ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 249 (2007) 75–91
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Free orbit dimension of finite von Neumann algebras

Don Hadwin *, Junhao Shen [1]

Department of Mathematics and Statistics, University of New Hampshire, Durham, NH 03824, USA

Received 24 May 2006; accepted 10 April 2007

Available online 7 June 2007

Communicated by D. Voiculescu

Abstract

We introduce a new free entropy invariant, which yields improvements of most of the applications of free entropy to finite von Neumann algebras, including those with Cartan subalgebras, simple masas, property T, property Γ, nonprime factors, and thin factors.

Keywords: Von Neumann algebras; Free entropy dimension; Free orbit dimension

1. Introduction

The theory of free probability and free entropy was introduced by Voiculescu in the 1980s. In papers [18,19], Voiculescu introduced the concept of free entropy dimension and used it to provide the first example of separable type II_1 factor that does not have Cartan subalgebras, which solved a long-standing problem. Later, using the theory of free entropy, Ge in [5] showed that the free group factors are not prime, i.e., are not a tensor product of two infinite-dimensional von Neumann algebras. This also answered an old open question. In [8], Ge and the second author computed free entropy dimension for a large class of finite von Neumann algebras including some II_1 factors with property T.

Here we introduce a new invariant, the upper free orbit-dimension of a finite von Neumann algebra, which is closely related to Voiculescu's free entropy dimension. Suppose that $\mathcal{M}$ is a finitely generated von Neumann algebra with a faithful normal tracial state τ and $\mathcal{M}$ can be faith-

* Corresponding author.
E-mail addresses: don@math.unh.edu (D. Hadwin), jog2@cisunix.unh.edu (J. Shen).
1 Research supported in part by an NSF grant.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.04.008

fully embedded into the ultrapower of the hyperfinite II_1 factor. Roughly speaking, if $x_1, \ldots, x_n$ generates $\mathcal{M}$, Voiculescu's free entropy dimension $\delta_0(x_1, \ldots, x_n)$ is obtained by considering the covering numbers of certain sets by ω-balls, and letting ω approach 0. The upper free orbit-dimension $\mathfrak{K}_2(x_1, \ldots, x_n)$ is obtained by considering the covering numbers of the same sets by ω-neighborhoods of unitary orbits (see the definitions in Section 2), and taking the supremum over ω, $0 < \omega < 1$. It is easily shown that

$$\delta_0(x_1, \ldots, x_n) \leqslant 1 + \mathfrak{K}_2(x_1, \ldots, x_n)$$

always holds.

The upper free orbit-dimension has many useful properties, mostly in the case when $\mathfrak{K}_2(x_1, \ldots, x_n) = 0$. The key property is that if $\mathfrak{K}_2(x_1, \ldots, x_n) = 0$ for some generating set for $\mathcal{M}$, then $\mathfrak{K}_2(y_1, \ldots, y_p) = 0$ for every generating set $y_1, \ldots, y_p$ of $\mathcal{M}$. This fact allows us to show that the class of finite von Neumann algebras $\mathcal{M}$ with $\mathfrak{K}_2(\mathcal{M}) = 0$ is closed under certain operations that enlarge the algebra:

(1) If $\mathfrak{K}_2(\mathcal{N}_1) = \mathfrak{K}_2(\mathcal{N}_2) = 0$ and $\mathcal{N}_1 \cap \mathcal{N}_2$ is diffuse, then $\mathfrak{K}_2((\mathcal{N}_1 \cup \mathcal{N}_2)'') = 0$.
(2) If $\mathcal{M} = \{\mathcal{N}, u\}''$ where $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ with $\mathfrak{K}_2(\mathcal{N}) = 0$ and u is a unitary element in $\mathcal{M}$ satisfying, for a sequence $\{v_n\}$ of Haar unitary elements in $\mathcal{N}$, $\mathrm{dist}_{\|\ \|_2}(uv_nu^*, \mathcal{N}) \to 0$, then $\mathfrak{K}_2(\mathcal{M}) = 0$.
(3) If $\{\mathcal{N}_i\}_{i=1}^{\infty}$ is an ascending sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\mathfrak{K}_2(\mathcal{N}_i) = 0$ for all $i \geqslant 1$ and $\mathcal{M} = \overline{\bigcup_i \mathcal{N}_i}^{SOT}$, then $\mathfrak{K}_2(\mathcal{M}) = 0$.

Using these closure operations as building blocks, and the fact that $\mathfrak{K}_2(\mathcal{M}) = 0$ whenever $\mathcal{M}$ is hyperfinite, we can show that $\mathfrak{K}_2(\mathcal{M}) = 0$ for a large class of von Neumann algebras. As a corollary we recapture most of the old results. In particular, we extend results in [5,7–10,19,20].

Using free orbit dimension, we also obtain some general results on the decompositions of type II_1 factors (see Theorem 6). As a corollary, we extend the results in [3,4,6,12,14–16].

The organization of the paper is as follows. In Section 2, we give the definitions of free orbit dimension and upper free orbit dimension. Key properties of upper free orbit dimension are discussed in Section 3. The values of upper free orbit dimension for finite von Neumann algebras are computed in Section 4. Some results on the decompositions of type II_1 factors are obtained in Section 5.

In the paper, we only consider the finite von Neumann algebras $\mathcal{M}$ that can be faithfully embedded into the ultrapower of the hyperfinite II_1 factor.

After the completion of this work, we were informed that K. Jung [13] introduced a notion of a "strongly 1-bounded" set of generators of a finite von Neumann algebra and proved some results similar to ours. It seems that these two concepts are closely related to each other.

2. Definitions

Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}\mathrm{Tr}$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}(k)$ denote the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\|\cdot\|_2$ denote the trace norm induced by τ_k on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \ldots, A_n)\|_2^2 = \tau_k(A_1^*A_1) + \cdots + \tau_k(A_n^*A_n)$$

for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-ball $Ball(B_1, \ldots, B_n; \omega)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that $\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\|_2 < \omega$.

For every $\omega > 0$, we define the ω-orbit-ball $\mathcal{U}(B_1, \ldots, B_n; \omega)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\left\|(A_1, \ldots, A_n) - \left(WB_1W^*, \ldots, WB_nW^*\right)\right\|_2 < \omega.$$

For every $R > 0$, we define $(\mathcal{M}_k(\mathbb{C})^n)_R$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all these $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that $\max_{1 \leqslant j \leqslant n} \|A_j\| \leqslant R$.

Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal tracial state τ, and $x_1, \ldots, x_n$ be elements in $\mathcal{M}$. We now define our new invariants. For any positive R and ϵ, and any m, k in $\mathbb{N}$, let $\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon)$ be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that $(A_1, \ldots, A_n)$ is contained in $(\mathcal{M}_k(\mathbb{C})^n)_R$, and

$$\left|\tau_k\left(A_{i_1}^{\eta_1} \cdots A_{i_q}^{\eta_q}\right) - \tau\left(x_{i_1}^{\eta_1} \cdots x_{i_q}^{\eta_q}\right)\right| < \epsilon,$$

for all $1 \leqslant i_1, \ldots, i_q \leqslant n$, all $\eta_1, \ldots, \eta_q$ in $\{1, *\}$, and all q with $1 \leqslant q \leqslant m$.

For $\omega > 0$, we define the ω-orbit covering number $\nu(\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon), \omega)$ to be the minimal number of ω-orbit-balls that cover $\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon)$ with the centers of these ω-orbit-balls in $(\mathcal{M}_k(\mathbb{C})^n)_R$.

Now we define, successively,

$$\begin{aligned}
\mathfrak{K}(x_1, \ldots, x_n; \omega, R) &= \inf_{m \in \mathbb{N}, \epsilon > 0} \limsup_{k \to \infty} \frac{\log(\nu(\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon), \omega))}{-k^2 \log \omega}, \\
\mathfrak{K}(x_1, \ldots, x_n; \omega) &= \sup_{R > 0} \mathfrak{K}(x_1, \ldots, x_n; \omega, R), \\
\mathfrak{K}_1(x_1, \ldots, x_n) &= \limsup_{\omega \to 0} \mathfrak{K}(x_1, \ldots, x_n; \omega), \\
\mathfrak{K}_2(x_1, \ldots, x_n) &= \sup_{0 < \omega < 1} \mathfrak{K}(x_1, \ldots, x_n; \omega),
\end{aligned}$$

where $\mathfrak{K}_1(x_1, \ldots, x_n)$ is called the *free orbit-dimension* of $x_1, \ldots, x_n$ and $\mathfrak{K}_2(x_1, \ldots, x_n)$ is called the *upper free orbit-dimension* of $x_1, \ldots x_n$.

