Cayley digraphs with normal adjacency matrices

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A R T I C L E   I N F O

Article history:
Received 22 March 2004
Accepted 23 March 2007
Available online 14 March 2009

Keywords:
Cayley digraph
Abelian group
Quaternion group
Adjacency matrix
Normal matrix

A B S T R A C T

In this paper we give a criterion for the adjacency matrix of a Cayley digraph to be normal in terms of the Cayley subset $S$. It is shown with the use of this result that the adjacency matrix of every Cayley digraph on a finite group $G$ is normal iff $G$ is either abelian or has the form $Q_8 \times \mathbb{Z}_n^2$ for some non-negative integer $n$, where $Q_8$ is the quaternion group and $\mathbb{Z}_n^2$ is the abelian group of order $2^n$ and exponent 2.

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1. Introduction

Let $G$ be a finite group and $S$ be a Cayley subset of $G$ (i.e. any two of its elements are distinct and different from the identity element $e$). The Cayley digraph $D(G, S)$ is the directed graph whose vertices are the elements of $G$ and whose arc-set is $\{(g, sg) : g \in G, s \in S\}$. By definition, the $(g, h)$th element of the adjacency matrix of $D(G, S)$ is the number of arcs from $g$ to $h$ in $D(G, S)$. Cayley (undirected) graphs and their spectra are studied in many works (see, for example, [2–4, 15]). In our paper we deal with Cayley digraphs with normal adjacency matrices. Such digraphs are called normal in [10, 14, 22, 23]. However, we do not adopt this convention, since according to a more standard definition, the Cayley digraph $D(G, S)$ is normal if the group of right translations by elements of $G$ is a normal subgroup in the full automorphism group of $D(G, S)$ (see [27]).

Notice that one can also define the arc-set as $\{(g, gs) : g \in G, s \in S\}$. It is well known that the two digraphs need not be isomorphic to each other, but they have many properties in common. In particular, their adjacency matrices have the same spectrum and the same number of ones. By definition, a matrix $A$ with real entries is normal if it commutes with its transpose $A^T$. This is equivalent to the following condition on $A$ (see [7]): the sum of squares of the absolute values of its eigenvalues equals the sum of squares of the absolute values of its entries. Thus, if the adjacency matrix of one of the introduced Cayley digraphs is normal, then the same holds for the adjacency matrix of the other.

Obviously, the Cayley digraph $D(G, S)$ can be considered as an undirected graph if and only if $S^{-1} = S$. From this it follows that the adjacency matrix of every Cayley digraph on $G$ is symmetric iff $G$ has the form $\mathbb{Z}_2^n = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where $n$ belongs to the set $\mathbb{Z}_+$ of non-negative integers (if $n = 0$, we assume that $\mathbb{Z}_2^n$ consists only of the identity element). In Section 2 we show that the adjacency matrix of $D(G, S)$ is normal iff $SS^{-1} = S^{-1}S$. Using this criterion and Dedekind’s theorem on Hamiltonian groups, it is proved in Section 3 that the adjacency matrix of every Cayley digraph on $G$ is normal iff either $G$ is abelian or $G$ has the form $Q_8 \times \mathbb{Z}_n^2$, where $n \in \mathbb{Z}_+$ and $Q_8$ is the quaternion group. This theorem is the main result of our paper. Finally, in Section 4 we describe groups on which there exists a strongly connected Cayley digraph of degree two whose adjacency matrix is normal.

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doi:10.1016/j.disc.2007.03.086
2. Criteria for the adjacency matrix of a Cayley digraph to be normal and their consequences

Let $A(G, S)$ be the adjacency matrix of the Cayley digraph on $G$ with respect to $S$. The following proposition is a criterion for $A(G, S)$ to be normal in terms of $S$.

**Lemma 1.** Let $G$ be a finite group and $S$ be a Cayley subset of $G$. Then $A(G, S)$ is normal if and only if

$$S^{-1}S = SS^{-1}.$$  

**Proof.** The adjacency matrix of a digraph is normal iff for any two vertices $g$ and $h$, the number of their common in-neighbors equals the number of their common out-neighbors. For a Cayley digraph $D(G, S)$, the number of common out-neighbors of $g$ and $h$ equals the multiplicity of $gh^{-1}$ in $S^{-1}S$. Similarly, the number of common in-neighbors of $g$ and $h$ equals the multiplicity of $gh^{-1}$ in the multiset $SS^{-1}$. When both $g$ and $h$ run over $G$, the element $gh^{-1}$ also runs over $G$. Hence $S^{-1}S = SS^{-1}$ as multisets. Repeating the above arguments in reverse order we obtain that the condition $S^{-1}S = SS^{-1}$ is also sufficient for $A(G, S)$ to be normal. The lemma is proved.  

