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The lower estimate for the linear combinations of Bernstein–Kantorovich operators☆

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Abstract

In this paper we obtain a new strong type of Steckin inequality for the linear combinations of Bernstein–Kantorovich operators, which gives the optimal approximation rate. On the basis of this inequality, we further obtain the lower estimate for these operators.

Keywords: Bernstein–Kantorovich operator; Combination; Strong type of Steckin inequality; Modulus of smoothness

1. Introduction

Many mathematicians have investigated the approximation behavior of Bernstein–Kantorovich operators on $L^p[0,1]$, $1 \le p \le \infty$ with $L^\infty[0,1] = C[0,1]$, defined by

$$K_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\mathrm{d}t, \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

However, they are found not to be applicable to approximating functions with higher approximation degree. Butzer (see [1]) introduced the linear combinations of these operators that

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have higher approximation degree. To generalize Butzer's approach one introduces the following linear combinations (see [2, p. 116]):

$$K_{n,r}(f,x)=\sum_{i=0}^{r-1}c_i(n)K_{n_i-1}(f,x),$$

where n_i and $c_i(n)$ satisfy

$$\text{(a) } n=n_0<\cdots<n_{r-1}\le Kn;\quad \text{(b) } \sum_{i=0}^{r-1}|c_i(n)|\le C;$$

$$\text{(c) } \sum_{i=0}^{r-1}c_i(n)=1;\qquad \text{(d) } \sum_{i=0}^{r-1}c_i(n)n_i^{-\rho}=0,\quad \rho=1,2,\ldots,r-1.$$

It was shown in [2] that for $1\le p<\infty$

$$\|K_{n,r}(f)-f\|_p\le C\left(\omega_\varphi^{2r}\left(f,n^{-1/2}\right)_p+n^{-r}\|f\|_p\right), \tag{1.1}$$

and for $1\le p\le\infty$ and $0<\alpha<2r$

$$\|K_{n,r}(f)-f\|_p=\mathcal{O}(n^{-\alpha/2})\Leftrightarrow \omega_\varphi^{2r}(f,t)_p=\mathcal{O}(t^\alpha),$$

where $\omega_\varphi^{2r}(f,t)_p$ is the modulus of smoothness with the step-weight function $\varphi(x)=\sqrt{x(1-x)}$ and $\|f\|_p=\|f\|_{L_p[0,1]}$(see [2, p. 117]).

In [2,3,6,7,9] one may find some results concerning the approximation rate and the saturation for these operators with $r\ge 1$. However, the saturation problem for all $r\ge 1$ was first solved in [5]. Some notations are necessary to be mentioned. For $k=1,2,\ldots$ let

$$a_{j,k}=ja_{j,k-1}+(k-1)a_{j-1,k-2} \tag{1.2}$$

with

$$a_{0,k}=0,\qquad a_{1,k}=1,\qquad a_{k,2k}=(2k-1)!!,$$

where $1<j<[k/2]$ if k is even and $1<j\le[k/2]$ otherwise. The differential operators needed are given by

$$\begin{aligned}P_r(D)=&\frac{1}{(r+1)!}a_{1,r+1}(1-2x)^{\delta_r}D^r+\sum_{j=1}^{r}\left(\frac{1}{(r+j)!}a_{j,r+j}\right.\\&\left.+\frac{j+1}{(r+j+1)!}\delta_{j,r+j}a_{j+1,r+j+1}\right)(x(1-x))^j(1-2x)^{\delta_{r+j}}D^{r+j},\end{aligned}$$

where $\delta_j=0$ if j is even and $\delta_j=1$ otherwise, and $\delta_{j,r+j}=1$ for $1\le j\le r-1$ and $\delta_{r,2r}=0$. We use these differential operators to define the K-functional, namely,

$$K(f,r,t)_p=\inf_g\left\{\|f-g\|_p+t^{2r}\|P_r(D)g\|_p+t^{2r+1}\|\varphi^{2r+1}g^{(2r+1)}\|_p\right\},$$

where, as usual, $\varphi(x) = \sqrt{x(1-x)}$, $g^{(2r)} \in A.C._{\text{loc}}$ and $1 \le p \le \infty$. Let further

$$\sigma(x) = \frac{1}{(r+1)!}a_{1,r+1} + \sum_{j=1}^{r}\left(\frac{1}{(r+1)!}a_{j,r+j} + \frac{j+1}{(r+j+1)!}\delta_{j,r+j}a_{j+1,r+j+1}\right) \times x(x-1)\cdots(x-j+1).$$

