{2}$	$\bar{0}$	$\bar{3}$	6	$\bar{4}$	5	7
7	2	3	1	4	$\bar{0}$	5	$\bar{6}$	7		7	$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	0	$\bar{5}$	6	7
7	3	4	2	5	$\bar{1}$	6	$\bar{0}$	7		7	$\bar{3}$	$\bar{4}$	$\bar{2}$	$\bar{5}$	1	$\bar{6}$	0	7
7	4	5	3	6	$\bar{2}$	0	$\bar{1}$	7		7	$\bar{4}$	$\bar{5}$	$\bar{3}$	$\bar{6}$	2	$\bar{0}$	1	7
7	5	6	4	0	$\bar{3}$	1	$\bar{2}$	7		7	$\bar{5}$	$\bar{6}$	$\bar{4}$	$\bar{0}$	3	$\bar{1}$	2	7
7	6	0	5	1	$\bar{4}$	2	$\bar{3}$	7		7	$\bar{6}$	$\bar{0}$	$\bar{5}$	$\bar{1}$	4	$\bar{2}$	3	7

The proof is completed. □

With the completion of this proof, we are ready to prove Theorem 1.

5. Theorem 1 and related results

Theorem 1. *If m and n are positive integers and H is any graph of order 2n, then*

(i) $\gamma(\overline{K_{2m}} + H) = (m - 1)(n - 1)$, *provided* $m \geqslant 2(n - 1)$, *and*

(ii) $\gamma(mK_2 + H) = (m - 1)(n - 1)$, provided $m \geqslant 2n$.

Proof. As was mentioned earlier, it suffices to show that this theorem is true for $H = K_{2n}$. This follows from the simple fact that if $G_1 \subseteq G_2 \subseteq G_3$ and if $\gamma(G_1) = \gamma(G_3)$ then $\gamma(G_1) = \gamma(G_2) = \gamma(G_3)$. In part (i) G_3 is $\overline{K_{2m}} + K_{2n}$, in part (ii) G_3 is $mK_2 + K_{2n}$, and in both parts G_1 is $K_{2m,2n}$.

Since $K_{2m,2n}$ is a subgraph of both $\overline{K_{2m}} + K_{2n}$ and $mK_2 + K_{2n}$, $\gamma(K_{2m,2n}) = (m-1)(n-1)$ is a lower bound for $\gamma(\overline{K_{2m}} + K_{2n})$ and $\gamma(mK_2 + K_{2n})$. That $(m-1)(n-1)$ is also an upper bound for the genera of these two graphs will be established by imbedding them in the surface $S(K_{m,n})$ whose genus is $(m-1)(n-1)$. Denote the partite sets of the graph $K_{m,n}$ (which underlies the surface) by $U = \{u_0, u_1, u_2, \ldots, u_{m-1}\}$ and $V = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$.

For part (i), imbed the vertices of $\overline{K_{2m}}$ as the poles of the spheres $S(u_0)$, $S(u_1), \ldots, S(u_{m-1})$ and imbed the vertices of K_{2n} as the poles of the spheres $S(v_0), S(v_1), \ldots, S(v_{n-1})$. Choose a one-to-one correspondence between the spheres $S(u_0)$, $S(u_1), \ldots, S(u_{2n-2})$ and the proposition chains in the complete collection of cyclic n-proposition chains constructed in the proof of Lemma 7. Now imbed the edges of $K_{2m,2n}$ as in the proof of Lemma 2, ordering and orienting the tubes incident with these $2n-2$ spheres according to the proposition chain to which it corresponds. (The order of attachment around the equators of all other spheres as well as the orientations of all other tubes are irrelevant and can be chosen at random.) Each Couple formed in the collection of proposition chains corresponds to the long diagonal of a region of the imbedding. Because the collection of proposition chains is complete, the long diagonals of all regions centered on the spheres $S(u_0), S(u_1), S(u_2), \ldots, S(u_{2n-2})$ are precisely the edges of K_{2n}. Imbedding these edges as diagonals through these regions completes the imbedding of $\overline{K_{2m}} + K_{2n}$ in $S(K_{m,n})$ and thus completes the proof of part (i).