**Remark 1.** A similar criterion was obtained in [10] for a Cayley digraph to be a normally regular digraph.

For a two-element Cayley subset $S = \{s_1, s_2\}$, the condition $S^{-1}S = SS^{-1}$ can be rewritten as $\{s_1^{-1}s_2, s_2^{-1}s_1\} = \{s_1s_2^{-1}, s_2s_1^{-1}\}$. In this case the criterion given by Lemma 1 takes the following very simple form.

**Corollary 1.** The adjacency matrix $A(G, \{s_1, s_2\})$ is normal if and only if either $s_1s_2 = s_2s_1$ or $s_1^{-1}s_2 = s_2^{-1}s_1$.

It is interesting to compare this criterion with analogous results involving Cayley digraphs of degree two. For instance, in [20] it is shown that the Cayley digraph with respect to the Cayley subset $\{s_1, s_2\}$ is the pattern of a unitary matrix iff $s_1s_2^{-1} = s_2s_1^{-1}$. Moreover, in [24] all commutative weakly distance-regular digraphs of degree two are determined in terms of Cayley digraphs on abelian groups.

If $G$ is an abelian group, then Lemma 1 implies that the adjacency matrix of every Cayley digraph on $G$ is normal (the combinatorial proof of this statement for cyclic groups is given in [21]). This fact is very useful and often helps to show that some regular digraphs cannot be represented as Cayley digraphs on abelian groups. For instance, according to [5], a strongly-regular digraph $D$ with parameters $(n, k, \mu, \lambda, \tau)$ is a regular digraph on $n$ vertices with degree $k$ whose adjacency matrix $A$ satisfies the equation

$$A^2 = tE + \lambda A + \mu (J - E - A),$$

where $E$ is the identity matrix and $J$ is the matrix with all entries equal to one. It was proved therein that $A$ has integer (and, therefore, real) eigenvalues when $0 < t < k$. Assume that the parameter $t$ satisfies this condition and suppose that $A$ is a normal matrix. Then $A$ must be symmetric. In particular, the corresponding digraph $D$ must be undirected and therefore $t = k$. This contradiction shows that a strongly-regular digraph with parameters $k$ and $t$ such that $0 < t < k$ cannot be a Cayley digraph on an abelian group (recall that its adjacency matrix must be normal). This observation was first proved in [11] (see Theorem 5). The proof therein is based on other ideas and uses some results from [12].

We say that two subsets $S_1$ and $S_2$ commute if any element of $S_1$ commutes with every element of $S_2$. In what follows, we shall need the following criterion.

**Lemma 2.** Let $S_1$ and $S_2$ be commuting Cayley subsets of $G$ which have no common elements. Assume that $A(G, S_1)$ is normal. Then $A(G, S_1 \cup S_2)$ is normal if and only if $A(G, S_2)$ is normal.

**Proof.** Denote $S_1 \cup S_2$ by $S$. It is not difficult to show that

$$SS^{-1} = S_1S_1^{-1} \cup S_1S_2^{-1} \cup S_2S_1^{-1} \cup S_2S_2^{-1}$$

and

$$S^{-1}S = S_1^{-1}S_1 \cup S_1^{-1}S_2 \cup S_2^{-1}S_1 \cup S_2^{-1}S_2.$$  

The inverse element $s^{-1}$ is a power of the original element $s$. Since $S_1$ and $S_2$ commute, we have $S_1S_2^{-1} = S_2^{-1}S_1$ and $S_2S_1^{-1} = S_1^{-1}S_2$. Moreover, assume that $S_1S_1^{-1} = S_1^{-1}S_1$. Then $SS^{-1} = S^{-1}S$ if $S_2S_2^{-1} = S_2^{-1}S_2$. The lemma is proved.  

The statement of Corollary 1 can be generalized to Cayley subsets containing more than two elements.

**Corollary 2.** Let $G$ be a finite group and $S$ be a Cayley subset of $G$. Assume that $S$ can be partitioned into commuting subsets of elements having the same square. Then the adjacency matrix $A(G, S)$ is normal.

