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An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings

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ABSTRACT

In this paper, we propose an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a strict pseudo-contraction mapping in the setting of real Hilbert spaces. We establish some weak and strong convergence theorems of the sequences generated by our proposed scheme. Our results combine the ideas of Marino and Xu's result [G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007) 336–346], and Takahashi and Takahashi's result [S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–515]. In particular, necessary and sufficient conditions for strong convergence of our iterative scheme are obtained.

1. Introduction

Let H be a real Hilbert space with inner product $\langle ., .\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and $\Phi : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* (for short, EP) is to find $x \in C$ such that

$$\Phi(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(\Phi)$. Given a mapping $T : C \to H$, let $\Phi(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $x \in EP(\Phi)$ if and only if $x \in K$ is a solution of the variational inequality $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the EP is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of EP; see, for example [1,6,8,9] and references therein. Some solution methods have been proposed to solve the EP; see, for example, [3–5,14,15] and references therein. Motivated by the work in [4,12,14], Takahashi and Takahashi [15] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the EP (1.1) and the set of fixed points of a nonexpansive mapping in the setting of Hilbert

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spaces. They also studied the strong convergence of the sequences generated by their algorithm for a solution of the EP which is also a fixed point of a nonexpansive mapping defined on a closed convex subset of a Hilbert space.

Recall, a mapping T with domain $D(T)$ and range $R(T)$ in H is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in D(T).$$

We denote by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{x \in D(T) : Tx = x\}$. If $C \subset H$ is nonempty, bounded, closed and convex and T is a nonexpansive self-mapping of C, then $F(T)$ is nonempty; see, for example, [7].

The mapping T is said to be a *strict pseudo-contraction* if there exists a constant $0 \le \kappa < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D(T).$$

If

$$\|Tx - Ty\|^2 \le \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D(T),$$

then T is called a *κ-strict pseudo-contraction* mapping.

Note that the class of strict pseudo-contraction mappings strictly includes the class of nonexpansive mappings. Clearly, T is nonexpansive if and only if T is a 0-strict pseudo-contraction. Construction of fixed points of nonexpansive mappings via Mann's algorithm [10] has extensively been investigated in the literature; See, for example [2,10,13,16–19] and references therein. If T is a nonexpansive self-mapping of C, then Mann's algorithm generates, initializing with an arbitrary $x_1 \in C$, a sequence according to the recursive manner

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \ge 1,$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a real control sequence in the interval (0, 1).

If $T : C \to C$ is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm converges weakly to a fixed point of T. Reich [13] showed that the conclusion also holds good in the setting of uniformly convex Banach spaces with a Fréchet differentiable norm. It is well known that Reich's result is one of the fundamental convergence results. Very recently, Marino and Xu [11] extended Reich's result [13] to strict pseudo-contraction mappings in the setting of Hilbert spaces.

Motivated and inspired by the research work of Marino and Xu [11] and Takahashi and Takahashi [15], in this paper, we propose a new implicit iterative scheme for finding a common element of the set of solutions of EP (1.1) and the set of fixed points of a strict pseudo-contraction mapping defined in the setting of real Hilbert spaces. We establish some weak and strong convergence theorems for our iterative scheme. These results are connected with Marino and Xu's result [11], and Takahashi and Takahashi's result [15]. In particular, necessary and sufficient conditions for strong convergence of our iterative scheme are obtained. Since our iterative scheme involves strict pseudo-contraction mappings, the proofs of our results are very different from Takahashi and Takahashi's one [15]. Moreover, our requirements on the iterative parameters are much weaker than those in [15].

2. Preliminaries

Throughout the paper, unless otherwise specified, we consider H is a real Hilbert space with inner product $\langle ., .\rangle$ and norm $\|\cdot\|$, C is a nonempty closed convex subset of H and we use the following notations:

(i) When $\{x_n\}$ is a sequence in H, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) denotes strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x;

(ii) For a given sequence $\{x_n\} \subset H$, $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\}.$$

We assume that the bifunction $\Phi : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $\Phi(x, x) = 0, \forall x \in C$;

(A2) Φ is monotone, that is, $\Phi(x, y) + \Phi(y, x) \le 0, \forall x, y \in C$;

(A3) For all $x, y, z \in C$,

$$\lim_{t\downarrow 0} \Phi(tz + (1 - t)x, y) \le \Phi(x, y);$$

(A4) For each fixed $x \in C$, the function $y \mapsto \Phi(x, y)$ is convex and lower semicontinuous.