The proof of part (ii) is obtained by making a few minor changes to the proof of part (i). Instead of assigning the members of a complete collection of cyclic n-proposition chains to the spheres $S(u_0), S(u_1), S(u_2), \ldots, S(u_{2n-2})$, assign the members of a complete collection of linear n-proposition chains to the spheres $S(u_0), S(u_1)$, $S(u_2), \ldots, S(u_{2n})$. Imbedding long diagonals corresponding to all Couples formed by this collection of proposition chains results in an imbedding of $\overline{K_{2m}} + K_{2n}$ in $S(K_{m,n})$. For each of the spheres $S(u_0), S(u_1), S(u_2), \ldots, S(u_{2n})$ there is a region centered on this sphere in which a diagonal has not been imbedded. (This is the region between the tubes which correspond to the end links of the proposition chain.) Imbed a short diagonal in each of these regions. Also, for each of the spheres $S(u_{2n+1}), S(u_{2n+2}), S(u_{2n+3}), \ldots, S(u_{m-1})$ choose a region centered on this sphere and imbed a short diagonal in it. These short diagonals are the edges of mK_2. This completes the imbedding of $mK_2 + K_{2n}$ in $S(K_{m,n})$ and the proof of the theorem. □

The first corollary of this theorem is offered as an example of its application.

Corollary 1. *If a, b, and c are positive integers with $b + c$ even and $a \geqslant b + c - 2$ then*

$$\gamma(K_{2a,b,c}) = \tfrac{1}{2}(a-1)(b+c-2).$$

Proof. Observe that $K_{2a,b,c} = \overline{K_{2a}} + K_{b,c}$ and apply part (i) of Theorem 1. □

The next corollary points to the fact that Theorem 1 gives an upper bound for an even order graph which has a large independent set of vertices, where 'large' is interpreted as containing approximately two-thirds of all vertices in the graph.

Corollary 2. *If G is a graph of positive even order 2n having independence number α and if $\lfloor \alpha/2 \rfloor \geqslant 2n/3$ then $\gamma(G) \leqslant (\lfloor \alpha/2 \rfloor - 1)(n - \lfloor \alpha/2 \rfloor - 1)$.*

Proof. This follows immediately, via Theorem 1, from the observation that the graph G, as described, is a subgraph of $\overline{K_{2\lfloor \alpha/2 \rfloor}} + K_{2n-2\lfloor \alpha/2 \rfloor}$, in which the order of the empty factor is at least twice that of the complete factor. □

In proving Theorem 1, the join of an empty graph with a complete graph was shown to have the same genus as the complete bipartite spanning subgraph provided the empty graph has roughly twice as many vertices as the complete graph. We will now show that this lower bound on the order of the empty graph can be reduced by restricting the second factor. In particular, we will show that if the second factor is a regular bipartite graph or any spanning subgraph of such a graph, the empty factor in the join need only have the same or larger order. Collections of proposition chains will be constructed to suit this purpose.

If each edge of the complete graph K_n is replaced by two oppositely directed arcs (or directed edges), the result is the complete symmetric digraph of order n which is denoted K_n^*. Let us briefly consider the problem of decomposing K_n^* into spanning directed paths and cycles as was done in the undirected case. When n is even, the decomposition of K_n into spanning paths can be used to decompose K_n^* into spanning directed paths. Simply create the obvious pair of directed paths from each undirected path. Similarly, the decomposition of K_n into spanning cycles can be used to decompose K_n^* into spanning directed cycles by creating two directed cycles from each undirected cycle. The remaining two cases were almost completed by Tillson [10] with the following result.

Lemma 8. *For $2n \geqslant 8$,*

(i) *K_{2n}^* can be decomposed into $2n - 1$ directed spanning cycles and*

(ii) *K_{2n-1}^* can be decomposed into $2n - 1$ directed spanning paths.*

Bermond and Faber [2] then completed the task (from our perspective) by showing that (i) and (ii) of this lemma are false when $2n = 4$ or 6. Thus our latest decomposition problem has the following solution.