In the spirit as in Voiculescu's definition of free entropy dimension, we shall also define free orbit-dimension and upper free orbit-dimension of $x_1, \ldots, x_n$ in the presence of $y_1, \ldots, y_p$ for all $x_1, \ldots, x_n, y_1, \ldots, y_p$ in the von Neumann algebra $\mathcal{M}$ as follows. Let $\Gamma_R(x_1, \ldots, x_n : y_1, \ldots, y_p; m, k, \epsilon)$ be the image of the projection of $\Gamma_R(x_1, \ldots, x_n, y_1, \ldots, y_p; m, k, \epsilon)$ onto the first n components, i.e.,

$$(A_1, \ldots, A_n) \in \Gamma_R(x_1, \ldots, x_n : y_1, \ldots, y_p; m, k, \epsilon)$$

if there are elements $B_1, \ldots, B_p$ in $\mathcal{M}_k(\mathbb{C})$ such that

$$(A_1, \ldots, A_n, B_1, \ldots, B_p) \in \Gamma_R(x_1, \ldots, x_n, y_1, \ldots, y_p; m, k, \epsilon).$$

Then we define, successively,

$$\begin{aligned}
&\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_p;\omega,R),\\
&\quad=\inf_{m\in\mathbb{N},\epsilon>0}\limsup_{k\to\infty}\frac{\log(\nu(\Gamma_R(x_1,\ldots,x_n:y_1,\ldots,y_p;m,k,\epsilon),\omega))}{-k^2\log\omega},\\
&\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_p;\omega)=\sup_{R>0}\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_p;\omega,R),\\
&\mathfrak{K}_1(x_1,\ldots,x_n:y_1,\ldots,y_p)=\limsup_{\omega\to0}\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_p;\omega),\\
&\mathfrak{K}_2(x_1,\ldots,x_n:y_1,\ldots,y_p)=\sup_{0<\omega<1}\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_p;\omega).
\end{aligned}$$

Definition 1. Suppose $\mathcal{M}$ is a finitely generated von Neumann algebra with a faithful normal tracial state τ. Then the *free orbit-dimension* $\mathfrak{K}_1(\mathcal{M})$ of $\mathcal{M}$ is defined by

$$\mathfrak{K}_1(\mathcal{M})=\sup\bigl\{\mathfrak{K}_1(x_1,\ldots,x_n)\bigm| x_1,\ldots,x_n \text{ generate } \mathcal{M} \text{ as a von Neumann algebra}\bigr\},$$

and the *upper free orbit-dimension* $\mathfrak{K}_2(\mathcal{M})$ of $\mathcal{M}$ is defined by

$$\mathfrak{K}_2(\mathcal{M})=\sup\bigl\{\mathfrak{K}_2(x_1,\ldots,x_n)\bigm| x_1,\ldots,x_n \text{ generate } \mathcal{M} \text{ as a von Neumann algebra}\bigr\}.$$

Here, we quote a useful proposition from [9, Theorem 2.1], which is an extension of [18, Lemma 4.3].

Proposition 1. *Suppose $\mathcal{R}$ is a hyperfinite von Neumann algebra with a faithful normal tracial state τ. Suppose that $x_1,\ldots,x_n$ is a family of generators of $\mathcal{R}$. Then, for every $\delta>0$, $R>\max_{1\leqslant j\leqslant n}\|x_j\|$, there are a positive integer m_0 and a positive number ϵ_0 such that the following hold: for $k\geqslant 1$, if $A_1,\ldots,A_n,B_1,\ldots,B_n$ in $\mathcal{M}_k(\mathbb{C})$ satisfying,*

(a) $0\leqslant\|A_j\|,\|B_j\|\leqslant R$ *for all* $1\leqslant j\leqslant n$;
(b)

$$\bigl|\tau_k\bigl(A_{i_1}^{\eta_1}\cdots A_{i_p}^{\eta_p}\bigr)-\tau\bigl(x_{i_1}^{\eta_1}\cdots x_{i_p}^{\eta_p}\bigr)\bigr|<\epsilon_0,$$
$$\bigl|\tau_k\bigl(B_{i_1}^{\eta_1}\cdots B_{i_p}^{\eta_p}\bigr)-\tau\bigl(x_{i_1}^{\eta_1}\cdots x_{i_p}^{\eta_p}\bigr)\bigr|<\epsilon_0,$$

for all $1\leqslant i_1,\ldots,i_p\leqslant n$, $\{\eta_j\}_{j=1}^p\subset\{*,1\}$ *and* $1\leqslant p\leqslant m_0$,

then there exists a unitary matrix U in $\mathcal{U}(k)$ such that

$$\left(\sum_{j=1}^n\|U^*A_jU-B_j\|_2^2\right)^{1/2}<\delta.$$

Proof. Suppose on the contrary that the following holds: there is some $\delta_0>0$ such that for every $m\geqslant 1$, there are some $k_m\geqslant 1$ and some $A_{1,m},\ldots,A_{n,m},B_{1,m},\ldots,B_{n,m}$ in $\mathcal{M}_{k_m}(\mathbb{C})$ satisfying:

(a) $$\max_{1\leqslant j\leqslant n}\bigl\{\|A_{j,m}\|,\|B_{j,m}\|\bigr\}\leqslant R;$$

(b) $$\left|\tau_k\left(A_{i_1,m}^{\epsilon_1}\cdots A_{i_p,m}^{\epsilon_p}\right)-\tau\left(x_{i_1}^{\epsilon_1}\cdots x_{i_p}^{\epsilon_p}\right)\right|<\frac{1}{m};$$
$$\left|\tau_k\left(B_{i_1,m}^{\epsilon_1}\cdots B_{i_p,m}^{\epsilon_p}\right)-\tau\left(x_{i_1}^{\epsilon_1}\cdots x_{i_p}^{\epsilon_p}\right)\right|<\frac{1}{m};$$
for all $1\leqslant i_1,\dots,i_p\leqslant n$, $\{\epsilon_1,\dots,\epsilon_p\}\subset\{*,1\}$ and $1\leqslant p\leqslant m$; and

(c) for all U in $\mathcal{U}(k_m)$, $\sum_{j=1}^n\|UA_{j,m}U^*-B_{j,m}\|_2^2>\delta_0^2$.

Let ω be a free filter in $\beta(\mathbb{N})\setminus\mathbb{N}$. Denote by $\mathcal{M}_{k_m}(\mathbb{C})^\omega$ the ultrapower of $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^\infty$ along the filter ω. So $\mathcal{M}_{k_m}(\mathbb{C})^\omega$ is a type II_1 factor.

Let ρ, or σ, be the mapping from $\mathcal{R}$ into $\mathcal{M}_{k_m}(\mathbb{C})^\omega$ induced by sending each x_j, for $1\leqslant j\leqslant n$, to $[(A_{j,m})_m]$, or $[(B_{j,m})_m]$, respectively. It is not hard to see that ρ and σ are two trace-preserving embeddings of $\mathcal{R}$ into $\mathcal{M}_{k_m}(\mathbb{C})^\omega$. For $\delta_0>0$, there exist a finite dimensional subalgebra $\mathcal{A}_0$ of $\mathcal{R}$ and $\{\tilde{x}_1,\dots,\tilde{x}_n\}$ in $\mathcal{A}_0$ such that $(\sum_{1\leqslant j\leqslant n}\|x_j-\tilde{x}_j\|_2^2)^{1/2}\leqslant\delta_0/8$. Since $\mathcal{M}_{k_m}(\mathbb{C})^\omega$ is a type II_1 factor and ρ and σ are two trace-preserving embedding of $\mathcal{A}_0$ into $\mathcal{M}_{k_m}(\mathbb{C})^\omega$, there is some unitary element u in $\mathcal{M}_{k_m}(\mathbb{C})^\omega$ such that $\rho(y)=u\sigma(y)u^*$ for all $y\in\mathcal{A}_0$. Let $[(U_m)_m]$ be a representative of u in $\mathcal{M}_{k_m}(\mathbb{C})^\omega$. We can further assume that each U_m is a unitary element in $\mathcal{U}(k_m)$ for all $m\geqslant1$. It follows that $(\lim_{m\to\omega}\sum_{1\leqslant j\leqslant n}\|U_mA_{j,m}U_m^*-B_{j,m}\|_2^2)^{1/2}\leqslant\delta_0/2$, which contradicts with the assumption that $(\sum_{1\leqslant j\leqslant n}\|UA_{j,m}U^*-B_{j,m}\|_2^2)^{1/2}>\delta_0$ for all unitary matrix U in $\mathcal{U}(k_m)$. Therefore, the statement of the proposition is true. □

3. Key properties of $\mathfrak{K}_2$

In this section, we are going to study the properties of upper free orbit dimension. By using an equivalent packing number formulation of free entropy dimension due to Jung [11] or the fractal free entropy dimension defined by Dostál and Hadwin [2], we have the following lemma.

Lemma 1. *Let $x_1,\dots,x_n$ be self-adjoint elements in a von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ. Let $\delta_0(x_1,\dots,x_n)$ be Voiculescu's modified free entropy dimension. Then*

$$\delta_0(x_1,\dots,x_n)\leqslant\mathfrak{K}_1(x_1,\dots,x_n)+1\leqslant\mathfrak{K}_2(x_1,\dots,x_n)+1.$$

Proof. The first inequality follows from [2, Theorem 14] or [11], and the second inequality is obvious. □

Lemma 2. *Let $x_1,\dots,x_n,y_1,\dots,y_p$ be elements in a von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ. If $y_1,\dots,y_p$ are in the von Neumann subalgebra generated by $x_1,\dots,x_n$ in $\mathcal{M}$, then, for every $0<\omega<1$,*

$$\mathfrak{K}(x_1,\dots,x_n;\omega)=\mathfrak{K}(x_1,\dots,x_n:y_1,\dots,y_p;\omega).$$

Proof. It is a straightforward adaptation of the proof of [19, Proposition 1.6]. Given $R>\max_{1\leqslant j\leqslant n}\|x_j\|+\max_{1\leqslant j\leqslant p}\|y_j\|$, $m\in\mathbb{N}$ and $\epsilon>0$, we can find $m_1\in\mathbb{N}$ and $\epsilon_1>0$ such that, for all $k\in\mathbb{N}$,

$$\begin{aligned}\Gamma_R(x_1,\ldots,x_n;m_1,k,\epsilon_1)&\subset\Gamma_R(x_1,\ldots,x_n:y_1,\ldots,y_p;m,k,\epsilon)\\&\subset\Gamma_R(x_1,\ldots,x_n;m,k,\epsilon).\end{aligned}$$

Hence

$$\begin{aligned}\nu\big(\Gamma_R(x_1,\ldots,x_n;m_1,k,\epsilon_1),\omega\big)&\leqslant\nu\big(\Gamma_R(x_1,\ldots,x_n:y_1,\ldots,y_p;m,k,\epsilon),\omega\big)\\&\leqslant\nu\big(\Gamma_R(x_1,\ldots,x_n;m,k,\epsilon),\omega\big),\end{aligned}$$

for all $0<\omega<1$. The rest follows from the definitions. $\square$

The following key theorem shows that, in some cases, the upper free orbit-dimension $\mathfrak{K}_2$ is a von Neumann algebra invariant, i.e., it is independent of the choice of generators.