Definition 2.1 (*[7]*). Let K be a nonempty closed subset of a Banach space E. A mapping $T : K \to K$ is said to be *semicompact* if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ (as $n \to \infty$), there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in K$ (as $i \to \infty$).

Let us recall the following definitions and results which will be used in the sequel.

Lemma 2.1 (*[11], Lemma 1.1*). *For a real Hilbert space H, the following identities hold:*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$, $\forall x, y \in H$;
(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, $\forall t \in [0,1]$, $\forall x, y \in H$;
(iii) *If $\{x_n\}$ is a sequence in H weakly convergent to z, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2, \quad \forall y \in H.$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x-y\|, \quad \forall y \in C.$$

Such a P_C is called the *metric projection of H onto C*. It is known that P_C is nonexpansive.

Lemma 2.2 (*[11], Lemma 1.3*). *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if*

$$\langle x-z, y-z\rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.3 (*[11], Proposition 2.1*). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C.*

(i) *If S is a κ-strict pseudo-contraction mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+\kappa}{1-\kappa}\|x-y\|, \quad \forall x, y \in C.$$

(ii) *If S is a κ-strict pseudo-contraction mapping, then the mapping $I-S$ is demiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-S)x_n \to 0$, then $(I-S)\tilde{x} = 0$.*
(iii) *If S is a κ-(quasi-)strict pseudo-contraction, then the fixed point set $F(S)$ of S is closed and convex so that the projection $P_{F(S)}$ is well defined.*

The following lemma appeared implicitly in [1].

Lemma 2.4 (*See also [4,15]*). *Let C be a nonempty closed convex subset of H and let $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)–(A4) *. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0 \quad \text{for all } y \in C.$$

The following lemma is established in [4].

Lemma 2.5 (*[4]*). *Assume that $\Phi : C \times C \to \mathbb{R}$ satisfies* (A1)–(A4) *. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : \Phi(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C\}, \quad \forall x \in H.$$

Then,

(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, that is, $\forall x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle;$$

(3) *$F(T_r) = EP(\Phi)$;*
(4) *$EP(\Phi)$ is nonempty, closed and convex.*

3. Iterative scheme and convergence results

We propose an iterative scheme for finding a common element of the set of solutions of EP (1.1) and the set of fixed points of a strict pseudo-contraction mapping in the setting of real Hilbert spaces. We also prove the strong and weak convergences of the sequences generated by our iterative scheme.

Theorem 3.1. *Let C be a nonempty closed convex subset of H, $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)–(A4) *and $S : C \to C$ be a κ-strict pseudo-contraction mapping for some $0 \le \kappa < 1$ such that $F(S) \cap EP(\Phi) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} \Phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n) S u_n, \quad \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (\kappa, 1)$;
(ii) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_n r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(S) \cap EP(\Phi)$.

Proof. We divide the proof into four steps.
Step 1. We claim that the following statements hold:

(i) $\lim_n \|x_n - q\|$ exists for each $q \in F(S) \cap EP(\Phi)$;
(ii) $\lim_n \|u_n - Su_n\| = 0$;
(iii) $\lim_n \|x_n - Sx_n\| = 0$.

Indeed, let q be an arbitrary element of $F(S) \cap EP(\Phi)$. Then from the definition of T_r in Lemma 2.5, we have $u_n = T_{r_n} x_n$, and therefore

$$\|u_n - q\| = \|T_{r_n} x_n - T_{r_n} q\| \le \|x_n - q\|, \quad \forall n \ge 1.$$

Since S is a κ-strict pseudo-contraction, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n u_n + (1-\alpha_n) S u_n - q\|^2 \\ &= \alpha_n \|u_n - q\|^2 + (1-\alpha_n)\|Su_n - q\|^2 - \alpha_n(1-\alpha_n)\|u_n - Su_n\|^2 \\ &\le \alpha_n \|u_n - q\|^2 + (1-\alpha_n)\left(\|u_n - q\|^2 + \kappa\|u_n - Su_n\|^2\right) - \alpha_n(1-\alpha_n)\|u_n - Su_n\|^2 \\ &= \|u_n - q\|^2 - (\alpha_n - \kappa)(1-\alpha_n)\|u_n - Su_n\|^2 \\ &\le \|x_n - q\|^2 - (\alpha_n - \kappa)(1-\alpha_n)\|u_n - Su_n\|^2. \end{aligned} \tag{3.1}$$