We proved in [5] the following

Theorem 1.1. *Let $1 \le p \le \infty$ and $r \ge 1$. If $-1/p \notin \{Re\, x : \sigma(x) = 0\}$, then there holds for $f \in L^p[0,1]$*

$$\|K_{n,r}(f) - f\|_p = \mathcal{O}(n^{-r}) \Longleftrightarrow K(f,r,t)_p = \mathcal{O}(t^{2r}).$$

In [5] we studied also the problem that one can replace $K(f,r,t)_p = \mathcal{O}(t^{2r})$ by $\omega_\varphi^{2r}(f,t)_p = \mathcal{O}(t^{2r})$. We have (see [5]).

Theorem 1.2. *Let $1 \le p < \infty$ and $r \ge 1$. If $-1/p \notin \{Re\ x : \sigma(x) = 0\}$, then there holds for $f \in L^p[0,1]$*

$$K(f,r,t)_p = \mathcal{O}(t^{2r}) \Longleftrightarrow \omega_\varphi^{2r}(f,t)_p = \mathcal{O}(t^{2r}).$$

Furthermore, the restriction $1 \le p < \infty$ in Theorem 1.2 cannot be replaced by $1 \le p \le \infty$. In fact, calculation shows that for $p = \infty$ and $f(x) = x\ln x$, one has $\omega_\varphi^2(f,t)_\infty = \mathcal{O}\left(t^2\right)$ but $K(f,1,t)_\infty \ne \mathcal{O}\left(t^2\right)$.

Let Π_n be a set of algebraic polynomials with degree n, and

$$E_n(f)_p = \inf_{P\in\Pi_n} \|f - P\|_p.$$

In this paper we will prove a strong type of Steckin inequality for $K_{n,r}$, i.e.,

Theorem 1.3. *For $1 \le p \le \infty$ and $r \ge 1$, there is a constant $C > 0$ such that for $f \in L^p[0,1]$ and $n = 1, 2, \ldots$*

$$\|K_{n,r}(f) - f\|_p \le C\left(K(f,r,n^{-1/2})_p + n^{-r}E_r(f)_p\right) \tag{1.3}$$

and

$$K(f,r,n^{-1/2})_p \le C\left(n^{-r-1/2}\sum_{k=1}^{n} k^{r-1/2}\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p\right). \tag{1.4}$$

We know that the classic Steckin inequality for operators does not give an optimal approximation rate, while (1.3) and (1.4) imply the result of Theorem 1.1. The following result improves Theorem 1.2.

Theorem 1.4. *Let $1 \le p < \infty$ and $r \ge 1$. If $-1/p \notin \{Re\ x : \sigma(x) = 0\}$, then there holds for $f \in L^p[0,1]$*

$$\max_{k\ge n}\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p \asymp \omega_\varphi^{2r}(f,n^{-1/2})_p + n^{-r}E_r(f)_p,$$

where the symbol $X \asymp Y$ means that there exists a positive constant M independent of n and f such that $M^{-1}Y \le X \le MY$.

In Section 2, we will present some needed lemmas. The proofs of Theorems 1.3 and 1.4 will be given in Section 3. Throughout this paper, C denotes a positive constant independent of n and x, whose value may be different in different places.

2. Lemma

We will need the following lemmas.

Lemma 2.1. *For $P_n \in \Pi_n$ satisfying $\|P_n - f\|_p \le CE_n(f)_p$, we have*

$$\|f - P_n\|_p + n^{-2r}\|P_r(D)P_n\|_p + n^{-2r-1}\|\varphi^{2r+1}P_n^{(2r+1)}\|_p \asymp K(f, r, n^{-1})_p. \tag{2.1}$$