Lemma 9. *For $n \geqslant 2$, K_n^* can be decomposed into*
(i) *$n - 1$ directed spanning cycles, except when $n = 4$ or 6, and*
(ii) *n directed spanning paths, except when $n = 3$ or 5.*

Each of these decompositions corresponds to a collection of n-chains in which each couple appears exactly twice — once with each possible order. What do these represent if interpreted as n-proposition chains with all links positively oriented? If the two links are $\ell_i = (a_i, b_i)$ and $\ell_j = (a_j, b_j)$, then the Couples formed by $\ldots \ell_i \ell_j \ldots$ and $\ldots \ell_j \ell_i \ldots$ are $b_i a_j$ and $b_j a_i$ respectively. If we call the coordinates of a link its poles, then these collections of proposition chains form all opposite-pole Couples (i.e., $a_i b_j$ and $b_i a_j$) and no same-pole Couples (i.e., neither $a_i a_j$ nor $b_i b_j$). If we replace the complete collections of n-proposition chains in the proof of Theorem 2 with the ones just constructed, we will have proved the following. (Note, the partite sets of H are the set of north poles and the set of south poles of the cluster of spheres that hold the vertices of H.)

Corollary 3. *If m and n are positive integers and H is any spanning subgraph of $K_{n,n}$, then*
(i) *$\gamma(\overline{K_{2m}} + H) = (m - 1)(n - 1)$, provided $m \geqslant n - 1$ and $n \neq 4, 6$ and*
(ii) *$\gamma(mK_2 + H) = (m - 1)(n - 1)$, provided $m \geqslant n$ and $n \neq 3, 5$.*

By setting $m = n$, in part (ii) of Corollary 3, and assigning the same collection of proposition chains to the spheres holding the vertices of mK_2, and imbedding long diagonals in all remaining regions (centered on these spheres), we have a result first proved by Jungerman [6] using current graphs.

Corollary 4. *If n is a positive integer, with $n \neq 3, 5$, and G and H are spanning subgraphs of $K_{n,n}$ then $\gamma(G + H) = (n - 1)^2$. In particular, $\gamma(K_{n,n,n,n}) = (n - 1)^2$ for $n \neq 3, 5$.*

(Jungerman also proved that $\gamma(K_{3,3,3,3}) \geqslant 5$ and that $\gamma(K_{5,5,5,5}) = 16$. White [13] used a voltage graph to complete the proof that $\gamma(K_{3,3,3,3}) = 5$.)

Although the argument given prior to the statement of Corollary 4 is sufficient proof, a few comments are in order. This last result on the genus of joins is the only one in which the first factor has been allowed to be *interesting*. In the construction, envision the surface as two clusters of spheres joined together by a bundle of tubes. We use the fact that the order of attachment of these tubes to the spheres in one cluster is independent of the order of attachment to the spheres of the other cluster. Just as important in imbeddings which include long diagonals in regions centered in both clusters is the issue of consistent tube orientations. When accomplishing adjacencies among the poles within one cluster of spheres, not only must the order of attachment to the spheres of the other cluster be specified, the orientation of the tubes must also be specified. To accomplish adjacencies among the poles of the other cluster, we are free to specify the order of tube attachment to spheres in the first cluster but the orientations of the tubes have already been fixed. What made the proof of Corollary 4 go through was the fact that all tubes were positively oriented. Unfortunately, this will not be the case in Theorem 2. Thus, we are left with the task of first studying the pattern of orientations within tube bundles and how modifications of our collections of proposition chains affect these patterns.

6. Theorem 2 and related results

Theorem 2. *If G is a nontrivial, connected (p,q)-graph with minimum degree at least 2, and H is a graph of positive even order $2n$, then $\gamma(G[H]) \leqslant n^2q - np + 1$, with equality if G is triangle-free.*

Proof. The first step toward proving this result was taken with Lemma 4, in which $G[\overline{K_{2n}}]$ was shown to have a quadrilateral imbedding in the surface $S(G[\overline{K_n}])$ which, by Lemma 1, has genus $n^2q - np + 1$. By specifying attachment orders and orientations of tubes, similar to what was done in proving Theorem 1, we will be able to imbed diagonals in some of the quadrilateral regions to obtain an imbedding of the graph $G[K_{2n}]$ in the same surface. Since $G[\overline{K_{2n}}] \subseteq G[H] \subseteq G[K_{2n}]$ we will, by so doing, have shown that $n^2q - np + 1$ is an upper bound for $\gamma(G[H])$. Furthermore, if G is triangle-free, so is $G[\overline{K_{2n}}]$. This implies that $\gamma(G[\overline{K_{2n}}]) = n^2q - np + 1$ which, in turn, implies that $\gamma(G[H]) = n^2q - np + 1$. Thus, the only bit of work remaining is to imbed $G[K_{2n}]$ in $S(G[\overline{K_n}])$ beginning with the quadrilateral imbedding of $G[\overline{K_{2n}}]$ in this surface.