Since $\kappa < \alpha \le \alpha_n \le \beta < 1$ for all $n \ge 1$, we get $\|x_{n+1} - q\| \le \|x_n - q\|$, that is, the sequence $\{\|x_n - q\|\}$ is decreasing. Thus, $\lim_n \|x_n - q\|$ exists and hence $\{x_n\}$ is bounded. Also, from (3.1) it follows that

$$\begin{aligned} (\alpha - \kappa)(1-\beta)\|u_n - Su_n\|^2 &\le (\alpha_n - \kappa)(1-\alpha_n)\|u_n - Su_n\|^2 \\ &\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2. \end{aligned} \tag{3.2}$$

This implies that

$$\lim_{n\to\infty} \|u_n - Su_n\| = 0. \tag{3.3}$$

Note that

$$u_n - x_{n+1} = (1-\alpha_n)(u_n - Su_n).$$

Now, we compute

$$\begin{aligned} \|x_{n+1} - Sx_{n+1}\|^2 &= \|\alpha_n(u_n - Sx_{n+1}) + (1-\alpha_n)(Su_n - Sx_{n+1})\|^2 \\ &= \alpha_n\|u_n - Sx_{n+1}\|^2 + (1-\alpha_n)\|Su_n - Sx_{n+1}\|^2 - \alpha_n(1-\alpha_n)\|u_n - Su_n\|^2 \\ &\le \alpha_n\|(u_n - x_{n+1}) + (x_{n+1} - Sx_{n+1})\|^2 - \alpha_n(1-\alpha_n)\|u_n - Su_n\|^2 \\ &\quad + (1-\alpha_n)\left[\|u_n - x_{n+1}\|^2 + \kappa\|(u_n - Su_n) - (x_{n+1} - Sx_{n+1})\|^2\right] \\ &= \alpha_n(\|u_n - x_{n+1}\|^2 + \|x_{n+1} - Sx_{n+1}\|^2 \\ &\quad + 2\langle u_n - x_{n+1}, x_{n+1} - Sx_{n+1}\rangle) - \alpha_n(1-\alpha_n)\|u_n - Su_n\|^2 + (1-\alpha_n)[\|u_n - x_{n+1}\|^2 \\ &\quad + \kappa(\|u_n - Su_n\|^2 + \|x_{n+1} - Sx_{n+1}\|^2 - 2\langle u_n - Su_n, x_{n+1} - Sx_{n+1}\rangle)] \\ &= \|u_n - x_{n+1}\|^2 + \alpha_n\|x_{n+1} - Sx_{n+1}\|^2 \\ &\quad + 2\alpha_n\langle u_n - x_{n+1}, x_{n+1} - Sx_{n+1}\rangle - \alpha_n(1-\alpha_n)\|u_n - Su_n\|^2 \\ &\quad + \kappa(1-\alpha_n)(\|u_n - Su_n\|^2 + \|x_{n+1} - Sx_{n+1}\|^2 - 2\langle u_n - Su_n, x_{n+1} - Sx_{n+1}\rangle) \\ &= (1-\alpha_n)^2\|u_n - Su_n\|^2 + \alpha_n\|x_{n+1} - Sx_{n+1}\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle u_n - Su_n, x_{n+1} - Sx_{n+1}\rangle - \alpha_n(1-\alpha_n)\|u_n - Su_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&\quad + \kappa(1-\alpha_n)(\|u_n - Su_n\|^2 + \|x_{n+1} - Sx_{n+1}\|^2 - 2\langle u_n - Su_n, x_{n+1} - Sx_{n+1}\rangle) \\
&= [\alpha_n + \kappa(1-\alpha_n)]\|x_{n+1} - Sx_{n+1}\|^2 \\
&\quad + (1-\alpha_n)(1+\kappa-2\alpha_n)\|u_n - Su_n\|^2 + 2(\alpha_n - \kappa)(1-\alpha_n)\langle u_n - Su_n, x_{n+1} - Sx_{n+1}\rangle \\
&\le [\alpha_n + \kappa(1-\alpha_n)]\|x_{n+1} - Sx_{n+1}\|^2 \\
&\quad + (1-\alpha_n)(1+\kappa-2\alpha_n)\|u_n - Su_n\|^2 + 2(\alpha_n - \kappa)(1-\alpha_n)\|u_n - Su_n\|\|x_{n+1} - Sx_{n+1}\|.
\end{aligned}
$$