Theorem 1. *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful normal tracial state τ and is generated by a family of elements $\{x_1,\ldots,x_n\}$ as a von Neumann algebra. If*

$$\mathfrak{K}_2(x_1,\ldots,x_n)=0,$$

then

$$\mathfrak{K}_2(\mathcal{M})=0.$$

Proof. Suppose that $y_1,\ldots,y_p$ are elements in $\mathcal{M}$ that generate $\mathcal{M}$ as a von Neumann algebra. For every $0<\omega<1$, there exists a family of noncommutative polynomials $\psi_i(x_1,\ldots,x_n)$, $1\leqslant i\leqslant p$, such that

$$\sum_{i=1}^{p}\big\|y_i-\psi_i(x_1,\ldots,x_n)\big\|_2^2<\left(\frac{\omega}{4}\right)^2.$$

For such a family of polynomials $\psi_1,\ldots,\psi_p$, and every $R>0$ there always exists a constant $D\geqslant 1$, depending only on $R,\psi_1,\ldots,\psi_n$, such that

$$\left(\sum_{i=1}^{p}\big\|\psi_i(A_1,\ldots,A_n)-\psi_i(B_1,\ldots,B_n)\big\|_2^2\right)^{1/2}\leqslant D\big\|(A_1,\ldots,A_n)-(B_1,\ldots,B_n)\big\|_2,$$

for all $(A_1,\ldots,A_n),(B_1,\ldots,B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$, all $k\in\mathbb{N}$, satisfying $\|A_j\|,\|B_j\|\leqslant R$, for $1\leqslant j\leqslant n$.

For $R>1$, m sufficiently large, ϵ sufficiently small and k sufficiently large, every $(H_1,\ldots,H_p,A_1,\ldots,A_n)$ in $\Gamma_R(y_1,\ldots,y_p,x_1,\ldots,x_n;m,k,\epsilon)$ satisfies

$$\left(\sum_{i=1}^{p}\big\|H_i-\psi_i(A_1,\ldots,A_n)\big\|_2^2\right)^{1/2}\leqslant\frac{\omega}{4}.$$

It is obvious that such an $(A_1,\ldots,A_n)$ is also in $\Gamma_R(x_1,\ldots,x_n;m,k,\epsilon)$. On the other hand, by the definition of the orbit covering number, we know there exists a set $\{\mathcal{U}(B_1^\lambda,\ldots,B_n^\lambda;\frac{\omega}{4D})\}_{\lambda\in\Lambda_k}$

of $\frac{\omega}{4D}$-orbit-balls that cover $\Gamma_R(x_1,\dots,x_n;m,k,\epsilon)$ with the cardinality of Λ_k satisfying $|\Lambda_k| = \nu(\Gamma_R(x_1,\dots,x_n;m,k,\epsilon),\frac{\omega}{4D})$. Thus for such $(A_1,\dots,A_n)$ in $\Gamma_R(x_1,\dots,x_n;m,k,\epsilon)$, there exists some $\lambda\in\Lambda_k$ and $W\in\mathcal{U}(k)$ such that

$$\left\|(A_1,\dots,A_n)-\left(WB_1^\lambda W^*,\dots,WB_n^\lambda W^*\right)\right\|_2\leqslant\frac{\omega}{4D}.$$

It follows that

$$\sum_{i=1}^p\left\|H_i-W\psi_i\left(B_1^\lambda,\dots,B_n^\lambda\right)W^*\right\|_2^2=\sum_{i=1}^p\left\|H_i-\psi_i\left(WB_1^\lambda W^*,\dots,WB_n^\lambda W^*\right)\right\|_2^2\leqslant\left(\frac{\omega}{2}\right)^2,$$

for some $\lambda\in\Lambda_k$ and $W\in\mathcal{U}(k)$, i.e.,

$$(H_1,\dots,H_p)\in\mathcal{U}\left(\psi_1\left(B_1^\lambda,\dots,B_n^\lambda\right),\dots,\psi_p\left(B_1^\lambda,\dots,B_n^\lambda\right);\frac{\omega}{2}\right).$$

Hence, by the definition of the free orbit-dimension, we get

$$\begin{aligned}0&\leqslant\mathfrak{K}(y_1,\dots,y_p:x_1,\dots,x_n;\omega,R)\leqslant\inf_{m\in\mathbb{N},\epsilon>0}\limsup_{k\to\infty}\frac{\log(|\Lambda_k|)}{-k^2\log\omega}\\&=\inf_{m\in\mathbb{N},\epsilon>0}\limsup_{k\to\infty}\frac{\log(\nu(\Gamma_R(x_1,\dots,x_n;m,k,\epsilon),\frac{\omega}{4D}))}{-k^2\log\omega}\\&=0,\end{aligned}$$

since $\mathfrak{K}_2(x_1,\dots,x_n)=0$. Therefore $\mathfrak{K}(y_1,\dots,y_p:x_1,\dots,x_n;\omega)=0$. Now it follows from Lemma 2 that

$$\mathfrak{K}(y_1,\dots,y_p;\omega)=\mathfrak{K}(y_1,\dots,y_p:x_1,\dots,x_n;\omega)=0;$$

whence $\mathfrak{K}_2(y_1,\dots,y_p)=0$ and $\mathfrak{K}_2(\mathcal{M})=0$. □

Theorem 2. *If $\mathcal{M}$ is a hyperfinite von Neumann algebra with a faithful normal tracial state τ, then $\mathfrak{K}_2(\mathcal{M})=0$.*

Proof. When $\mathcal{M}$ is an abelian von Neumann algebra, the result follows from [18, Lemma 4.3]. Generally, it is a direct consequence of Proposition 1, that, for each $0<\omega<1$,

$$\nu\left(\Gamma_R(x_1,\dots,x_n,m,\varepsilon,k),\omega\right)=1$$

whenever m is sufficiently large and ε is sufficiently small. □

The proof of next theorem, being a slight modification of that of Theorem 1, will be omitted.

Theorem 3. *Suppose that $\mathcal{M}$ is a finitely generated von Neumann algebra with a faithful normal tracial state τ. Suppose that $\{\mathcal{N}_i\}_{i=1}^\infty$ is an ascending sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\mathfrak{K}_2(\mathcal{N}_i)=0$ for all $i\geqslant 1$ and $\mathcal{M}=\overline{\bigcup_i\mathcal{N}_i}^{SOT}$. Then $\mathfrak{K}_2(\mathcal{M})=0$.*

Definition 2. A unitary matrix U in $\mathcal{M}_k(\mathbb{C})$ is a *Haar unitary matrix* if $\tau_k(U^m) = 0$ for all $1 \leqslant m < k$ and $\tau_k(U^k) = 1$.

The proof of following lemma can be found in [8] (see also [20]). For the sake of completeness, we also sketch its proof here.

Lemma 3. *Let V_1, V_2 be two Haar unitary matrices in $\mathcal{M}_k(\mathbb{C})$. For every $\delta > 0$, let*

$$\Omega(V_1, V_2; \delta) = \big\{U \in \mathcal{U}(k) \bigm| \|UV_1 - V_2U\|_2 \leqslant \delta\big\}.$$

Then, for every $0 < \delta < r$, there exists a set $\{Ball(U_\lambda; \frac{4\delta}{r})\}_{\lambda \in \Lambda}$ of $\frac{4\delta}{r}$-balls in $\mathcal{U}(k)$ that cover $\Omega(V_1, V_2; \delta)$ with the cardinality of Λ satisfying $|\Lambda| \leqslant (\frac{3r}{2\delta})^{4rk^2}$.

Sketch of Proof. Let D be a diagonal unitary matrix, $\mathrm{diag}(\lambda_1, \ldots, \lambda_k)$, where λ_j is the jth root of unity 1. Since V_1, V_2 are Haar unitary matrices, there exist W_1, W_2 in $\mathcal{U}(k)$ such that $V_1 = W_1 D W_1^*$ and $V_2 = W_2 D W_2^*$. Let $\tilde{\Omega}(\delta) = \{U \in \mathcal{U}(k) \mid \|UD - DU\|_2 \leqslant \delta\}$. Clearly $\Omega(V_1, V_2; \delta) = \{W_2^* U W_1 \mid U \in \tilde{\Omega}(\delta)\}$; whence $\tilde{\Omega}(\delta)$ and $\Omega(V_1, V_2; \delta)$ have the same covering numbers.