Visualize the surface $S(G[\overline{K_n}])$ as being constructed from G by replacing each of its p vertices with a cluster of n spheres and each edge uv with a bundle of tubes joining each sphere in the u-cluster to every sphere in the v-cluster. In particular, in forming $G[\overline{K_n}]$ from G, each vertex $u \in V(G)$ is replaced with n vertices denoted $u_0, u_1, u_2, \ldots, u_{n-1}$. The *u-cluster* refers to the spheres $S(u_0)$, $S(u_1)$, $S(u_2)$, $\ldots$, $S(u_{n-1})$. For each edge $uv \in E(G)$, the *uv-bundle* refers to the set of tubes $\{T(u_iv_j)\colon i, j \in Z_n\}$. Each of these tubes carries the four edges which join the poles of one of its end spheres to the poles of the other. This is the quadrilateral imbedding of $G[\overline{K_{2n}}]$. Now, for each cluster, we must construct an imbedding which contains a collection of regions in which to imbed all edges (as diagonals of these regions) necessary to join all poles within the cluster to each other. Furthermore, this must be done so as not to interfere with the same process taking place for all other clusters.

The first step is to separate tube bundles. For each vertex $u \in V(G)$ carry out the following procedure. Let $\deg_G u = d$ and recall that, by virtue of the minimum degree requirement on G, $d \geqslant 2$. Attached around the equator of each sphere $S(u_i)$ in the u-cluster are the tubes of d incident bundles. Partition each such equator into

d connected sections and assign the incident bundles to these sections; i.e., the tubes in a particular bundle are all attached to $S(u_i)$ within the same section and tubes in different bundles are attached to $S(u_i)$ within different sections of its equator.

As a beginning to the process of imbedding the necessary additional edges, choose any point on the equator of $S(u_i)$ which separates two sections and imbed a short diagonal through the region (centered on $S(u_i)$) which contains this point. Note that we will not need to imbed a long diagonal through this region as such a diagonal would join poles of spheres in different clusters. It remains to imbed edges which join poles of different spheres within the u-cluster. These must be long diagonals of regions centered on spheres within clusters adjacent to the u-cluster.

Choose two vertices $v, w \in V(G)$ which are adjacent to u. Together, the v- and w-clusters contain $2n$ spheres. If we were allowed to ignore the part of the surface other than these three sphere clusters and the portion of the composition graph imbedded thereon, our problem would be one that has already been solved. That is, order and orient the tubes in the uv and uw-bundles by assigning each of the spheres in the v- and w-clusters an element of a complete collection of linear n-proposition chains. The result would be the imbedding of $\overline{K_{4n}} + K_{2n}$, constructed in part (ii) of Theorem 1, with the vertices of the complete factor residing at the poles of the u-cluster and the vertices of the empty factor residing at the poles of the v- and w-clusters. Unfortunately, the tube orientations so assigned may not be those we might wish to assign these same tubes in attempting to imbed the edges joining the poles of the v-cluster to one another or those of the w-cluster to one another. This is the problem of conflicting orientation patterns mentioned above. In order to overcome this obstacle, we will reexamine our complete collections of linear n-proposition chains and modify them, if necessary, to meet the following criteria: that all bundles have the same orientation pattern and that this pattern be symmetric as defined below.

In the surface $S(G[\overline{K_n}])$, in which the graph $G[\overline{K_{2n}}]$ has been imbedded, each tube is oriented by edges of the imbedded graph. Observe that each sphere cluster can be seen to represent Z_n under the natural correspondence. For example, in the u-cluster this correspondence is $i \leftrightarrow S(u_i)$. For each edge $uv \in V(G)$, the negatively oriented edges of the uv-bundle can be used to define two relations R_{uv} and R_{vu} on the set Z_n. Namely, $(i, j) \in R_{uv}$ if and only if the tube $T(u_i, v_j)$ is negatively oriented. Similarly define R_{vu}. We will say the uv-bundle has a *symmetric orientation pattern*, or that it is *symmetric* if and only if the relation R_{uv} is symmetric. Thus, to say that the uv-bundle is symmetric is to say that $R_{uv} = R_{vu}$.