Proof. Clearly, we need only to verify the following three inequalities:

$$\|f - P_n\|_p \le MK(f, r, n^{-1})_p, \qquad \|\varphi^{2r+1}P_n^{(2r+1)}\|_p \le Mn^{2r+1}K(f, r, n^{-1})_p$$

and

$$\|P_r(D)P_n\|_p \le Mn^{2r}K(f, r, n^{-1})_p. \tag{2.2}$$

The first two are evident as

$$\omega_\varphi^{2r+1}(f, t)_p \le CK(f, r, t)_p, \qquad E_n(f)_p \le C\omega_\varphi^{2r+1}(f, n^{-1})_p$$

and

$$\|\varphi^{2r+1}P_n^{(2r+1)}\|_p \le Cn^{2r+1}\omega_\varphi^{2r+1}(f, n^{-1})_p,$$

which can be deduced immediately from the definition of $K(f, r, t)_p$, (7.2.2) and (7.3.1) in [2]. To prove (2.2) we choose $g^{(2r)} \in A.C \cdot_{\text{loc}}$, such that

$$\|f - g\|_p + n^{-2r}\|P_r(D)g\|_p + n^{-2r-1}\|\varphi^{2r+1}g^{(2r+1)}\|_p \le 2K(f, r, n^{-1})_p. \tag{2.3}$$

We may assume $n = 2^m$, $P_{2^j} \in \Pi_{2^j}$, $j = m + 1, \ldots$, and

$$\|P_{2^j} - g\|_p = E_{2^j}(g)_p, \quad j = m + 1, \ldots.$$

Thus we have

$$g - P_{2^m} = \sum_{j=m}^{\infty}(P_{2^{j+1}} - P_{2^j}).$$

From Theorem 7.2.1 in [2] and (2.3), we conclude

$$\begin{aligned}\|P_{2^{j+1}} - P_{2^j}\|_p &\le C(2^{-j})^{2r+1}\|\varphi^{2r+1}g^{(2r+1)}\|_p \\ &\le C(2^{-j})^{2r+1}(2^m)^{2r+1}K(f, r, 2^{-m})_p, \quad j = m + 1, \ldots.\end{aligned}$$

Obviously, this estimate holds also for $j = m$. Hence, by using the Bernstein inequality (see e.g. [2]), we obtain finally

$$\begin{aligned}\|P_r(D)(g - P_{2^m})\|_p &\le C\sum_{j=m}^{\infty} 2^{2rj}2^{-2rj-j}2^{2rm+m}K(f, r, 2^{-m})_p \\ &\le C2^{2rm}K(f, r, 2^{-m})_p,\end{aligned}$$

which obviously implies (2.2). □

Denote $E_n = [a/n, 1 - a/n]$ for fixed $a > 0$. For the moments of the operator $K_{n,r}$, we have shown in [5]:

Lemma 2.2. *Let $x \in E_n$, $C(n) = \sum_{i=0}^{r-1} c_i(n) n_i^{-r}$ and $a_{j,k}$ given by (1.2). Then for some $\epsilon_k \in Lip1$ satisfying $\epsilon_k(0) = \epsilon_k(1) = 0$, $\epsilon_{2r-1}(x) \equiv 0$ and $\epsilon_{2r}(x) \equiv 0$, we have*

$$K_{n,r}((\cdot - x)^r, x) = C(n)(1-2x)^{\delta_r} \frac{1}{r+1} a_{1,r+1},$$

and for $r + 1 \le k \le 2r$

$$K_{n,r}((\cdot - x)^k, x) = C(n)(x(1-x))^{k-r}(1-2x)^{\delta_k} \times \left(a_{k-r,k} + \frac{k-r+1}{k+1}\delta_{k-r,k} a_{k-r+1,k+1} + \epsilon_k(x)\right) + \mathcal{O}\left(\frac{\varphi^{2(k-r-1)}(x)}{n^{r+1}}\right),$$

where $\delta_{k-r,k} = 1$ for $r + 1 \le k \le 2r - 1$ and $\delta_{r,2r} = 0$.

Moreover, following the notations of Lemma 2.2, we have (see [5]).