Putting $a_n = \|x_{n+1} - Sx_{n+1}\|$ and $b_n = \|u_n - Su_n\|$ for each $n \ge 1$, we obtain that

$$(1-\alpha_n)(1-\kappa)a_n^2 \le (1-\alpha_n)(1+\kappa-2\alpha_n)b_n^2 + 2(1-\alpha_n)(\alpha_n-\kappa)a_n b_n.$$

Since $1-\alpha_n > 0$ and since we may assume $b_n > 0$, we can divide the last inequality by $(1-\alpha_n)b_n^2$ and also set $\gamma_n = a_n/b_n$ to get the quadratic inequality for γ_n,

$$(1-\kappa)\gamma_n^2 - 2(\alpha_n - \kappa)\gamma_n - (1+\kappa-2\alpha_n) \le 0.$$

Solving this inequality, we obtain

$$\gamma_n \le \frac{\alpha_n - \kappa + \sqrt{(\alpha_n-\kappa)^2 + (1-\kappa)(1+\kappa-2\alpha_n)}}{1-\kappa} = 1.$$

Therefore, $a_n \le b_n$ and hence $\|x_{n+1} - Sx_{n+1}\| \le \|u_n - Su_n\|$. Since $\lim_n \|u_n - Su_n\| = 0$, by (3.3) we find

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.4}$$

Step 2. We claim that $\lim_n \|x_n - u_n\| = 0$.

Indeed, let q be an arbitrary element of $F(S) \cap EP(\Phi)$. Then as above $u_n = T_{r_n}x_n$ and we have

$$
\begin{aligned}
\|u_n - q\|^2 &= \|T_{r_n}x_n - T_{r_n}q\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}q, x_n - q\rangle \\
&= \langle u_n - q, x_n - q\rangle \\
&= \frac{1}{2}\left(\|u_n - q\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2\right)
\end{aligned}
$$

and hence

$$\|u_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - u_n\|^2.$$

Therefore, from (3.1), we have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n(u_n - q) + (1-\alpha_n)(Su_n - q)\|^2 \\
&\le \|u_n - q\|^2 - (\alpha_n - \kappa)(1-\alpha_n)\|u_n - Su_n\|^2 \\
&\le \|u_n - q\|^2 \\
&\le \|x_n - q\|^2 - \|x_n - u_n\|^2
\end{aligned}
$$

and hence

$$\|x_n - u_n\|^2 \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

So, from the existence of $\lim_n \|x_n - q\|$, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.5}$$

Step 3. We claim that $\omega_w(x_n) \subset F(S) \cap EP(\Phi)$, where

$$\omega_w(x_n) = \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{n_i\} \text{ of } \{n\}\}.$$

Indeed, since $\{x_n\}$ is bounded and H is reflexive, $\omega_w(x_n)$ is nonempty. Let $w \in \omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence x_{n_i} of $\{x_n\}$ converges weakly to w. Hence, from (3.5) we know that $u_{n_i} \rightharpoonup w$. As $\|Su_n - u_n\| \to 0$, we obtain that $Su_{n_i} \rightharpoonup w$. Let us show $w \in EP(\Phi)$. Since $u_n = T_{r_n}x_n$, we have

$$\Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge \Phi(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq \Phi\left(y, u_{n_i}\right).$$

Since $\frac{u_{n_i}-x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, from (A4) we have

$$0 \geq \Phi(y, w), \quad \forall y \in C.$$

For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $\Phi(y_t, w) \leq 0$. So, from (A1) and (A4) we have

$$0 = \Phi(y_t, y_t) \leq t\Phi(y_t, y) + (1-t)\Phi(y_t, w) \leq t\Phi(y_t, y)$$

and hence $0 \leq \Phi(y_t, y)$. From (A3), we have

$$0 \leq \Phi(w, y), \quad \forall y \in C$$

and hence $w \in EP(\Phi)$.