Before launching into a reconfiguration of collections of proportion chains, let us spend a little more time discussing relations. A relation on Z_n can be given by (vertically) listing all elements of Z_n, with each of these followed by a (horizontal) list of all elements of Z_n related to it. Also, such a relation can be diagrammed using a bipartite graph with the partite sets representing the domain and codomain of the relation and the edges representing the related elements. Such a diagram is very much a picture of the orientation pattern of the corresponding tube bundle. Fig. 5 shows an example of a symmetric relation on Z_6:

0:
1: 1
2: 2, 4
3: 3, 4, 5
4: 2, 3
5: 3

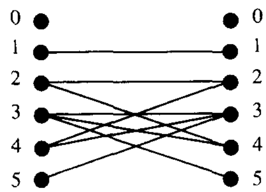

Fig. 5.

If R is a relation on Z_n, and $i \in Z_n$, define the neighborhood of i by $N_i = \{j: (i,j) \in R\}$. For instance, in the example just given, $N_2 = \{2, 4\}$. It should be clear that reassigning the six neighborhoods to the elements of Z_6 will result in an entirely different relation on Z_6 which may not be symmetric.

We are now in a position to complete the proof. Recall that our goal is to find, for each positive integer n, a complete collection of linear n-proposition chains together with an assignment of these to the spheres in the v- and w-clusters, so that the uv- and uw-bundles have identical symmetric orientation patterns.

Case 1 (*n odd*): Refer to the collection of proposition chains constructed in case 3 of the proof of Lemma 7. Recall that we began with a double cover of K_n, oriented the links of each of the n elements of this double cover by positively orienting the first half of the links in each chain and alternating orientations of the remaining links, and finally created another set of n proposition chains from this first set by changing the orientations of all links. An alternate approach would have been to create the second set from the first, not by changing all orientations, but by reversing the order of links in all the proposition chains. That the second collection is complete, i.e., that the two collections are equivalent, can be verified by observing that the first n proposition chains of these collections are identical and that the second n are contrapositives. We will use this alternate collection by assigning the first n proposition chains to the spheres in the v-cluster so that the resulting orientation pattern for the uv-bundle is symmetric. Then the second set of proposition chains, whose links have exactly the same orientations as their counterparts, can be assigned to the corresponding spheres in the w-cluster to result in the uw-bundle having the same symmetric orientation pattern. Thus, we need only be concerned with making the uv-bundle symmetric.

For each proposition chain, the list of negatively oriented links will be the neighborhood of the sphere (i.e., the corresponding element of Z_n) to which it is assigned. Therefore the first n proposition chains in the complete collection supply us with n neighborhoods which will result in a symmetric relation when properly assigned to the elements of Z_n. In the first proposition chain of the collection, the negative links are $\lceil (n-1)/4 \rceil$ consecutive elements of Z_n. Since the remaining members of the collection are obtained by repeatedly adding 1 (mod n), $n-1$ times, the set of neighborhoods from which our relation will be built consists of all possible sets of $\lceil (n-1)/4 \rceil$ consecutive elements of Z_n. If $\lceil (n-1)/4 \rceil$ is odd (i.e., if $n \equiv 3$ or $5 \pmod 8$)

assign each of these neighborhoods to the element of Z_n which is the center of the neighborhood.

For example, the relation for $n = 11$ would be

0: 10, 0, 1	6: 5, 6, 7
1: 0, 1, 2	7: 6, 7, 8
2: 1, 2, 3	8: 7, 8, 9
3: 2, 3, 4	9: 8, 9, 10
4: 3, 4, 0	10: 9, 10, 0
5: 4, 5, 6	

If, on the other hand $\lceil (n-1)/4 \rceil$ is even (i.e., if $n \equiv 1$ or 7 (mod 8)) assign each of these neighborhoods to the element of Z_n which is the center of the complement of the neighborhood.