**Proof.** Let $S = S_1 \cup \ldots \cup S_r$ be a partition of the set $S$ into its subsets. Assume that for every $i = 1, \ldots, r$, any two elements $s$ and $s'$ in $S_i$ have the same square. This condition is equivalent to $ss^{-1} = s's'$ and therefore implies $S_iS_i^{-1} = S_i^{-1}S_i$. By Lemma 1, the matrix $A(G, S_i)$ is normal for every $i = 1, \ldots, r$. Moreover, if any two $S_i$ and $S_j$ commute, then the matrix $A(G, S)$ is also normal (see Lemma 2). The corollary is proved.  

**Remark 2.** If the Cayley subset $S$ in the condition of Corollary 2 generates the group $G$, then $G$ is a quotient of the direct product $(S_1) \times \cdots \times (S_r)$, where the group $(S_i)$ is generated by the set $S_i$.  

*References*

By definition, $S$ is a minimal generating subset of $G$ if $S$ generates $G$ and every element in $S$ is not a product of powers of the other elements of $S$ (in particular, $S$ is an independent set in $D(G, S)$). In this case the Cayley digraph $D(G, S)$ is said to be minimal (for example, see [17]). The following statement shows that if $S$ is minimal and $A(G, S)$ is normal, then the subsets of elements having the same square are the building blocks of $S$ (cp. with Lemma 1 [16]).

**Proposition 1.** Let $G$ be a finite group and $S$ be a minimal generating subset of $G$. Then $A(G, S)$ is normal if and only if the set $S$ can be partitioned into commuting subsets of elements having the same square.

**Proof.** By Corollary 2, we must only show the necessity of the conditions. Assume that $S = \{s_1, \ldots, s_d\}$ is minimal and $A(G, S)$ is normal. By Lemma 1, in this case $SS^{-1} = S^{-1}S$. If $s_j S_j^{-1}$ is equal to $s_i S_i^{-1}$, then the (unordered) pair of the indices $i$ and $j$ coincides with the pair of the indices $k$ and $\ell$. Indeed, in the opposite case some element in the set $\{s_i, s_j, s_k, s_\ell\}$ is a product of powers of the others, a contradiction to the definition of a minimal Cayley subset. So, either $s_j S_j^{-1} = s_j S_j^{-1}$ or $s_i S_i^{-1} = s_i S_i^{-1}$. These equalities are equivalent to the relations $s_j s_i = s_i s_j$ and $s_j^2 = s_i^2$, respectively. Thus, any two non-commuting elements of $S$ must have the same square. The proposition is proved.

Let $D(G, S)$ be an arbitrary Cayley digraph (not necessarily, with normal adjacency matrix). It is not difficult to show that if $S$ is a minimal generating subset of $G$, then any two elements $s_i$ and $s_j$ of $S$ have at most two common out-neighbors in $D(G, S)$. More precisely, only the vertices $s_j s_i$ ($= s_i s_j$) and $s_j^2$ ($= s_i^2$) are their possible common out-neighbors (the arguments here are the very same as in the proof of Proposition 1). This simple observation allows us to reformulate Proposition 1 in pure graph theoretic terms.

**Corollary 3.** Let $G$ be a finite group and $S$ be a minimal generating subset of $G$. Then $A(G, S)$ is normal if and only if any two elements of $S$ have at least one common out-neighbor in $D(G, S)$.

**Remark 3.** If any two elements of $S$ have exactly two common out-neighbors in $D(G, S)$, then $G = \mathbb{Z}_2^{|S|} \times \mathbb{Z}_{2m}$ for some integer $m \geq 1$. In particular, if $G = \mathbb{Z}_2^{|S|}$, then $D(G, S)$ coincides with the $|S|$-dimensional hypercube $Q_{|S|}$ (this graph is isomorphic to the Cartesian product of $|S|$ copies of $K_2$).

Obviously, not every group admits a minimal Cayley digraph whose adjacency matrix is normal. Indeed, it is not difficult to show that if the order of $G$ is odd, then there are no two distinct elements $s_i$ and $s_j$ of $G$ with $s_i^2 = s_j^2$. This fact and Proposition 1 imply the following statement (cp. with Lemma 3.1 [17]).

**Corollary 4.** Let $G$ be a finite group of odd order and $S$ be a minimal generating subset of $G$. Then $A(G, S)$ is normal if and only if $G$ is abelian.