Let $\{e_{st}\}_{s,t=1}^k$ be the canonical system of matrix units of $\mathcal{M}_k(\mathbb{C})$. Let

$$\mathcal{S}_1 = \mathrm{span}\big\{e_{st} \bigm| |\lambda_s - \lambda_t| < r\big\}, \qquad \mathcal{S}_2 = M_k(\mathbb{C}) \ominus S_1.$$

For every $U = \sum_{s,t=1}^k x_{st} e_{st}$ in $\tilde{\Omega}(\delta)$, with $x_{st} \in \mathbb{C}$, let $T_1 = \sum_{e_{st} \in \mathcal{S}_1} x_{st} e_{st} \in \mathcal{S}_1$ and $T_2 = \sum_{e_{st} \in \mathcal{S}_2} x_{st} e_{st} \in \mathcal{S}_2$. But

$$\begin{aligned}\delta^2 &\geqslant \|UD - DU\|_2^2 = \sum_{s,t=1}^k \big|(\lambda_s - \lambda_t) x_{st}\big|^2 \geqslant \sum_{e_{st} \in \mathcal{S}_2} \big|(\lambda_s - \lambda_t) x_{st}\big|^2 \\ &\geqslant r^2 \sum_{e_{st} \in \mathcal{S}_2} |x_{st}|^2 = r^2 \|T_2\|_2^2.\end{aligned}$$

Hence $\|T_2\|_2 \leqslant \frac{\delta}{r}$. Note that $\|T_1\|_2 \leqslant \|U\|_2 = 1$ and $\dim_{\mathbb{R}} \mathcal{S}_1 \leqslant 4rk^2$. By standard arguments on covering numbers, we know that $\tilde{\Omega}(\delta)$ can be covered by a set $\{Ball(A^\lambda; \frac{2\delta}{r})\}_{\lambda \in \Lambda}$ of $\frac{2\delta}{r}$-balls in $\mathcal{M}_k(\mathbb{C})$ with $|\Lambda| \leqslant (\frac{3r}{2\delta})^{4rk^2}$. Because $\tilde{\Omega}(\delta) \subset \mathcal{U}(k)$, after replacing A^λ by a unitary U^λ in $Ball(A^\lambda, \frac{2\delta}{r})$, we obtain that the set $\{Ball(U_\lambda; \frac{4\delta}{r})\}_{\lambda \in \Lambda}$ of $\frac{4\delta}{r}$-balls in $\mathcal{U}(k)$ that cover $\tilde{\Omega}(\delta)$ with the cardinality of Λ satisfying $|\Lambda| \leqslant (\frac{3r}{2\delta})^{4rk^2}$. The same result holds for $\Omega(V_1, V_2; \delta)$. □

Definition 3. Suppose that $\mathcal{M}$ is a diffuse von Neumann algebra with a faithful normal tracial state τ. Then a unitary element u in $\mathcal{M}$ is called a *Haar unitary* if $\tau(u^m) = 0$ when $m \neq 0$.

Theorem 4. *Suppose $\mathcal{M}$ is a diffuse von Neumann algebra with a faithful normal tracial state τ. Suppose $\mathcal{N}$ is a diffuse von Neumann subalgebra of $\mathcal{M}$ and u is a unitary element in $\mathcal{M}$ such that $\mathfrak{K}_2(\mathcal{N}) = 0$ and $\{\mathcal{N}, u\}$ generates $\mathcal{M}$ as a von Neumann algebra. If there exist Haar unitary elements $v_1, v_2, \ldots$ and $w_1, w_2, \ldots$ in $\mathcal{N}$ such that $\|v_p u - u w_p\|_2 \to 0$ as $p \to \infty$, then*

$\mathfrak{K}_2(\mathcal{M}) = 0$. In particular, if there are Haar unitary elements v, w in $\mathcal{N}$, such that $vu = uw$, then $\mathfrak{K}_2(\mathcal{M}) = 0$.

Proof. Suppose that $\{x_1, \ldots, x_n\}$ is a family of generators of $\mathcal{N}$. We know that $\{x_1, \ldots, x_n, u\}$ is a family of generators of $\mathcal{M}$.

For every $0 < \omega < 1$, $0 < r < 1$, there exist an integer $p > 0$ and two Haar unitary elements v_p, w_p in $\mathcal{N}$ such that

$$\|v_p u - u w_p\|_2 < \frac{r\omega}{65}.$$

Note that $\{x_1, \ldots, x_n, v_p, w_p\}$ is also a family of generators of $\mathcal{N}$.

For $R > 1$, $m \in \mathbb{N}$, $\epsilon > 0$ and $k \in \mathbb{N}$, by the definition of the orbit covering number, there exists a set $\{\mathcal{U}(B_1^\lambda, \ldots, B_n^\lambda, V^\lambda, W^\lambda; \frac{r\omega}{64})\}_{\lambda \in \Lambda_k}$ of $\frac{r\omega}{64}$-orbit-balls in $\mathcal{M}_k(\mathbb{C})^{n+2}$ that cover $\Gamma_R(x_1, \ldots, x_n, v_p, w_p; m, k, \epsilon)$, where the cardinality of Λ satisfies $|\Lambda_k| = \nu(\Gamma_R(x_1, \ldots, x_n, v_p, w_p; m, k, \epsilon), \frac{r\omega}{64})$. When m is sufficient large, ϵ is sufficient small, by Proposition 1 we can assume that all V^λ, W^λ are Haar unitary matrices in $\mathcal{M}_k(\mathbb{C})$.

For m sufficiently large and ϵ sufficiently small, when $(A_1, \ldots, A_n, V, W, U)$ is contained in $\Gamma_R(x_1, \ldots, x_n, v_p, w_p, u; m, k, \epsilon)$ then, by Proposition 1, there exists a unitary element U_1 in $\mathcal{U}(k)$ so that

$$\|U_1 - U\|_2 < \frac{r\omega}{64} \quad \text{and} \quad \|VU_1 - U_1 W\|_2 < \frac{r\omega}{64}.$$

It is easy to see that $(A_1, \ldots, A_n, V, W)$ is also in $\Gamma_R(x_1, \ldots, x_n, v_p, w_p; m, k, \epsilon)$. Since $\Gamma_R(x_1, \ldots, x_n, v_p, w_p; m, k, \epsilon)$ is covered by the set $\{\mathcal{U}(B_1^\lambda, \ldots, B_n^\lambda, V^\lambda, W^\lambda; \frac{r\omega}{64})\}_{\lambda \in \Lambda_k}$ of $\frac{r\omega}{64}$-orbit-balls, there exist some $\lambda \in \Lambda_k$ and $X \in \mathcal{U}(k)$ such that

$$\left\|(A_1, \ldots, A_n, V, W) - \left(XB_1^\lambda X^*, \ldots, XB_n^\lambda X^*, XV^\lambda X^*, XW^\lambda X^*\right)\right\|_2 \leqslant \frac{r\omega}{64}.$$

Hence,

$$\left\|V^\lambda X^* U_1 X - X^* U_1 X W^\lambda\right\|_2 = \left\|XV^\lambda X^* U_1 - U_1 X W^\lambda X^*\right\|_2 \leqslant \frac{r\omega}{16}.$$

Note that V^λ, W^λ were chosen to be Haar unitary matrices in $\mathcal{M}_k(\mathbb{C})$. From Lemma 3, it follows that there exists a set $\{Ball(U_{\lambda,\sigma}; \frac{\omega}{4})\}_{\sigma \in \Sigma_k}$ of $\frac{\omega}{4}$-balls in $\mathcal{U}(k)$ that cover $\Omega(V^\lambda, W^\lambda; \frac{r\omega}{16})$ with $|\Sigma_k| \leqslant (\frac{24}{\omega})^{4rk^2}$, i.e., there exists some $U_{\lambda,\sigma}$ in $\{U_{\lambda,\sigma}\}_{\sigma \in \Sigma_k}$ such that

$$\left\|X^* U_1 X - U_{\lambda,\sigma}\right\|_2 = \left\|U_1 - XU_{\lambda,\sigma} X^*\right\|_2 \leqslant \frac{\omega}{4}.$$

Thus for such an $(A_1, \ldots, A_n, V, W, U)$ in $\Gamma_R(x_1, \ldots, x_n, v_p, w_p, u; m, k, \epsilon)$, there exist some $(B_1^\lambda, \ldots, B_n^\lambda, V^\lambda, W^\lambda)$ and $U_{\lambda,\sigma}$ such that

$$\left\|(A_1, \ldots, A_n, U) - \left(XB_1^\lambda X^*, \ldots, XB_n^\lambda X^*, XU_{\lambda,\sigma} X^*\right)\right\|_2 \leqslant \frac{\omega}{2},$$

for some $X \in \mathcal{U}(k)$, i.e.,

$$(A_1, \ldots, A_n, U) \in \mathcal{U}\big(B_1^\lambda, \ldots, B_n^\lambda, U_{\lambda,\sigma}; \omega\big).$$

Hence, by the definition of the free orbit-dimension, we have shown

$$\begin{aligned}
0 \leqslant \mathfrak{K}(x_1, \ldots, x_n, u : v_p, w_p; \omega, R) &\leqslant \inf_{m\in\mathbb{N}, \epsilon>0} \limsup_{k\to\infty} \frac{\log(|\Lambda_k||\Sigma_k|)}{-k^2\log\omega} \\
&\leqslant \inf_{m\in\mathbb{N}, \epsilon>0} \limsup_{k\to\infty} \left(\frac{\log(|\Lambda_k|)}{-k^2\log\omega} + \frac{\log(\frac{24}{\omega})^{4rk^2}}{-k^2\log\omega} \right) \\
&\leqslant 0 + 4r \cdot \frac{\log 24 - \log\omega}{-\log\omega},
\end{aligned}$$

since $\mathfrak{K}_2(x_1, \ldots, x_n, v_p, w_p) \leqslant \mathfrak{K}_2(\mathcal{N}) = 0$. Thus, by Lemma 2,

$$0 \leqslant \mathfrak{K}(x_1, \ldots, x_n, u; \omega) = \mathfrak{K}(x_1, \ldots, x_n, u : v_p, w_p; \omega) \leqslant 4r \cdot \frac{\log 24 - \log\omega}{-\log\omega}.$$

Because r is an arbitrarily small positive number, we have $\mathfrak{K}(x_1, \ldots, x_n, u; \omega) = 0$; whence, $\mathfrak{K}_2(x_1, \ldots, x_n, u) = 0$. By Theorem 1, $\mathfrak{K}_2(\mathcal{M}) = 0$. □

Theorem 5. *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful normal tracial state τ. Suppose $\mathcal{M}$ is generated by von Neumann subalgebras $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{M}$. If $\mathfrak{K}_2(\mathcal{N}_1) = \mathfrak{K}_2(\mathcal{N}_2) = 0$ and $\mathcal{N}_1 \cap \mathcal{N}_2$ is a diffuse von Neumann subalgebra of $\mathcal{M}$, then $\mathfrak{K}_2(\mathcal{M}) = 0$.*

Proof. Suppose that $\{x_1, \ldots, x_n\}$ is a family of generators of $\mathcal{N}_1$ and $\{y_1, \ldots, y_p\}$ a family of generators of $\mathcal{N}_2$. Since $\mathcal{N}_1 \cap \mathcal{N}_2$ is a diffuse von Neumann subalgebra, we can find a Haar unitary u in $\mathcal{N}_1 \cap \mathcal{N}_2$.