For example, the relation for $n = 7$ would be

0: 3, 4	4: 0, 1
1: 4, 5	5: 1, 2
2: 5, 6	6: 2, 3
3: 6, 0	

If $i, j \in Z_n$ and $d \leqslant n/2$, then any relation R defined on Z_n by '$(i, j) \in R$ if and only if $|i - j| < d$' is clearly symmetric. The relation given for $n \equiv 3$ or 5 (mod 8) has this form and the one for $n \equiv 1$ or 7 (mod 8) is the complement of such a relation. Thus both are symmetric.

Fig. 6 shows a diagram of the uv- and uw-bundles when $n = 7$. The three columns of vertices represent, from left to right, the v-, u-, and w-clusters of spheres. In each cluster, the spheres are numbered $0, 1, 2, \ldots, 6$ from the top down. The assignments of proposition chains to the spheres in the v- and w-clusters are given to the far left and right, respectively. The edges represent the negatively oriented tubes. The missing edges are the positively oriented tubes.

Case 2 (*n even*): Although case 1 in the proof of Lemma 7 was the easiest of the four cases, the complete collection constructed there cannot be used here since no amount of rearranging the corresponding neighborhoods will result in a symmetric relation. However, a few modifications will suffice.

Recall that we began with a decomposition of K_n into $n/2$ spanning paths corresponding to $n/2$ linear n-chains. For each of these, the links were oriented in four different ways yielding four linear n-proposition chains. These orientations were (i) all positive, (ii) all negative, (iii) alternating with odd terms positive and even terms negative, and (iv) alternating with even terms positive and odd terms negative. (Also recall, in (iii) and (iv), *even* and *odd* refer to the position of the link in the chain.) As our first modification, replace all proposition chains having type (ii) and (iv) orientations with their contrapositives. Call these replacement orientations (ii′) and (iv′) and observe that, unlike the original four types, these involve a change in order from the initial collection of chains. A type (ii′) proposition chain can be obtained from a type (i)

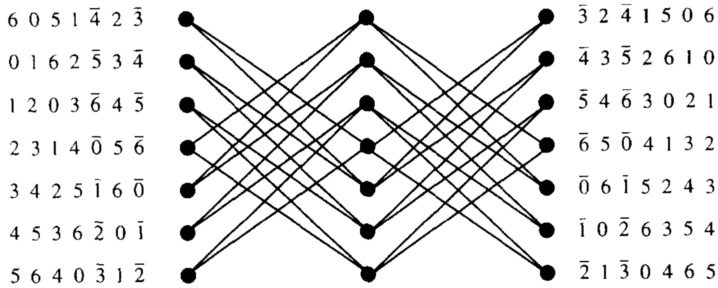

Fig. 6.

proposition chain by writing the oriented links in the opposite order. The same relationship holds for type (iii) and (iv′) proposition chains. Thus the collection of all type (i) proposition chains together with all type (iii) proposition chains represents the same set of (relation) neighborhoods as does the collection of all proposition chains with the modified orientations. We now concentrate on further modifying the (i) & (iii) collection with a view toward rendering the corresponding relation symmetric. Success in this endeavor will also render the relation corresponding to the (ii′) & (iv′) collection symmetric as well, under analogous modifications.

The following is an example of the various types of proposition chains derived from the 6-chain 0 1 5 2 4 3:

(i) $0\,1\,5\,2\,4\,3$ (ii) $\bar{0}\,\bar{1}\,\bar{5}\,\bar{2}\,\bar{4}\,\bar{3}$ (ii′) $3\,4\,2\,5\,1\,0$

(iii) $0\,\bar{1}\,5\,\bar{2}\,4\,\bar{3}$ (iv) $\bar{0}\,1\,\bar{5}\,2\,\bar{4}\,3$ (iv′) $\bar{3}\,4\,\bar{2}\,5\,\bar{1}\,0$

Consider the type (i) and (iii) proposition chains in the preceding example. The associated neighborhoods are ø and $\{1, 2, 3\}$. Notice that the set of Couples formed by these two proposition chains can be formed by any two proposition chains with the same underlying chain provided the links 0, 5, and 4 are positively oriented in both and the links 1, 2, and 3 are oppositely oriented in the two proposition chains. For example, $0\,1\,5\,\bar{2}\,4\,3$ and $0\,\bar{1}\,5\,2\,4\,\bar{3}$ form the same Couples as do $0\,1\,5\,2\,4\,3$ and $0\,\bar{1}\,5\,\bar{2}\,4\,\bar{3}$. Thus, in modifying the (i) & (iii) collection in constructing a symmetric relation, the neighborhood N from a type (iii) proposition chain and the empty neighborhood from the corresponding type (i) proposition chain can be replaced by any pair of disjoint sets whose union is N.