Let us now return to the case of an arbitrary Cayley subset $S$. Another useful criterion for $A(G, S)$ to be normal may be formulated as follows: the adjacency matrix $A(G, S)$ is normal iff

$$|gS \cap S| = |S \cap Sg|$$

for every $g \in G$ (here $|\cdot|$ is the number of elements in the set). This criterion is often more convenient than Lemma 1. For instance, according to Wang and Xu [26], $D(G, S)$ is called a quasialbelian Cayley digraph if $S$ is a normal Cayley subset, i.e. $S$ is a union of the conjugacy classes of $G$ (we refer the reader to [29] for an extensive list of references to works on quasialbelian Cayley digraphs and their applications). In this case $gS \cap S = S$ and therefore $gS = Sg$ for every $g \in G$. This fact and the above criterion, which involves all the elements of $G$, imply directly that the adjacency matrix of every quasialbelian Cayley digraph is normal (by the way, quasialbelian Cayley digraphs were also called normal Cayley digraphs in [13, 18]). This simple observation and the results of [9, 31], in which the eigenvalues of $D(G, S)$ were expressed via the characters of $G$, show that the spectral properties of Cayley digraphs with respect to normal Cayley subsets are close to those of Cayley digraphs on abelian groups and therefore the term “quasialbelian” is acceptable for them, indeed.

### 3. Groups on which every Cayley digraph of degree two has normal adjacency matrix

The quaternion group $Q_8$ is usually presented by means of the defining relations:

$$Q_8 = \langle a, b \mid a^4 = e, b^2 = a^2, b^{-1}ab = a^{-1} \rangle.$$

It is also well known that $1, -1, i, j, k$ and their inverses $-i$, $-j$, $-k$ form $Q_8$ (one can take any two non-commuting quaternion units, for example, $i$ and $j$, as the generating elements $a$ and $b$). Obviously, for any element $g \in Q_8$ not belonging to the center $Z(Q_8) = \{1, -1\}$, we have $g^2 = -1$. Moreover, a central element coinciding with the square of every non-central element also exists for the group $Q_8 \times \mathbb{Z}_{2^n}$, where $n \in \mathbb{Z}_{+}$, and can be represented as the direct product of $-1$ and the identity element of $\mathbb{Z}_{2^n}$. By Corollary 2, if all the elements of a Cayley subset have the same square, then the corresponding adjacency matrix is normal. Moreover, checking whether $A(G, S)$ is normal or not, one can remove central elements from the Cayley subset $S$ (see Lemma 2). This implies that the adjacency matrix of every Cayley digraph on $Q_8 \times \mathbb{Z}_{2^n}$ is normal. To prove that there are no other examples of such non-abelian groups, we need the following simple result on Cayley digraphs of degree two.
Lemma 3. Let \( x \) and \( y \) be any two elements of \( G \). Assume that both \( A(G, \{x, y\}) \) and \( A(G, \{y, x\}) \) are normal. Then
\[
either y^{-1}xy = x \quad or \quad y^{-1}xy = x^{-1}.
\]

Proof. Assume that \([x, y] \neq e\). Since \( A(G, \{x, y\}) \) is normal, we have \( x^2 = y^2 \) by Corollary 1. The fact that \( A(G, \{y, x\}) \) is normal provides \( yxy = x \) (note that \([y, x] = [y, x] \neq e\)). Multiplying both sides of this identity by \( y^{-2} \) from the left and recalling that \( y^{-2} = x^{-2} \), we obtain the desired relation. The lemma is proved. \( \square \)

Lemma 3 implies that if the adjacency matrix of every Cayley digraph of degree two on \( G \) is normal, then any subgroup of \( G \) is normal in it (so, there is a relation between the notions of “normality” in linear algebra and group theory). Such a non-abelian group is called Hamiltonian. By Dedekind’s theorem (see Satz 7.12 on p. 308 in [8]), every finite Hamiltonian group is a direct product of the form \( Q_8 \times Z_2^n \times O \), where \( n \in Z_+ \) and \( O \) is an abelian group of odd order (see also Chapter IV Section 6 of [28]). Lemma 3 also implies that the conjugacy class for every non-central element of \( G \) consists of the element itself and its inverse. In this case we have \(|O| = 1\). This gives us the following statement whose independent proof is also presented below.

Theorem 1. Let \( G \) be a finite group. Assume that the adjacency matrix of every Cayley digraph of degree two on \( G \) is normal. Then either \( G \) is abelian or \( G \equiv Q_8 \times Z_2^n \) for some \( n \in Z_+ \).