For every $R > 1 + \max_{1\leqslant i\leqslant n, 1\leqslant j\leqslant p}\{\|x_i\|, \|y_j\|\}$, $0 < \omega < \frac{1}{2n}$, $0 < r < 1$ and $m \in \mathbb{N}$, $\epsilon > 0$, $k \in \mathbb{N}$, there exists a set $\{\mathcal{U}(B_1^\lambda, \ldots, B_n^\lambda, U_\lambda; \frac{r\omega}{24R})\}_{\lambda\in\Lambda_k}$ of $\frac{r\omega}{24R}$-orbit-balls in $\mathcal{M}_k(\mathbb{C})^{n+1}$ covering $\Gamma_R(x_1, \ldots, x_n, u; m, k, \epsilon)$ with $|\Lambda_k| = \nu(\Gamma_R(x_1, \ldots, x_n, u; m, k, \epsilon), \frac{r\omega}{24R})$.

Also there exists a set $\{\mathcal{U}(D_1^\sigma, \ldots, D_p^\sigma, U_\sigma; \frac{r\omega}{24R})\}_{\sigma\in\Sigma_k}$ of $\frac{r\omega}{24R}$-orbit-balls in $\mathcal{M}_k(\mathbb{C})^{p+1}$ that cover $\Gamma_R(y_1, \ldots, y_p, u; m, k, \epsilon)$ with $|\Sigma_k| = \nu(\Gamma_R(y_1, \ldots, y_p, u; m, k, \epsilon), \frac{r\omega}{24R})$. When m is sufficiently large and ϵ is sufficiently small, by Proposition 1 we can assume all U_λ, U_σ to be Haar unitary matrices in $\mathcal{M}_k(\mathbb{C})$.

For each $(A_1, \ldots, A_n, C_1, \ldots, C_p, U)$ in $\Gamma_R(x_1, \ldots, x_n, y_1, \ldots, y_p, u; m, k, \epsilon)$, we know that $(A_1, \ldots, A_n, U)$ is contained in $\Gamma_R(x_1, \ldots, x_n, u; m, k, \epsilon)$ and $(C_1, \ldots, C_p, U)$ is contained in $\Gamma_R(y_1, \ldots, y_p, u; m, k, \epsilon)$. Note $\Gamma_R(x_1, \ldots, x_n, u; m, k, \epsilon)$ is covered by the set $\{\mathcal{U}(B_1^\lambda, \ldots, B_n^\lambda, U_\lambda; \frac{r\omega}{24R})\}_{\lambda\in\Lambda_k}$ of $\frac{r\omega}{24R}$-orbit-balls and $\Gamma_R(y_1, \ldots, y_p, u; m, k, \epsilon)$ is covered by the set $\{\mathcal{U}(D_1^\sigma, \ldots, D_p^\sigma, U_\sigma; \frac{r\omega}{24R})\}_{\sigma\in\Sigma_k}$ of $\frac{r\omega}{24R}$-orbit-balls. Hence, there exist some $\lambda \in \Lambda_k$, $\sigma \in \Sigma_k$ and W_1, W_2 in $\mathcal{U}(k)$ such that

$$\begin{aligned}
\big\|(A_1, \ldots, A_n, U) - \big(W_1 B_1^\lambda W_1^*, \ldots, W_1 B_n^\lambda W_1^*, W_1 U_\lambda W_1^*\big)\big\|_2 &\leqslant \frac{r\omega}{24R}, \\
\big\|(C_1, \ldots, C_p, U) - \big(W_2 D_1^\sigma W_2^*, \ldots, W_2 D_p^\sigma W_2^*, W_2 U_\sigma W_2^*\big)\big\|_2 &\leqslant \frac{r\omega}{24R}.
\end{aligned}$$

Hence,

$$\left\| W_2^* W_1 U_\lambda - U_\sigma W_2^* W_1 \right\|_2 = \left\| W_1 U_\lambda W_1^* - W_2 U_\sigma W_2^* \right\|_2 \leqslant \frac{r\omega}{12R}.$$

From our assumption that U_λ, U_σ are Haar unitary matrices in $\mathcal{M}_k(\mathbb{C})$, by Lemma 3 we know that there exists a set $\{Ball(U_{\lambda\sigma\gamma}; \frac{\omega}{3R})\}_{\gamma\in\mathcal{I}_k}$ of $\frac{\omega}{3R}$-balls in $\mathcal{U}(k)$ that cover $\Omega(U_\lambda, U_\sigma; \frac{r\omega}{12R})$ with the cardinality of $\mathcal{I}_k$ never exceeding $(\frac{18R}{\omega})^{4rk^2}$. Then there exists some $\gamma \in \mathcal{I}_k$ such that $\| W_2^* W_1 - U_{\lambda\sigma\gamma} \|_2 \leqslant \frac{\omega}{3R}$. This in turn implies

$$\begin{aligned}&\Big\|(A_1,\ldots,A_n,C_1,\ldots,C_p,U) - \big(W_2 U_{\lambda\sigma\gamma} B_1^\lambda U_{\lambda\sigma\gamma}^* W_2^*, \ldots, W_2 U_{\lambda\sigma\gamma} B_n^\lambda U_{\lambda\sigma\gamma}^* W_2^*,\\ &\quad W_2 D_1^\sigma W_2^*, \ldots, W_2 D_p^\sigma W_2^*, W_2 U_\sigma W_2^*\big)\Big\|_2 \leqslant n\omega\end{aligned}$$

for some $\lambda \in \Lambda_k, \sigma \in \Sigma_k, \gamma \in \mathcal{I}_k$ and $W_2 \in \mathcal{U}(k)$, i.e.,

$$(A_1,\ldots,A_n,C_1,\ldots,C_p,U) \in \mathcal{U}\big(U_{\lambda\sigma\gamma} B_1^\lambda U_{\lambda\sigma\gamma}^*, \ldots, U_{\lambda\sigma\gamma} B_n^\lambda U_{\lambda\sigma\gamma}^*, D_1^\sigma, \ldots, D_p^\sigma, U_\sigma; 2n\omega\big).$$

Hence, by the definition of the free orbit-dimension we get

$$\begin{aligned}&\mathfrak{K}(x_1,\ldots,x_n,y_1,\ldots,y_p,u; 2n\omega, R)\\ &\quad\leqslant \inf_{m\in\mathbb{N},\epsilon>0} \limsup_{k\to\infty} \frac{\log(|\Lambda_k||\Sigma_k||\mathcal{I}_k|)}{-k^2\log(2n\omega)}\\ &\quad\leqslant \inf_{m\in\mathbb{N},\epsilon>0} \limsup_{k\to\infty}\left(\frac{\log(|\Lambda_k|)}{-k^2\log(2n\omega)} + \frac{\log(|\Sigma_k|)}{-k^2\log(2n\omega)} + \frac{\log(|\mathcal{I}_k|)}{-k^2\log(2n\omega)}\right)\\ &\quad\leqslant 0 + \inf_{m\in\mathbb{N},\epsilon>0} \limsup_{k\to\infty} \frac{\log(\frac{18R}{\omega})^{4rk^2}}{-k^2\log(2n\omega)}\\ &\quad\leqslant 4r\cdot\frac{\log(18R) - \log\omega}{-\log(2n\omega)},\end{aligned}$$

since $\mathfrak{K}_2(N_1) = \mathfrak{K}_2(N_2) = 0$. Since r is an arbitrarily small positive number, we get that $\mathfrak{K}(x_1,\ldots,x_n,y_1,\ldots,y_p,u; 2n\omega, R) = 0$; whence $\mathfrak{K}_2(x_1,\ldots,x_n,y_1,\ldots,y_p,u) = 0$. By Theorem 1, $\mathfrak{K}_2(\mathcal{M}) = 0$. $\square$

4. Applications

In this section, we discuss a few applications of the results from the last section. (We only consider finite von Neumann algebra $\mathcal{M}$ that can be faithfully embedded into the ultrapower of the hyperfinite II_1 factor.) Let $L(F_n)$ denote the free group factor on n generators. By Voiculescu's fundamental result in [18], we know $\delta_0(L(F_n)) \geqslant n$, where δ_0 is Voiculescu's modified free entropy dimension. By combining Theorems 1–5, we obtain the results in [5,7,8,19,20]. Here are a few sample applications improving earlier results.

The following lemma can be proved using [1, Theorem 5.3].

Lemma 4. *If $\mathcal{M}$ is a* II_1 *factor with property Γ with the tracial state τ, then there are a hyperfinite* II_1 *factor $\mathcal{R}$ and a sequence $\{u_n\}$ of Haar unitary elements of $\mathcal{R}$ such that*

$$\|u_n x - x u_n\|_2 \to 0$$

for every $x \in \mathcal{M}$.