We show that $w \in F(S)$. Since S is a κ-strict pseudo-contraction mapping, by Lemma 2.3 (ii) we know that the mapping $I-S$ is demiclosed at zero. Note that $\|u_n - Su_n\| \to 0$ and $u_{n_i} \rightharpoonup w$. Thus, $w \in F(S)$. Consequently, we deduce that $w \in F(S) \cap EP(\Phi)$. Since w was an arbitrary element, we conclude that $\omega_w(x_n) \subset F(S) \cap EP(\Phi)$.

Step 4. We claim that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(S) \cap EP(\Phi)$.

Indeed, to verify that the assertion is valid, it is sufficient to show that $\omega_w(x_n)$ is a single-point set. We take $w_1, w_2 \in \omega_w(x_n)$ arbitrarily and let $\{x_{k_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{k_i} \rightharpoonup w_1$ and $x_{m_j} \rightharpoonup w_2$, respectively. Since $\lim_n \|x_n - q\|$ exists for each $q \in F(S) \cap EP(\Phi)$ and since $w_1, w_2 \in F(S) \cap EP(\Phi)$, by Lemma 2.1(iii), we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - w_1\|^2 &= \lim_{j\to\infty} \|x_{m_j} - w_1\|^2 \\
&= \lim_{j\to\infty} \|x_{m_j} - w_2\|^2 + \|w_2 - w_1\|^2 \\
&= \lim_{i\to\infty} \|x_{k_i} - w_2\|^2 + \|w_2 - w_1\|^2 \\
&= \lim_{i\to\infty} \|x_{k_i} - w_1\|^2 + 2\|w_2 - w_1\|^2 \\
&= \lim_{n\to\infty} \|x_n - w_1\|^2 + 2\|w_2 - w_1\|^2.
\end{aligned}$$

Hence $w_1 = w_2$. This shows that $\omega_w(x_n)$ is a single-point set. This completes the proof. □

As direct consequences of Theorem 3.1, we derive the following results.

Corollary 3.1. *Let C be a nonempty closed convex subset of H and $S : C \to C$ be a κ-strict pseudo-contraction mapping for some $0 \leq \kappa < 1$ such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated initially by an arbitrary element $x_1 \in H$ and then by*

$$x_{n+1} = \alpha_n P_C x_n + (1-\alpha_n) S P_C x_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (\kappa, 1)$. Then, $\{x_n\}$ converges weakly to an element of $F(S)$.

Proof. Put $\Phi(x, y) = 0\ \forall x, y \in C$ and $r_n = 1$ for all $n \geq 1$ in Theorem 3.1. Then, by Lemma 2.2 we have $u_n = P_C x_n$. So, from Theorem 3.1, the sequence $\{x_n\}$ generated initially by $x_1 \in H$ and then by

$$x_{n+1} = \alpha_n P_C x_n + (1-\alpha_n) S P_C x_n, \quad \forall n \geq 1,$$

converges weakly to an element of $F(S)$. □

Corollary 3.2. *Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap EP(\Phi) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} \Phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n) S u_n, & \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) *$\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (0, 1)$;*
(ii) *$\{r_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$.*

Then, $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(S) \cap EP(\Phi)$.

Proof. Since each nonexpansive mapping is a κ-strict pseudo-contraction mapping with $\kappa = 0$, the conclusion follows immediately from Theorem 3.1. $\square$

Theorem 3.2. *Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)–(A4) *and $S : C \to C$ be a κ-strict pseudo-contraction mapping for some $0 \le \kappa < 1$ such that $F(S) \cap EP(\Phi) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} \Phi(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) S u_n, & \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (\kappa, 1)$*;*
(ii) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_n r_n > 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of $F(S) \cap EP(\Phi)$ if and only if $\liminf_n d\,(x_n, F(S) \cap EP(\Phi)) = 0$, where $d\,(x_n, F(S) \cap EP(\Phi))$ denotes the metric distance from the point x_n to $F(S) \cap EP(\Phi)$.

Proof. From the proof of Theorem 3.1, we know that $\lim_{n\to\infty} \|x_n - q\|$ exists for each $q \in F(S) \cap EP(\Phi)$ and $\lim_n \|u_n - x_n\| = 0$. Hence $\{x_n\}$ is bounded.