Proof. Assume that there exist \( a \) and \( b \) such that \( ab \neq ba \). By Corollary 1, we have \( a^2 = b^2 \). Moreover, Lemma 3 implies \( b^{-1}ab = a^{-1} \). Hence, \( a^2 = b^{-1}ab = (b^{-1}ab)(b^{-1}ab) = a^{-2} \) and therefore \( a^4 = e \). In this case, the subgroup \( Q \) generated by \( a \) and \( b \) is isomorphic to the quaternion group \( Q_8 \). If \( q \in Q \) and \( g \in G \), then either \( g^{-1}qg = q \) or \( g^{-1}qg = q^{-1} \in Q \) (see Lemma 3). Thus, \( Q \) is a normal subgroup of \( G \).

Assume that some \( x \in G \) commutes with none of \( a \) and \( b \). By Lemma 3, we have \( a^{-1}xa = x^{-1} \) and \( b^{-1}xb = x \) (we note that if \( bx \neq xb \), then \( bx^{-1} \neq x^{-1}b \)). Then \( (ab)^{-1}xab = b^{-1}a^{-1}xab = b^{-1}xb = x \). This shows that any \( x \in G \) commutes with at least one of \( a \), \( b \).

Assume that \( ax = xa \) (the other two cases are similar to the considered one). If \( bx = xb \), then the fact that the adjacency matrix of the Cayley digraph with \( S = \{ax, b\} \) is normal and Corollary 1 imply \( a^2x^2 = (ax) = (bx) = b^2 = a^2 \) (we note that if \( bx = xb \), then \( [ax, b] = [a, b] \neq e \)). Hence, \( x^2 = e \) and therefore \( x \) lies in the center of \( G \) (note that if \( x \) in the condition of Lemma 3 is an involution, then it commutes with \( y \) in both cases). In the case \( bx = xb \), we have \( x^2 = b^2 = a^2 \) and therefore \( (ax)^2 = a^2x^2 = a^4 = e \). Hence \( ax \) lies in the center. Thus, the group \( G \) is generated by its subgroup \( Q \) and \( Z(G) : G = \{Q, Z(G)\} \).

We have seen above that \( Z(G) \) has exponent 2 (see the case of \( ax = xa \) and \( bx = xb \)) and \( Z(G) \cap Q = \{e, a^2\} \). Choose a subgroup \( H \) of index 2 in \( Z(G) \) such that \( a^2 \notin H \). It is clear that \( G = \langle H, Q \rangle \) and \( H \) is a normal subgroup of \( G \). Moreover, we have \( H \cap Q = e \). Thus, \( G = Q \times H \). If \( Z(G) \) has exponent 2, then the same is true for its subgroup \( H \). This implies that \( H \cong Z_2^n \) for some \( n \in Z_+ \) and therefore \( G \cong Q_8 \times Z_2^n \). The theorem is proved. \( \square \)

Remark 4. From Theorem 1 it follows that if the adjacency matrix of every Cayley digraph of degree two on \( G \) is normal, then the same holds for a Cayley digraph of any degree on \( G \).

Assume that \( A(G, S) \) is normal for any two-element subset \( S \) of \( G \) and there are elements \( a \) and \( b \) in \( G \) such that \( ab \neq ba \). We have seen above (see the proof of Theorem 1) that if \( ax = xa \) and \( bx = xb \), then \( x^2 = e \). Moreover, if either \( ax \neq xa \) or \( bx \neq xb \), then \( x^2 = a^2 = b^2 \) (see Corollary 1). Thus, there is \( c \in Z(G) \) such that \( g^2 = c \) for any \( g \notin Z(G) \), and \( g^2 = e \) for any \( g \in Z(G) \). In this case the mapping \( \alpha : g \rightarrow g^{-1} \), where \( g \in G \), is a graph automorphism for every Cayley undirected graph on \( G \). It fixes the identity \( e \), but it is not a group automorphism of \( G \). This implies that every Cayley undirected graph on \( G \) is non-normal by Definition 1.4 in [27]. By Theorem 2.11 [27] (see [25] for the long proof of this theorem), any such \( G \) is either \( Z_4 \times Z_2 \) or \( Q_8 \times Z_2^n \) for some \( n \in Z_+ \). So, Theorem 1 can be considered as a consequence of the results in [27] and therefore shows that there is a relation between the notions of “normality” in linear algebra and graph theory.