Lemma 2.3. *Let $P \in \prod_m$ with $m \le \sqrt{n}$, then the following inequality is true for $1 \le p \le \infty$:*

$$\begin{aligned} &\Bigg\| K_{n,r}(P, x) - P(x) - C(n)\frac{a_{1,r+1}}{(r+1)!}(1-2x)^{\delta_r} P^{(r)}(x) \\ &\quad - C(n)\sum_{i=1}^{r} \frac{(x(1-x))^i}{(r+i)!}(1-2x)^{\delta_{r+i}} \\ &\quad \times \left(a_{i,r+i} + \frac{i+1}{r+i+1}\delta_{i,r+i} a_{i+1,r+i+1} + \epsilon_{r+i}(x)\right) P^{(r+i)}(x) \Bigg\|_p \\ &\le C_1 n^{-r-1/2}\left(\|\varphi^{2r+1} P^{(2r+1)}(x)\|_p + \|P\|_p\right), \end{aligned} \tag{2.4}$$

where C_1 is a positive constant independent of P and n.

Let q be a given algebraic polynomial, and $\bar{q} = \{\mathrm{Re}\, x : q(x) = 0\}$. We need also some results concerning the following differential operator. Let

$$P(D) = \sum_{i=0}^{l} \alpha_i(x)(x(1-x))^i D^{l+i},$$

where $\alpha_i \in \mathrm{Lip}\,\delta$, $i = 0, 1, \ldots, l$, for some $0 < \delta \le 1$ and $\alpha_l(x) \ne 0$ for $x \in [0, 1]$. Let further

$$\sigma_0(x) = \alpha_0(0) + \sum_{i=1}^{l} \alpha_i(0) x(x-1)\cdots(x-i+1)$$

and

$$\sigma_1(x) = \alpha_0(1) + \sum_{i=1}^{l} (-1)^i \alpha_i(1) x(x-1)\cdots(x-i+1).$$

We have (see [4] or [8]).

Lemma 2.4. *Let $1 \le p \le \infty$ and $\alpha \ge 0$. Then there is a constant $A > 0$ such that for $P \in \Pi_n$ and $n = 1, 2, \ldots$, there hold*

$$\|\varphi^{2l+2\alpha+1} P^{(2l+1)}\|_p \le An(\|\varphi^{2\alpha} P(D)P\|_p + \|\varphi^{2\alpha} P\|_p),$$

and if $-1/p - \alpha \notin \overline{\sigma}_0 \cup \overline{\sigma}_1$,

$$\|\varphi^{2l+2\alpha} P^{(2l)}\|_p \le A(\|\varphi^{2\alpha} P(D)P\|_p + \|\varphi^{2\alpha} P\|_p). \tag{2.5}$$

3. Proofs of Theorems 1.3 and 1.4

With the help of the lemmas shown in Section 2, we are now ready to prove our results.

Proof of Theorem 1.3. Let $P_m \in \Pi_m$ with $m = [\sqrt{n}]$ satisfy

$$\|P_m - f\|_p = E_m(f)_p.$$

By (7.2.2) and (7.3.1) in [2] we have

$$\|P_m - f\|_p \le C\omega_\varphi^{2r+1}(f, n^{-1/2})_p \tag{3.1}$$

and

$$\|\varphi^{2r+1} P_m^{(2r+1)}\|_p \le Cm^{2r+1}\omega_\varphi^{2r+1}(f, n^{-1/2})_p. \tag{3.2}$$

Recalling the definition of $K(f, r, t)_p$ we have

$$\omega_\varphi^{2r+1}(f, n^{-1/2})_p \le CK(f, r, n^{-1/2})_p. \tag{3.3}$$

Thus, from (3.1) and (3.3) we conclude

$$\begin{aligned}\|K_{n,r}(f) - f\|_p &\le 2\|f - P_m\|_p + \|K_{n,r}(P_m) - P_m\|_p \\ &\le CK(f, r, n^{-1/2})_p + \|K_{n,r}(P_m) - P_m\|_p. \end{aligned}\tag{3.4}$$

Using (2.4), (3.2) and (3.3), we obtain, for $x \in [0, 1]$,

$$\begin{aligned}&\|K_{n,r}(P_m, x) - P_m(x)\|_p \\ &\le |C(n)| \left\| \frac{a_{1,r+1}}{(r+1)!}(1-2x)^{\delta_r} P^{(r)}(x) + \sum_{i=1}^{r} \frac{\varphi^{2i}(x)}{(r+i)!}(1-2x)^{\delta_{r+i}} \right. \\ &\quad \left. \times \left(a_{i,r+i} + \frac{i+1}{r+i+1}\delta_{i,r+i} a_{i+1,r+i+1} + \epsilon_{r+i}(x)\right) P_m^{(r+i)}(x) \right\|_p \\ &\quad + C\left(K(f, r, n^{-1/2})_p + n^{-r-1/2}\|P_m\|_p\right). \end{aligned}\tag{3.5}$$