In general, the neighborhoods corresponding to the type (iii) proposition chains are $M_i = \{i, i+1, \ldots, k+i-1\}$ for $i = 1, 2, \ldots, k$ where $k = n/2$. By splitting each of these into two pieces and assigning them to the elements of Z_n, we have the following relation $R(k)$, defined in terms of its neighborhoods.

$$N_0 = \emptyset, \qquad N_1 = \{1\},$$

$$N_i = \{i\} \cup \{k+1, k+2, \ldots, k+i-1\} \quad \text{for } i = 2, 3, \ldots, k,$$

$$N_i = \{i-k+1, i-k+2, \ldots, k\} \quad \text{for } i = k+1, k+2, \ldots, n-1.$$

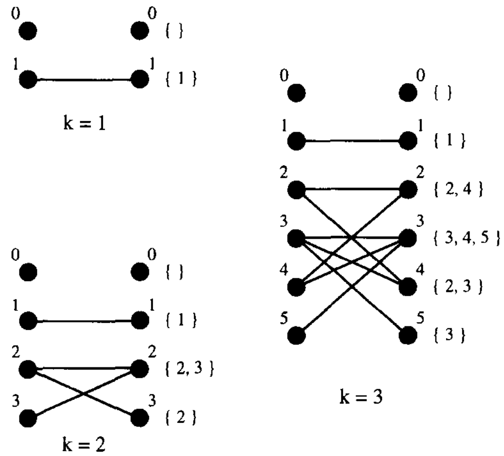

Fig. 7.

Observe that $N_0 \cup N_k = M_k$, and $N_i \cup N_{k+i} = M_i$ for $i = 1, 2, \dots, k-1$. Thus, we have demonstrated that we can construct a complete collection of linear n-proposition chains and divide it into two equal parts so that each part corresponds to the relation $R(k)$ just defined. We proceed by induction to show that $R(k)$ is symmetric for all positive integers k. First note that $R(1) = \{(1, 1)\}$ which is clearly symmetric. The reader can visually verify symmetry in the smallest three cases in Fig. 7.

Now let $n = 2k$ and assume that $R(k-1)$ is symmetric for some fixed $k \geqslant 2$. Define a relation $\varphi R(k-1)$ on Z_n as the image of $R(k-1)$ under the mapping $\varphi : Z_{n-1} \times Z_{n-1} \to Z_n \times Z_n$ defined by $\varphi : (i, j) \mapsto (i+1, j+1)$. Note that symmetry is preserved under this mapping. A careful comparison of $\varphi R(k-1)$ and $R(k)$ reveals that $\varphi R(k-1) \subseteq R(k)$ with $R(k) - \varphi R(k-1)$ consisting of $(1, 1)$ and the symmetric pairs $(i, k+i-1)$ and $(k+i-1, i)$ for $i = 2, 3, \dots, k$. Thus $R(k)$, as the disjoint union of symmetric relations, must also be symmetric.

As an example, Fig. 8 shows the uv- and uw-bundles when $n = 6$. The three columns of vertices represent, from left to right, the v-, u-, and w-clusters of spheres. In each cluster, the spheres are numbered $0, 1, 2, \dots, 5$ from the top down. The assignments of proposition chains to the spheres in the v- and w-clusters are given to the far left and right. The edges represent the negatively oriented tubes. The missing edges are the positively oriented tubes.

This completes case 2 and the proof of Theorem 2. □

A straightforward application of the famous theorem by Battle et al. [1] gives the following generalization of Theorem 2.