Corollary 1. *If $\mathcal{M}$ is a* II_1 *factor with property Γ, then $\mathfrak{K}_2(\mathcal{M}) = 0$.*

Proof. Choose a hyperfinite II_1 factor $\mathcal{R}$ and a sequence of Haar unitary elements $u_1, u_2, \ldots$ in $\mathcal{R}$ such that $\lim_{n\to\infty} \|x u_n - u_n x\|_2 = 4$ for every x in $\mathcal{M}$. Since $\mathcal{R}$ is hyperfinite, $\mathfrak{K}_2(\mathcal{R}) = 0$. If $\{v_1, v_2, \ldots\}$ is a sequence of Haar unitaries that generate $\mathcal{M}$, it inductively follows from Theorem 4 that, for each $n \geqslant 1$

$$\mathfrak{K}_2\big((\mathcal{R} \cup \{v_1, \ldots, v_n\})''\big) = 0.$$

Whence, by Theorem 3, $\mathfrak{K}_2(\mathcal{M}) = 0$. $\square$

A maximal abelian self-adjoint subalgebra (or, masa) $\mathcal{A}$ in a II_1 factor $\mathcal{M}$ is called a *Cartan subalgebra* if the *normalizer algebra* of $\mathcal{A}$,

$$\mathcal{N}_1(\mathcal{A}) = \big\{u \in \mathcal{U}(\mathcal{M}) \colon\ u^* \mathcal{A} u = \mathcal{A}\big\}''$$

equals $\mathcal{M}$. We define $\mathcal{N}_{k+1}(\mathcal{A}) = \mathcal{N}_1(\mathcal{N}_k(\mathcal{A}))$ for $k \geqslant 1$, and $\mathcal{N}_\infty(\mathcal{A}) = (\bigcup_{1 \leqslant k < \infty} \mathcal{N}_k(\mathcal{A}))''$. The following is a direct consequence of Theorems 4 and 3.

Corollary 2. *Suppose $\mathcal{M}$ is a finitely generated type* II_1 *factor, and $\mathcal{A}$ is a diffuse von Neumann subalgebra with $\mathfrak{K}_2(\mathcal{A}) = 0$. If $\mathcal{M} = \mathcal{N}_k(\mathcal{A})$ for some k, $1 \leqslant k \leqslant \infty$, then $\mathfrak{K}_2(\mathcal{M}) = 0$, and $\delta_0(\mathcal{M}) \leqslant 1$.*

Some applications of free entropy to finite von Neumann algebras (nonprime factors, some II_1 factors with property T) are consequences of a result of L. Ge and J. Shen [8], which states that if $\mathcal{M}$ is a II_1 von Neumann algebra generated by a sequence of Haar unitary elements $\{u_i\}_{i=1}^{\infty}$ in $\mathcal{M}$ such that each $u_{i+1} u_i u_{i+1}^*$ is in the von Neumann subalgebra generated by $\{u_1, \ldots, u_i\}$ in $\mathcal{M}$, then $\delta_0(\mathcal{M}) \leqslant 1$. This result is also a consequence of Theorem 4. Here is another result.

Corollary 3. *Suppose $\mathcal{M}$ is a finitely generated type* II_1 *factor that is generated by a family $\{u_{ij} \colon\ 1 \leqslant i, j < \infty\}$ of Haar unitary elements in $\mathcal{M}$ such that*:

(1) *for each i, j, $u_{i+1,j} u_{ij} u_{i+1,j}^*$ is in the von Neumann subalgebra generated by $\{u_{1j}, \ldots, u_{ij}\}$; and*
(2) *for each $j \geqslant 1$, $\{u_{1j}, u_{2j}, \ldots\} \cap \{u_{1,j+1}, u_{2,j+1}, \ldots\} \neq \emptyset$.*

Then $\mathfrak{K}_2(\mathcal{M}) = 0$, $\delta_0(\mathcal{M}) \leqslant 1$. Thus $\mathcal{M}$ is not $*$-isomorphic to any $L(F(n))$ for $n \geqslant 2$.

Remark 1. Suppose that G is a group generated by elements a, b, c such that $ab^2 = b^3 a$ and $ac^2 = c^3 a$. The group von Neumann algebra associated with G is a type II_1 factor, and the preceding corollary implies that $\mathfrak{K}_2(L(G)) = 0$ and $\delta_0(L(G)) \leqslant 1$.

The next two corollaries follows directly from Corollary 3.

Corollary 4. *Suppose $\mathcal{M}$ is a nonprime* II_1 *factor, i.e.* $\mathcal{M} \simeq \mathcal{N}_1 \otimes \mathcal{N}_2$ *for some* II_1 *subfactors* $\mathcal{N}_1, \mathcal{N}_2$. *Then* $\mathfrak{K}_2(\mathcal{M}) = 0$, $\delta_0(\mathcal{M}) \leqslant 1$. *Thus $\mathcal{M}$ is not *-isomorphic to any $L(F(n))$ for $n \geqslant 2$.*

Corollary 5. *If* $\mathcal{M} = L(SL(\mathbb{Z}, 2m+1))$ *is the group von Neumann algebra associated with* $SL(\mathbb{Z}, 2m+1)$ (*the special linear group with integer entries*) *for* $m \geqslant 1$, *then* $\mathfrak{K}_2(\mathcal{M}) = 0$, $\delta_0(\mathcal{M}) \leqslant 1$. *Thus $\mathcal{M}$ is not *-isomorphic to any $L(F(n))$ for $n \geqslant 2$.*

5. Decompositions of type II_1 factors

In [6] L. Ge and S. Popa defined a type II_1 factor to be *weakly n-thin*, if it contains hyperfinite subalgebras $\mathcal{R}_0, \mathcal{R}_1$ and n vectors $\xi_1, \ldots, \xi_n$ in $L^2(\mathcal{M}, \tau)$ such that $L^2(\mathcal{M}, \tau) = \overline{\mathrm{span}}^{\|\cdot\|_2}(\mathcal{R}_0\{\xi_1, \ldots, \xi_n\}\mathcal{R}_1)$. They showed that $L(F_m)$ is not weakly n-thin for $m > 2 + 2n$. In [16], Stefan extended the preceding result in [6] and showed that free group factors are not decomposable over nonprime subfactors and abelian subalgebras. Motivated by these facts, we have the following theorem.

Theorem 6. *Suppose that $\mathcal{M}$ is a finitely generated type* II_1 *factor with a tracial state τ. Suppose there exist von Neumann subalgebras $\mathcal{N}_0, \mathcal{N}_1$ of $\mathcal{M}$ with $\mathfrak{K}_2(\mathcal{N}_0) = \mathfrak{K}_2(\mathcal{N}_1) = 0$ and n vectors $\xi_1, \ldots, \xi_n$ in $L^2(\mathcal{M}, \tau)$ such that*

$$L^2(\mathcal{M}, \tau) = \overline{\mathrm{span}}^{\|\cdot\|_2}\mathcal{N}_0\{\xi_1, \ldots, \xi_n\}\mathcal{N}_1.$$

*Then $\mathfrak{K}_1(\mathcal{M}) \leqslant 1 + 2n$ and $\delta_0(\mathcal{M}) \leqslant 2 + 2n$. Thus $\mathcal{M}$ is not *-isomorphic to $L(F_m)$ for $m > 2 + 2n$.*

Proof. Suppose $x_1, \ldots, x_p$ is a family of self-adjoint elements in $\mathcal{M}$ that generate $\mathcal{M}$ as a von Neumann algebra. Note there exist von Neumann subalgebras $\mathcal{N}_0, \mathcal{N}_1$ of $\mathcal{M}$ with $\mathfrak{K}_2(\mathcal{N}_0) = \mathfrak{K}_2(\mathcal{N}_1) = 0$ and n vectors $\xi_1, \ldots, \xi_n$ in $L^2(\mathcal{M}, \tau)$ such that $\overline{\mathrm{span}}^{\|\cdot\|_2}\mathcal{N}_0\{\xi_1, \ldots, \xi_n\}\mathcal{N}_1 = L^2(\mathcal{M}, \tau)$. We can choose self-adjoint elements $y_1, y_2, \ldots, y_{2n-1}, y_{2n}$ in $\mathcal{M}$ to approximate $\mathrm{Re}\,\xi_1, \mathrm{Im}\,\xi_1, \ldots, \mathrm{Re}\,\xi_n, \mathrm{Im}\,\xi_n$, respectively. Hence, for any positive $\omega < 1$, there are a positive integer N, elements $\{a_{i,j,l}\}_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant N, 1 \leqslant l \leqslant 2n}$ in $\mathcal{N}_0$, $\{b_{i,j,l}\}_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant N, 1 \leqslant l \leqslant 2n}$ in $\mathcal{N}_1$, and self-adjoint elements $y_1, \ldots, y_{2n}$ in $\mathcal{M}$ such that

$$\sum_{i=1}^{p} \left\| x_i - \sum_{j=1}^{N} \sum_{l=1}^{2n} a_{i,j,l} y_l b_{i,j,l} \right\|_2^2 \leqslant \left(\frac{\omega}{8}\right)^2.$$

Without loss of generality, we can assume that $\{a_{i,j,l}\}_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant N, 1 \leqslant l \leqslant n}$ generates $\mathcal{N}_0$ and $\{b_{i,j,l}\}_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant N, 1 \leqslant l \leqslant n}$ generates $\mathcal{N}_1$ as von Neumann algebras. Otherwise we should add generators of $\mathcal{N}_0, \mathcal{N}_1$ into the families.

Let a be $\max_{1 \leqslant i \leqslant p}\{\|x_i\|_2\} + 2$. From now on the sequence $z_1, \ldots, z_s, \ldots, z_t$ is denoted by $(z_s)_{s=1,\ldots,t}$ or $(z_s)_s$ if there is no confusion arising from the range of index, where z_s is an element in $\mathcal{M}$ or a matrix in $\mathcal{M}_k(\mathbb{C})$.