We know that $\epsilon_{2r-1}(x) \equiv \epsilon_{2r}(x) \equiv 0$, and $\epsilon_{r+i} \in$ Lip1 with $\epsilon_{r+i}(0) = \epsilon_{r+i}(1) = 0$ for $i = 1, 2, \ldots, r-2$. Thus, there is $C > 0$ such that $|\epsilon_{r+i}(x)| \le C\varphi^2(x)$ for $x \in [0, 1]$ and $i = 1, 2, \ldots, r-2$. In what follows we should prove

$$\|\varphi^{2i+2} P_m^{(r+i)}\|_p \le C\left(\|P_r(D)P_m\|_p + \|P_m\|_p\right) \quad \text{for } i = 1, 2, \ldots, r-2. \tag{3.6}$$

Indeed, we may assume $m = 2^k$. The set $\overline{\sigma}_0 \cup \overline{\sigma}_1$ for $P_r(D)$ has only finite elements. We have $0 < \alpha < 1/2$ satisfying $-1/p - \alpha \notin \overline{\sigma}_0 \cup \overline{\sigma}_1$. Let $P_{2^j} \in \Pi_{2^j}$, $j = 0, 1, \ldots, k-1$, be the best approximation of P_m with the weight $\varphi^{2\alpha}$. Then, we have from Theorem 8.2.1 in [2]

$$\|\varphi^{2\alpha}(P_m - P_{2^j})\|_p \le C 2^{-2jr}(\|\varphi^{2\alpha+2r} P_m^{(2r)}\|_p + \|P_m\|_p), \quad j = 0, 1, \ldots, k-1.$$

Consequently, we conclude from (8.1.4) and (8.1.3) in [2]

$$\begin{aligned}
\|\varphi^{2i+2} P_m^{(r+i)}\|_p &\le \sum_{j=0}^{k-1} \|\varphi^{2i+2}(P_{2^{j+1}} - P_{2^j})^{(r+i)}\|_p \\
&\le C \sum_{j=0}^{k-1} 2^{(2r-1)j} \|\varphi(P_{2^{j+1}} - P_{2^j})\|_p \\
&\le C \sum_{j=0}^{k} 2^{-j}(\|\varphi^{2\alpha+2r} P_m^{(2r)}\|_p + \|P_m\|_p) \\
&\le C(\|\varphi^{2\alpha+2r} P_m^{(2r)}\|_p + \|P_m\|_p).
\end{aligned}$$

On the other hand, as $-1/p - \alpha \notin \overline{\sigma}_0 \cup \overline{\sigma}_1$ we obtain by (2.5) with $l = r$

$$\|\varphi^{2\alpha+2r} P_m^{(2r)}\|_p + \|P_m\|_p \le A(\|P_r(D) P_m\|_p + \|P_m\|_p).$$

Thus, (3.6) follows from the last two displays. From (3.6), we have

$$\begin{aligned}
&\Bigg\| \frac{a_{1,r+1}}{(r+1)!}(1-2x)^{\delta_r} P^{(r)}(x) + \sum_{i=1}^{r} \frac{\varphi^{2i}(x)}{(r+i)!}(1-2x)^{\delta_{r+i}} \\
&\quad \times \left(a_{i,r+i} + \frac{i+1}{r+i+1}\delta_{i,r+i} a_{i+1,r+i+1} + \epsilon_{r+i}(x) \right) P_m^{(r+i)}(x) \Bigg\|_p \\
&\le C(\|P_r(D) P_m\|_p + \|P_m\|_p).
\end{aligned}$$

Therefore, as $C(n) \asymp n^{-r}$, we conclude from (3.5) and (2.1)

$$\begin{aligned}
\|K_{n,r}(P_m) - P_m\|_p &\le C n^{-r}(\|P_r(D) P_m\|_p + \|P_m\|_p) + C K(f, r, n^{-1/2})_p \\
&\le C\left(K(f, r, n^{-1/2})_p + n^{-r}\|f\|_p \right).
\end{aligned}$$