In [30] it was observed that every Cayley undirected graph on the quaternion group \( Q_8 \) is quasiabelian (see the proof of Proposition 2.3 therein). From the definition given in the end of Section 2 it follows that every Cayley undirected graph on a group \( G \) is quasiabelian iff any non-central element of \( G \) together with its inverse form a conjugacy class of \( G \). By Dedekind’s theorem on Hamiltonian groups, any such non-abelian group \( G \) has the form \( Q_8 \times Z_2^n \) for some \( n \in Z_+ \). Thus, every Cayley undirected graph on \( G \) is quasiabelian iff either \( G \) is abelian or \( G \equiv Q_8 \times Z_2^n \) for some \( n \in Z_+ \). Note also that in our opinion, the fact that any conjugacy class of \( Q_8 \times Z_2^n \) consists of an element and its inverse explains why the groups \( Q_8 \times Z_2^n \) often appear together with abelian groups not only in algebraic graph theory, but also in combinatorial problems not involving Cayley graphs and digraphs at all (for example, see Theorem 7 in [1] devoted to the association schemes).

4. Groups on which there exists a strongly connected Cayley digraph of degree two whose adjacency matrix is normal

Let us consider two series of groups:
\[
T_{k, n} = (a, c|a^{2k} = e = c, a^{-1}ca = c^{-1}), \quad \text{where } k \geq 1 \quad \text{and} \quad n \geq 3;
\]
\[
H_{p, q} = (a, c|a^{2p} = e, c^{2q} = a^{2p}, a^{-1}ca = c^{-1}), \quad \text{where } p \geq 1 \quad \text{and} \quad q \geq 1.
\]
The following proposition shows that any two groups introduced above are not isomorphic to each other.

**Proposition 2.** For the groups $T_{k,n}$ and $H_{p,q}$, the following three statements hold:

1. $T_{k,n} \cong T_{k',n'}$ if and only if $k = k'$ and $n = n'$;
2. $H_{p,q} \cong H_{p',q'}$ if and only if $p = p'$ and $q = q'$;
3. $H_{p,q} \not\cong T_{k,n}$ for any $p$, $q$, $k$, $n$.

**Proof.** The relation $a^{-1}ca = c^{-1}$ implies that

$$Z(T_{k,n}) = \langle a^2, a^{-n}\rangle \quad \text{if } n \text{ is even},$$

$$Z(T_{k,n}) = \langle a^2 \rangle \quad \text{if } n \text{ is odd},$$

$$Z(H_{p,q}) = \langle a^2 \rangle .$$

In particular, $|Z(T_{k,n})| = 2k$ for even $n$, and $|Z(T_{k,n})| = k$ for odd $n$. Moreover, $|T_{k,n}| = 2kn$. From this it follows that only the isomorphism $T_{k,2n} \cong T_{2k,n}$, where $n$ is odd, is possible. Since $Z(T_{k,2n}) \cong \mathbb{Z}_k \times \mathbb{Z}_2$ and $Z(T_{2k,n}) \cong \mathbb{Z}_{2k}$ for odd $n$, the number $k$ must be odd, too. The order of any element of $T_{k,2n}$ is a divisor of $2kn$ and therefore is an odd integer or twice an odd integer. On the other hand, the order of $a$ in $T_{2k,n}$ is divisible by four. This implies that $T_{k,2n}$ is not isomorphic to $T_{2k,n}$. Thus, $T_{k,n} \cong T_{k',n'}$ if and only if $k = k'$ and $n = n'$. Moreover, since $|Z(H_{p,q})| = 2p$ and $|H_{p,q}| = 8pq$, the same result holds for the groups $H_{p,q}: H_{p,q} \cong H_{p',q'}$ if and only if $p = p'$ and $q = q'$.