For $R > a$, define mapping $\psi : (\mathcal{M}_k(\mathbb{C})^N)^{2n} \times \mathcal{M}_k(\mathbb{C})^{2n} \times (\mathcal{M}_k(\mathbb{C})^N)^{2n} \to \mathcal{M}_k(\mathbb{C})$ as follows,

$$\psi\big((D_{j,l})_{jl}, (E_l)_l, (F_{j,l})_{jl}\big) = \sum_{j=1}^{N}\sum_{l=1}^{2n} D_{j,l} E_l L_{j,l}.$$

Let $(\mathcal{M}_k(\mathbb{C}))_R$ be the collection of all A in $\mathcal{M}_k(\mathbb{C})$ such that $\|A\| \leqslant R$. Then there always exists a constant $D > 1$, not depending on k, such that

$$\begin{aligned}
&\big\|\big(\psi\big(\big(A^{(1)}_{1,j,l}\big)_{jl}, (Y_l)_l, \big(B^{(1)}_{1,j,l}\big)_{jl}\big), \ldots, \psi\big(\big(A^{(1)}_{p,j,l}\big)_{jl}, (Y_l)_l, \big(B^{(1)}_{p,j,l}\big)_{jl}\big)\big) \\
&\quad - \big(\psi\big(\big(A^{(2)}_{1,j,l}\big)_{jl}, (Y_l)_l, \big(B^{(2)}_{1,j,l}\big)_{jl}\big), \ldots, \psi\big(\big(A^{(2)}_{p,j,l}\big)_{jl}, (Y_l)_l, \big(B^{(2)}_{p,j,l}\big)_{jl}\big)\big)\big\|_2 \\
&\quad \leqslant D\big\|\big(\big(A^{(1)}_{i,j,l}\big)_{ijl}, \big(B^{(1)}_{i,j,l}\big)_{ijl}\big) - \big(\big(A^{(2)}_{i,j,l}\big)_{ijl}, \big(B^{(2)}_{i,j,l}\big)_{ijl}\big)\big\|_2,
\end{aligned} \tag{5.1}$$

for all

$$\big\{A^{(1)}_{i,j,l}, Y_l, B^{(1)}_{i,j,l}, A^{(2)}_{i,j,l}, B^{(2)}_{i,j,l}\big\}_{i,j,l} \subset \big(\mathcal{M}_k(\mathbb{C})\big)_R \quad \forall k \in N.$$

For m sufficiently large, ϵ sufficiently small and k sufficiently large, if

$$\big(X_1, \ldots, X_p, (A_{i,j,l})_{ijl}, (Y_l)_l, (B_{i,j,l})_{ijl}\big) \in \Gamma_R\big(x_1, \ldots, x_p, (a_{i,j,l})_{ijl}, (y_l)_l, (b_{i,j,l})_{ijl}; k, m, \epsilon\big),$$

then

$$\begin{aligned}
&\big\|(X_1, \ldots, X_p) - \big(\psi\big((A_{1,j,l})_{jl}, (Y_l)_l, (B_{1,j,l})_{jl}\big), \ldots, \psi\big((A_{p,j,l})_{jl}, (Y_l)_l, (B_{p,j,l})_{jl}\big)\big)\big\|_2 \\
&\quad = \left(\sum_{i=1}^{p}\left\|X_i - \sum_{j=1}^{N}\sum_{l=1}^{2n} A_{i,j,l} Y_l B_{i,j,l}\right\|_2^2\right)^{1/2} \leqslant \frac{\omega}{8},
\end{aligned} \tag{5.2}$$

and

$$\big((A_{i,j,l})_{ijl}\big) \in \Gamma_R\big((a_{i,j,l})_{ijl}; k, m, \epsilon\big), \quad \text{and} \quad \big((B_{i,j,l})_{ijl}\big) \in \Gamma_R\big((b_{i,j,l})_{ijl}; k, m, \epsilon\big).$$

On the other hand, from the definition of the orbit covering number, it follows there exists a set $\{\mathcal{U}((A^{\lambda}_{ijl})_{ijl}; \frac{\omega}{16D})\}_{\lambda \in \Lambda_k}$, or $\{\mathcal{U}((B^{\sigma}_{ijl})_{ijl}; \frac{\omega}{16D})\}_{\sigma \in \Sigma_k}$, of $\frac{\omega}{16D}$-orbit-balls that cover $\Gamma_R((a_{i,j,l})_{ijl}; k, m, \epsilon)$, or $\Gamma_R((b_{i,j,l})_{ijl}; k, m, \epsilon)$, respectively, with

$$|\Lambda_k| = \nu\left(\Gamma_R\big((a_{i,j,l})_{ijl}; k, m, \epsilon\big), \frac{\omega}{16D}\right), \qquad |\Sigma_k| = \nu\left(\Gamma_R\big((b_{i,j,l})_{ijl}; k, m, \epsilon\big), \frac{\omega}{16D}\right).$$

Therefore for such sequence $((A_{i,j,l})_{ijl}, (B_{i,j,l})_{ijl})$, there exist some $\lambda \in \Lambda_k$, $\sigma \in \Sigma_k$ and W_1, W_2 in $\mathcal{U}(k)$ such that

$$\big\|\big((A_{i,j,l})_{ijl}, (B_{i,j,l})_{ijl}\big) - \big(\big(W_1 A^{\lambda}_{i,j,l} W_1^*\big)_{ijl}, \big(W_2 B^{\sigma}_{i,j,l} W_2^*\big)_{ijl}\big)\big\|_2 \leqslant \frac{\omega}{8D}. \tag{5.3}$$

Thus, from (5.1), (5.2) and (5.3), it follows that

$$\begin{aligned}
&\Big\|(X_1,\dots,X_p)-\Big(\psi\big(\big(W_1A^{\lambda}_{1,j,l}W_1^*\big)_{jl},(Y_l)_l,\big(W_2B^{\sigma}_{1,j,l}W_2^*\big)_{jl}\big),\dots,\\
&\quad\psi\big(\big(W_1A^{\lambda}_{p,j,l}W_1^*\big)_{jl},(Y_l)_l,\big(W_2B^{\sigma}_{p,j,l}W_2^*\big)_{jl}\big)\Big)\Big\|_2\\
&\quad=\left(\sum_{1\leqslant i\leqslant p}\left\|X_i-\sum_{j=1}^{N}\sum_{l=1}^{2n}W_1A^{\lambda}_{i,j,l}W_1^*Y_lW_2B^{\sigma}_{i,j,l}W_2^*\right\|_2^2\right)^{1/2}\leqslant\frac{\omega}{4}.
\end{aligned}\tag{5.4}$$

Hence

$$\left(\sum_{1\leqslant i\leqslant p}\left\|W_1^*X_iW_1-\sum_{j=1}^{N}\sum_{l=1}^{2n}\big(A^{\lambda}_{i,j,l}W_1^*Y_lW_2B^{\sigma}_{i,j,l}\big)W_2^*W_1\right\|_2^2\right)^{1/2}\leqslant\frac{\omega}{4}.\tag{5.5}$$

By a result of Szarek [17], there exists a $\frac{\omega}{4ap}$-net $\{U_\gamma\}_{\gamma\in_k}$ in $\mathcal{U}(k)$ that cover $\mathcal{U}(k)$ with respect to the uniform norm such that the cardinality of $\mathcal{I}_k$ does not exceed $(\frac{4apC}{\omega})^{k^2}$, where C is a universal constant. Thus $\|W_2^*W_1-U_\gamma\|\leqslant\frac{\omega}{4ap}$, for some $\gamma\in\mathcal{I}_k$. Because of (4.5), we know

$$\left\|\sum_{j=1}^{N}\sum_{l=1}^{2n}A^{\lambda}_{i,j,l}W_1^*Y_lW_2B^{\sigma}_{i,j,l}\right\|_2\leqslant\|X_i\|_2+\omega<a.\tag{5.6}$$

From (5.5) and (5.6), we have

$$\left(\sum_{1\leqslant i\leqslant p}\left\|W_1^*X_iW_1-\left(\sum_{j=1}^{N}\sum_{l=1}^{2n}A^{\lambda}_{i,j,l}W_1^*Y_lW_2B^{\sigma}_{i,j,l}\right)U_\gamma\right\|_2^2\right)^{1/2}\leqslant\frac{\omega}{2}.\tag{5.7}$$

Define a linear mapping $\Psi_{\lambda\sigma\gamma}:\mathcal{M}_k(\mathbb{C})^{2n}\to\mathcal{M}_k(\mathbb{C})^p$ as follows:

$$\Psi_{\lambda\sigma\gamma}(S_1,\dots,S_{2n})=\left(\frac12\sum_{j=1}^{N}\sum_{l=1}^{2n}\big(A^{\lambda}_{i,j,l}S_lB^{\sigma}_{i,j,l}\big)U_\gamma+\big(\big(A^{\lambda}_{i,j,l}S_lB^{\sigma}_{i,j,l}\big)U_\gamma\big)^*\right)_{i=1,\dots,p}.$$