Combining this with (3.4) we obtain finally

$$\|K_{n,r}(f) - f\|_p \le C\left(K(f, r, n^{-1/2})_p + n^{-r}\|f\|_p \right).$$

This inequality implies (1.3), since for any $P \in \Pi_r$

$$\|K_{n,r}(f - P) - (f - P)\|_p = \|K_{n,r}(f) - f\|_p \quad \text{and}$$
$$K(f - P, r, n^{-1/2})_p = K(f, r, n^{-1/2})_p.$$

To show (1.4) we define the following differential operators

$$\begin{aligned}
\tilde{P}_r(D) = \frac{a_{1,r+1}}{(r+1)!}(1-2x)^{\delta_r} D^r + \sum_{i=1}^{r} \frac{\varphi^{2i}(x)}{(r+i)!}(1-2x)^{\delta_{r+i}} \Bigg(a_{i,r+i} \\
+ \frac{i+1}{r+i+1}\delta_{i,r+i} a_{i+1,r+i+1} + \epsilon_{r+i}(x) \Bigg) D^{r+i},
\end{aligned}$$

then the set $\overline{\sigma}_0 \cup \overline{\sigma}_1$ for $\tilde{P}_r(D)$ is the same as for $P_r(D)$. Consequently, (3.6) holds also for $\tilde{P}_r(D)$ instead of $P_r(D)$. Thus,

$$\left\| \sum_{i=1}^{r} \frac{\varphi^{2i}(x)}{(r+i)!} (1-2x)^{\delta_{r+i}} \epsilon_{r+i}(x) P_m^{(r+i)} \right\|_p \leq \left(\|\tilde{P}_r(D) P_m\|_p + \|P_m\|_p \right).$$

Therefore,

$$\|P_r(D) P_m\|_p \leq C \left(\|\tilde{P}_r(D) P_m\|_p + \|P_m\|_p \right). \tag{3.7}$$

On the other hand, we have

$$K(f, r, n^{-1/2})_p \leq C \left(\|f - P_m\|_p + n^{-r} \|P_r(D) P_m\|_p + n^{-r-1/2} \|\varphi^{2r+1} P_m^{(2r+1)}\|_p \right).$$

We know from (3.1) and (3.2)

$$\|f - P_m\|_p + n^{-r-1/2} \|\varphi^{2r+1} P_m^{(2r+1)}\|_p \leq C \omega_\varphi^{2r+1}(f, n^{-1/2})_p,$$

and from (3.7), (2.4), (3.1) and (3.2)

$$n^{-r} \|P_r(D) P_m\|_p \leq C \left(\|K_{n,r}(P_m) - P_m\|_p + \omega_\varphi^{2r+1}(f, n^{-1/2})_p + n^{-r} \|f\|_p \right).$$

Hence, there holds

$$K(f, r, n^{-1/2})_p \leq C \left(\|K_{n,r}(P_m) - P_m\|_p + \omega_\varphi^{2r+1}(f, n^{-1/2})_p + n^{-r} \|f\|_p \right).$$

Following from (3.1)

$$\|K_{n,r}(P_m) - P_m\|_p \leq \left(\|K_{n,r}(f) - f\|_p + \omega_\varphi^{2r+1}(f, n^{-1/2})_p \right)$$

and from Theorem 9.3.6 of [2]

$$\omega_\varphi^{2r+1}(f, n^{-1/2})_p \leq C n^{-r-1/2} \left(\sum_{k=1}^{n} k^{r-1/2} \|K_{k,r}(f) - f\|_p + \|f\|_p \right),$$

we have

$$K(f, r, n^{-1/2})_p \leq C \left(\|K_{n,r}(f) - f\|_p + n^{-r-1/2} \times \sum_{k=1}^{n} k^{r-1/2} \|K_{k,r}(f) - f\|_p + n^{-r} \|f\|_p \right).$$

By multiplying $n^{r-1/2}$ in the above inequality and summing from N to $2N$, we obtain by the monotonicity of $K(f, r, n^{-1/2})_p$

$$N^{r+1/2} K(f, r, (2N)^{-1/2})_p \leq C \left(\sum_{k=1}^{2N} k^{r-1/2} \|K_{k,r}(f) - f\|_p + N^{1/2} \|f\|_p \right),$$

which obviously implies (1.4). □

Next we should apply Theorem 1.3 and Lemma 2.4 to verifying Theorem 1.4.