Finally, suppose that $H_{p,q} \cong T_{k,n}$. Comparing the centers of these groups, we have $k = p$ if $n$ is even, and $k = 2p$ if $n$ is odd. Since $|T_{2p,n}| = 4pn$ and $|H_{p,q}| = 8pq$, we have $n = 2q$ in the last case. But this is impossible if $n$ is odd. Hence, $n$ is even and therefore $n = 4q$. The identity element $e$ and the elements $a^{2r+1}c^i$, $r = 0, \ldots , p - 1$ and $i = 0, \ldots , 4q - 1$, are solutions of the equation $x^{2p} = e$ on the group $H_{p,q}$. On the other hand, for the group $H_{p,q}$, we have $(a^{2r+1}c^i)^{2p} = a^{2p} = e$ for every $r = 0, \ldots , 2p - 1$ and $i = 0, \ldots , 2q - 1$. Hence, the equation $x^{2p} = e$ has at most $4pq$ solutions on the group $H_{p,q}$. From this it follows that $H_{p,q}$ is not isomorphic to $T_{p,q}$. Thus, $H_{p,q} \not\cong T_{k,n}$ for any $p$, $q$, $k$, $n$. The proposition is proved. □

**Remark 5.** By definition, a group $G$ is metacyclic if it has a cyclic normal subgroup $N$ such that the quotient group $G/N$ is also cyclic (see [19, p. 56]). A complete classification of such groups has been given by C.E. Hempel in [6]. It is not difficult to check that both $T_{k,n}$ and $H_{p,q}$ are metacyclic groups. In particular, $T_{k,n}$ is the dihedral group $D_n$ of order $2n$, and $H_{k,1}$ is the dicyclic group of order $8q$. Moreover, if $k$ is odd, then the subgroup $(a^k, c)$ of $T_{k,n}$ is isomorphic to $D_n$, and if $p$ is odd, then the subgroup $(a^p, c)$ of $H_{p,q}$ is isomorphic to the dicyclic group of order $8q$. So, from our point of view, it is natural to say that $T_{k,n}$ is a group of dihedral type, and $H_{p,q}$ is a group of dicyclic type.

By definition, a digraph $D$ is strongly connected if for any two vertices $v$ and $w$, there is a path from $v$ to $w$ in $D$. It is clear that the Cayley digraph on a finite group $G$ with respect to $S$ is strongly connected iff the Cayley subset $S$ generates $G$. The following theorem shows that the existence of at least one strongly connected Cayley digraph of degree two whose adjacency matrix is normal imposes very restrictive conditions on the group.

**Theorem 2.** Let $G$ be a finite group. Assume that there exists a strongly connected Cayley digraph $D(G, S)$ of degree two on $G$ whose adjacency matrix is normal. Then $G$ has one of the following forms:

1. $G$ is an abelian group of rank at most two;
2. $G \cong T_{k,n}$, where $k \geq 1$ and $n \geq 3$;
3. $G \cong H_{p,q}$, where $p \geq 1$ and $q \geq 1$.

**Proof.** Let $S = \{a, b\}$. Set $c = a^{-1}b$. It is clear that $G = \langle a, c \rangle$. Assume that $ab \neq ba$. By Corollary 1, we have $a^2 = b^2 = acac$ or, equivalently, $a^{-1}ca = c^{-1}$. If $c^{-1} = c$, then $a^{-1}ca = c$ and therefore $G$ must be abelian. Hence, the order of $c$ is greater than or equal to three.

Let $m$ be the order of $a$. Since $a^2 \in Z(G)$ and $a \not\in Z(G)$, we have $m = 2k$ for some $k$. Let $n$ be the smallest natural number such that $c^n = a$ for some $s \in \{0, \ldots , 2k - 1\}$. Then

$$c^n = a^{-1}c^na = (a^{-1}ca) \cdots (a^{-1}ca) = c^{-n}$$

and therefore $a^{2n} = c^{2n} = e$. Thus, either $s = 0$ or $s = k$.

Assume that $s = 0$. Then $G \cong T_{k,n}$. Consider now the case $s = k$. Then $G$ has the form

$$\langle a, c | a^{2k} = e, c^n = a^k, a^{-1}ca = c^{-1} \rangle .$$

Since $a^2 \in Z(G)$, $a^k \in Z(G)$, and $a \not\in Z(G)$, we have $k = 2p$ for some $p$. If $n$ is odd, then $(ca)^n = c^na^k = c^{2n} = e$. Moreover, $a^{-1}(ca)^n = c^{-1}a^k = a^{-1}c^{-1} = (ca)^{-1}$. This means that $G$ is isomorphic to $T_{k,n}$. Finally, if $n = 2q$ for some $q$, then $G = H_{p,q}$. The theorem is proved. □

**Acknowledgements**

The authors would like to thank both referees for their constructive suggestions and remarks which significantly improved the quality of presentation. The second author is grateful to the Russian Science Support Foundation for financially supporting his work in part.
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