Corollary 5. *Let G be a (p, q)-graph containing no vertices of degree 1 and let H be a graph of positive even order $2n$. If G contains m isolated vertices and has k nontrivial*

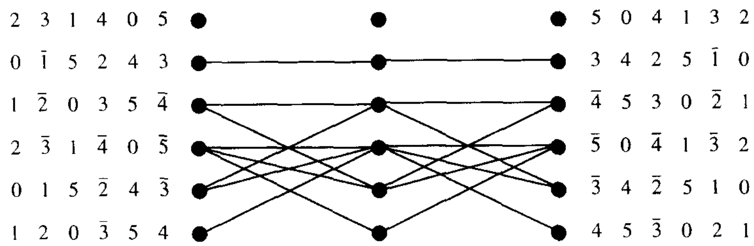

Fig. 8.

connected components then $\gamma(G[H]) \leqslant m\gamma(H) + n^2q - n(p-m) + k$ *with equality if* G *is triangle-free.*

The following three corollaries are offered as examples of the utility of Theorem 2.

Corollary 6. *For integers* $m \geqslant 4$ *and* $n \geqslant 3$,

$$\gamma(C_m[C_{2n}]) = n^2m - nm + 1.$$

Corollary 7. *For positive integers* m *and* n,

$$\gamma(Q_m[Q_n]) = m \cdot 2^{m+2n-3} - 2^{m+n-1} + 1.$$

Corollary 8. *For integers* m, n, a, *and* b *with* $m, n \geqslant 2$ *and* $a + b = 2k$,

$$\gamma(K_{m,n}[K_{a,b}]) = mnk^2 - 2(m+n)k + 1.$$

The next result follows from Theorem 2 in much the same way that Corollary 3 followed from Theorem 1, replacing the complete collection of proposition chains with the one used in part (ii) of Corollary 3—resulting in all positive tubes. The elimination of the minimum degree requirement (compared to Theorem 2) is due to the smaller order of H as well as the uniformity of tube orientations. It is presented without further proof.

Corollary 9. *If* G *is a nontrivial, connected* (p, q)*-graph, and* H *is any spanning subgraph of* $K_{n,n}$ *then* $\gamma(G[H]) \leqslant n^2q - np + 1$, *with equality if* G *is triangle-free.*

Our last result is simply a particular application of Corollary 9.

Corollary 10. *If* T *is a nontrivial tree of order* p *and* n *is any positive integer, then*

$$\gamma(T[K_{n,n}]) = n^2(p-1) - np + 1.$$

References

[1] J. Battle, F. Harary, Y. Kodama and J.W.T. Youngs, Additivity of the genus of a graph, Bull. Amer. Math. Soc. 68 (1962) 565–568.
[2] J.C. Bermond and V. Faber, Decomposition of the complete directed graph into k-circuits, J. Combin. Theory Ser. B 21 (1976) 146–155.
[3] G. Chartrand and L. Lesniak, Graphs and Digraphs (Wadsworth & Brooks/Cole, Monterey, CA, 2nd ed., 1986).
[4] D. Craft, Surgical techniques for constructing minimal orientable imbeddings of joins and compositions of graphs, Ph.D. Dissertation, Western Michigan University, Kalamazoo, MI, 1991.
[5] P.J. Heawood, Map-colour theorem, Quart. J. Pure Appl. Math. 24 (1890) 332–338.
[6] M. Jungerman, The genus of the symmetric quadripartite graph, J. Combin. Theory Ser. B 19 (1975) 181–187.
[7] G. Ringel, Das geschlecht des vollständigen paaren graphen, Abh. Math. Sem Univ. Hamburg. 28 (1965) 139–150.
[8] G. Ringel, Map color Theorem (Springer, Berlin, 1974).
[9] S. Stahl, The embeddings of a graph — a survey, J. Graph Theory 2 (1978) 275–298.
[10] T.W. Tillson, A hamiltonian decomposition of K^*_{2m}, $2m \geqslant 8$, J. Combin. Theory Ser. B 29 (1980) 68–74.
[11] A.T. White, On the genus of the composition of two graphs, Pacific J. Math. 41 (1972) 275–279.
[12] A.T. White, Graphs of groups on surfaces, in: C.P.J. Cameron, ed., Combinatorial Surveys: Proc. 6th British Combinatorial Conf. (1977) 165–197.
[13] A.T. White, Graphs, Groups, and Surfaces (North-Holland, New York, revised ed., 1984).