Proof of Theorem 1.4. For $P_m \in \Pi_m$ with $m = [\sqrt{n}]$ satisfying $\|f - P_m\|_p = E_m(f)_p$, we have

$$\omega_\varphi^{2r}(f, m^{-1})_p \le C\left(\|f - P_m\|_p + m^{-2r}\|\varphi^{2r}P_m^{(2r)}\|_p\right).$$

By (2.5) with $\alpha = 0$, and (2.1), we conclude

$$\begin{aligned}\omega_\varphi^{2r}(f, m^{-1})_p &\le C(\|f - P_m\|_p + m^{-2r}\|P_r(D)P_m\|_p + m^{-2r}\|f\|_p)\\ &\le C\left(K(f, r, m^{-1})_p + m^{-2r}\|f\|_p\right),\end{aligned}$$

which obviously implies

$$\omega_\varphi^{2r}(f, m^{-1})_p + m^{-2r}E_r(f)_p \le C(K(f, r, m^{-1})_p + m^{-2r}E_r(f))_p.$$

Therefore, it follows from (1.4) that

$$\omega_\varphi^{2r}(f, n^{-1/2})_p + n^{-r}E_r(f)_p \le Cn^{-r-1/2}\sum_{k=1}^{n} k^{r-1/2}(\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p).$$

Consequently, for $\tau = 0$, $1/4$, we obtain from the last display

$$\omega_\varphi^{2r}(f, n^{-1/2})_p + n^{-r}E_r(f)_p \le Cn^{-r-\tau}\max_{1\le k\le n} k^{r+\tau}(\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p).$$

On the other hand, let $J(f, t)_p = \omega_\varphi^{2r}(f, t)_p + t^{2r}E_r(f)_p$, then $J(f, \lambda t) \le C\lambda^{2r}J(f, t)$ for $\lambda \ge 1$. We conclude from (1.1)

$$n^{-r-\tau}\max_{1\le k\le n} k^{r+\tau}(\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p) \le CJ(f, n^{-1/2})_p.$$

Combining the last two displays, we obtain finally for $\tau = 0$ and $1/4$

$$n^{-r-\tau}\max_{1\le k\le n} k^{r+\tau}(\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p) \asymp J(f, n^{-1/2})_p. \tag{3.8}$$

Hence, there holds

$$\begin{aligned}&n^{-r}\max_{1\le k\le n} k^{r}(\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p)\\ &\quad\asymp n^{-r-1/4}\max_{1\le k\le n} k^{r+1/4}(\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p).\end{aligned}$$

Assuming $1 \le k_0 \le n$ satisfies

$$\begin{aligned}&\max_{1\le k\le n} k^{r+1/4}(\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p)\\ &\quad= {k_0}^{r+1/4}(\|K_{k_0,r}(f) - f\|_p + n^{-r}E_r(f)_p),\end{aligned}$$

we have for some $C_0 > 0$

$$\begin{aligned}&n^{-r}k_0^r(\|K_{k_0,r}(f) - f\|_p + n^{-r}E_r(f)_p)\\ &\quad\le n^{-r}\max_{1\le k\le n} k^{r}(\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p)\\ &\quad\le Cn^{-r-1/4}k_0^{r+1/4}(\|K_{k_0,r}(f) - f\|_p + n^{-r}E_r(f)_p),\end{aligned}$$

which gives $k_0 \geq nC_0^{-4}$. Therefore we have from (3.8)

$$\begin{aligned} J(f, n^{-1/2})_p &\leq Cn^{-r-1/4} \max_{1\leq k\leq n} k^{r+1/4}(\|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p) \\ &\leq Cn^{-r-1/4} {k_0}^{r+1/4}(\|K_{k_0,r}(f) - f\|_p + n^{-r}E_r(f)_p) \\ &\leq C\left(\max_{k\geq nC_0^{-4}} \|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p\right). \end{aligned}$$

The property of $J(f, n^{-1/2})$ implies

$$J(f, (nC_0^{-4})^{-1/2})_p \leq C\left(\max_{k\geq nC_0^{-4}} \|K_{k,r}(f) - f\|_p + n^{-r}E_r(f)_p\right).$$

The desired assertion follows from this estimate and (1.1). □

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