The necessity is apparent. We show the sufficiency. Suppose that

$$\liminf_{n\to\infty} d\,(x_n, F(S) \cap EP(\Phi)) = 0.$$

Since $x_{n+1} = \alpha_n u_n + (1 - \alpha_n) S u_n$, from (3.1) we have

$$\|x_{n+1} - q\| \le \|x_n - q\|. \tag{3.6}$$

Taking the infimum over all $q \in F(S) \cap EP(\Phi)$, from (3.6) we obtain

$$d\,(x_{n+1}, F(S) \cap EP(\Phi)) \le d\,(x_n, F(S) \cap EP(\Phi))$$

and hence $\lim_n d\,(x_n, F(S) \cap EP(\Phi))$ exists. Thus, we have

$$\lim_{n\to\infty} d\,(x_n, F(S) \cap EP(\Phi)) = \liminf_{n\to\infty} d\,(x_n, F(S) \cap EP(\Phi)) = 0.$$

Now, it follows from (3.6) that for all $q \in F(S) \cap EP(\Phi)$

$$\|x_{n+m} - x_n\| \le \|x_{n+m} - q\| + \|x_n - q\| \le 2\|x_n - q\|. \tag{3.7}$$

Taking the infimum over all $q \in F(S) \cap EP(\Phi)$, from (3.7) we obtain

$$\|x_{n+m} - x_n\| \le 2d\,(x_n, F(S) \cap EP(\Phi)).$$

Thus $\{x_n\}$ is a Cauchy sequence. Suppose $x_n \to \hat{x} \in H$. Then

$$d\,(\hat{x}, F(S) \cap EP(\Phi)) = \lim_{n\to\infty} d\,(x_n, F(S) \cap EP(\Phi)) = 0.$$

As S is a κ-strict pseudo-contraction mapping, we know from Lemma 2.3(iii) that $F(S)$ is closed and convex. Note that $EP(\Phi)$ is closed according to Lemma 2.5. Thus $F(S) \cap EP(\Phi)$ is closed. Consequently, $\hat{x} \in F(S) \cap EP(\Phi)$. In view of $\|u_n - x_n\| \to 0$, we conclude that both sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $\hat{x}$ of $F(S) \cap EP(\Phi)$. $\square$

Theorem 3.3. *Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)–(A4) *and $S : C \to C$ be a semicompact κ-strict pseudo-contraction mapping for some $0 \le \kappa < 1$ such that $F(S) \cap EP(\Phi) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} \Phi(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) S u_n, & \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (\kappa, 1)$*;*
(ii) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_n r_n > 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of $F(S) \cap EP(\Phi)$.

Proof. From the proof of Theorem 3.1, we know that $\lim_n \|x_n - q\|$ exists for each $q \in F(S) \cap EP(\Phi)$ and $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. Thus $\{x_n\}$ is bounded. Then from the semicompactness of S, we conclude that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \to p \in H \quad \text{as } i \to \infty.$$

Hence, $x_{n_i} \rightharpoonup p$. Clearly, repeating the same argument as in the proof of Theorem 3.1, we must have $p \in F(S) \cap EP(\Phi)$. This implies that $\lim_n \|x_n - p\|$ exists. Consequently, we have

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{i\to\infty} \|x_{n_i} - p\| = 0.$$

Since $\|u_n - x_n\| \to 0$, we deduce that both the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $p \in F(S) \cap EP(\Phi)$. $\square$

Utilizing Theorems 3.2 and 3.3, we immediately deduce the following corollaries.

Corollary 3.3. *Let C be a nonempty closed convex subset of H and $S : C \to C$ be a κ-strict pseudo-contraction mapping for some $0 \le \kappa < 1$ such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated initially by an arbitrary element $x_1 \in H$ and then by*

$$x_{n+1} = \alpha_n P_C x_n + (1 - \alpha_n) S P_C x_n, \quad \forall n \ge 1,$$

where $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (\kappa, 1)$. Then, $\{x_n\}$ converges strongly to an element of $F(S)$ if and only if $\liminf_n d(x_n, F(S)) = 0$, where $d(x_n, F(S))$ denotes the metric distance from the point x_n to $F(S)$.

Corollary 3.4. *Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $S : C \to C$ be a semicompact and nonexpansive mapping such that $F(S) \cap EP(\Phi) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} \Phi(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) S u_n, & \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (0, 1)$;
(ii) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_n r_n > 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of $F(S) \cap EP(\Phi)$.

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