Let $\mathfrak{F}_{\lambda\sigma\gamma}$ be the range of $\Psi_{\lambda\sigma\gamma}$ in $\mathcal{M}_k(\mathbb{C})^p$. It is easy to see that $\mathfrak{F}_{\lambda\sigma\gamma}$ is a real-linear subspace of $\mathcal{M}_k(\mathbb{C})^p$ whose real dimension does not exceed $2nk^2$. Therefore the bounded subset

$$\big\{(H_1,\dots,H_p)\in\mathfrak{F}_{\lambda\sigma\gamma}\ \big|\ \big\|(H_1,\dots,H_p)\big\|_2\leqslant ap\big\}\tag{5.8}$$

of $\mathcal{M}_k(\mathbb{C})^p$ can be covered by a set $\{(H_1^{\lambda\sigma\gamma,\rho},\dots,H_p^{\lambda\sigma\gamma,\rho})\}_{\rho\in\mathcal{S}_k}$ of ω-balls with the cardinality of $\mathcal{S}_k$ satisfying $|\mathcal{S}_k|\leqslant(\frac{3ap}{\omega})^{2nk^2}$. But we know from (5.6) that

$$
\begin{aligned}
&\left\| \left(\frac{1}{2} \sum_{j=1}^{N} \sum_{l=1}^{2n} \left(A_{i,j,l}^{\lambda} W_1^* Y_l W_2 B_{i,j,l}^{\sigma}\right) U_\gamma + \left(\left(A_{i,j,l}^{\lambda} W_1^* Y_l W_2 B_{i,j,l}^{\sigma}\right) U_\gamma \right)^* \right)_{i=1,\dots,p} \right\|_2 \\
&\quad = \left(\sum_{i=1}^{p} \left\| \frac{1}{2} \sum_{j=1}^{N} \sum_{l=1}^{2n} \left(A_{i,j,l}^{\lambda} W_1^* Y_l W_2 B_{i,j,l}^{\sigma}\right) U_\gamma + \left(\left(A_{i,j,l}^{\lambda} W_1^* Y_l W_2 B_{i,j,l}^{\sigma}\right) U_\gamma \right)^* \right\|_2^2 \right)^{1/2} \\
&\quad < ap, \qquad (5.9)
\end{aligned}
$$

and from (5.7) we have

$$
\begin{aligned}
&\left\| \left(W_1^* X_1 W_1, \dots, W_1^* X_p W_1\right) - \Psi_{\lambda\sigma\gamma}\left(W_1^* Y_1 W_2, \dots, W_1^* Y_{2n} W_2\right) \right\|_2 \\
&\quad = \left\| \left(W_1^* X_1 W_1, \dots, W_1^* X_p W_1\right) \right. \\
&\qquad \left. - \left(\frac{1}{2} \sum_{j=1}^{N} \sum_{l=1}^{2n} \left(A_{i,j,l}^{\lambda} W_1^* Y_l W_2 B_{i,j,l}^{\sigma}\right) U_\gamma + \left(\left(A_{i,j,l}^{\lambda} W_1^* Y_l W_2 B_{i,j,l}^{\sigma}\right) U_\gamma \right)^* \right)_{i=1,\dots,p} \right\|_2 \\
&\quad \leqslant \omega. \qquad (5.10)
\end{aligned}
$$

Thus, from (5.8), (5.9) and (5.10), there exists some $\rho \in \mathcal{S}_k$ such that

$$
\left\| \left(W_1^* X_1 W_1, \dots, W_1^* X_p W_1\right) - \left(H_1^{\lambda\sigma\gamma,\rho}, \dots, H_p^{\lambda\sigma\gamma,\rho}\right) \right\|_2 \leqslant 2\omega.
$$

By the definition of the free orbit-dimension, we know that

$$
\begin{aligned}
&\mathfrak{K}\left(x_1, \dots, x_p : (a_{ijl})_{ijl}, (y_l)_l, (b_{ijl})_{ijl}; 4\omega, R\right) \\
&\quad \leqslant \inf_{m\in\mathbb{N}, \epsilon>0} \limsup_{k\to\infty} \frac{\log(|\Lambda_k||\Sigma_k||\mathcal{I}_k||\mathcal{S}_k|)}{-k^2 \log(4\omega)} \\
&\quad \leqslant \inf_{m\in\mathbb{N}, \epsilon>0} \limsup_{k\to\infty} \left(\frac{\log|\Lambda_k|}{-k^2 \log(4\omega)} + \frac{\log|\Sigma_k|}{-k^2 \log(4\omega)} + \frac{\log(\frac{4apC}{\omega})^{k^2} (\frac{3ap}{\omega})^{2nk^2}}{-k^2 \log(4\omega)} \right) \\
&\quad = 0 + 0 + \frac{\log(4 \cdot (3ap)^{2n} \cdot apC) - (2n+1)\log\omega}{-\log(4\omega)},
\end{aligned}
$$

since $\mathfrak{K}_2(\mathcal{N}_0) = \mathfrak{K}_2(\mathcal{N}_1) = 0$. Thus, by Lemma 2

$$
\begin{aligned}
0 \leqslant \mathfrak{K}(x_1, \dots, x_p; 4\omega) &= \mathfrak{K}\left(x_1, \dots, x_p : (a_{ijl})_{ijl}, (y_l)_l, (b_{ijl})_{ijl}; 4\omega\right) \\
&\leqslant \frac{\log(4 \cdot (3ap)^{2n} \cdot apC) - (2n+1)\log\omega}{-\log(4\omega)}.
\end{aligned}
$$

By the definition of the free orbit-dimension, we obtain

$$
\mathfrak{K}_1(x_1, \dots, x_p) \leqslant \limsup_{\omega\to 0} \frac{\log(4 \cdot (3ap)^{2n} \cdot apC) - (2n+1)\log\omega}{-\log(4\omega)} \leqslant 1 + 2n.
$$

Hence, $\mathfrak{K}_1(\mathcal{M}) \leqslant 1 + 2n$ and $\delta_0(\mathcal{M}) \leqslant 2 + 2n$. □

Remark 2. The mapping $a \mapsto a^*$ extends from $\mathcal{M}$ to a unitary map on $L^2(\mathcal{M},\tau)$, so for $\xi \in L^2(\mathcal{M},\tau)$, it makes sense to talk about $\operatorname{Re}\xi = (\xi + \xi^*)/2$ and $\operatorname{Im}\xi = (\xi - \xi^*)/2i$. In particular, it makes sense to talk about self-adjoint elements of $L^2(\mathcal{M},\tau)$. If we have $\overline{\text{span}}^{\|\cdot\|_2}\mathcal{N}_0\{\xi_1,\ldots,\xi_n\}\mathcal{N}_1 = L^2(\mathcal{M},\tau)$ with $\xi_1,\ldots,\xi_n$ self-adjoint elements in $L^2(\mathcal{M},\tau)$, the proof of Theorem 6 yields $\mathfrak{K}_1(\mathcal{M}) \leqslant 1+n$ and $\delta_0(\mathcal{M}) \leqslant 2+n$.

Combining Theorem 6 and the preceding remark with Theorem 3, we have the following corollaries (see also [3,4,6,12,15,16]).

Corollary 6. *$L(F_n)$ has no simple maximal abelian self-adjoint subalgebra for $n \geqslant 4$.*

Corollary 7. *$L(F_n)$ is not a thin factor for $n \geqslant 4$.*

Remark 3. Another corollary of Theorem 6 is as follows. Suppose $\mathcal{M}$ is a II_1 factor with a tracial state τ. Suppose that $\mathcal{N}$ is a subfactor of $\mathcal{M}$ with finite index, i.e., $[\mathcal{M}:\mathcal{N}] = r < \infty$. If $\mathfrak{K}_2(\mathcal{N}) = 0$, then $\mathfrak{K}_1(\mathcal{M}) \leqslant 2[r]+3$ and $\delta_0(\mathcal{M}) \leqslant 2[r]+4$ where $[r]$ is the integer part of r.

References

[1] E. Christensen, F. Pop, A. Sinclair, R. Smith, Hochschild cohomology of factors with property Γ, Ann. of Math. (2) 158 (2) (2003) 635–659.
[2] M. Dostál, D. Hadwin, An alternative to free entropy for free group factors, in: International Workshop on Operator Algebra and Operator Theory, Linfen, 2001, Acta Math. Sin. (Engl. Ser.) 19 (3) (2003) 419–472.
[3] K. Dykema, Two applications of free entropy, Math. Ann. 308 (3) (1997) 547–558.
[4] L. Ge, Applications of free entropy to finite von Neumann algebras, Amer. J. Math. 119 (2) (1997) 467–485.
[5] L. Ge, Applications of free entropy to finite von Neumann algebras, II, Ann. of Math. (2) 147 (1) (1998) 143–157.
[6] L. Ge, S. Popa, On some decomposition properties for factors of type II_1, Duke Math. J. 94 (1) (1998) 79–101.
[7] L. Ge, J. Shen, Free entropy and property T factors, Proc. Natl. Acad. Sci. USA 97 (18) (2000) 9881–9885 (electronic).
[8] L. Ge, J. Shen, On free entropy dimension of finite von Neumann algebras, Geom. Funct. Anal. 12 (3) (2002) 546–566.
[9] D. Hadwin, Free entropy and approximate equivalence in von Neumann algebras, in: Operator Algebras and Operator Theory, Shanghai, 1997, in: Contemp. Math., vol. 228, Amer. Math. Soc., Providence, RI, 1998, pp. 111–131.
[10] K. Jung, The free entropy dimension of hyperfinite von Neumann algebras, Trans. Amer. Math. Soc. 355 (12) (2003) 5053–5089 (electronic).
[11] K. Jung, A free entropy dimension lemma, Pacific J. Math. 211 (2) (2003) 265–271.
[12] K. Jung, Some free entropy dimension inequalities for subfactors, arXiv: math.OA/0410594.
[13] K. Jung, Strongly 1-bounded von Neumann algebras, arXiv: math.OA/0510576.
[14] M. Stefan, The primality of subfactors of finite index in the interpolated free group factors, Proc. Amer. Math. Soc. 126 (8) (1998) 2299–2307.
[15] M. Stefan, An estimate of free entropy and applications, arXiv: math.OA/0402109.
[16] M. Stefan, Indecomposability of free group factors over nonprime subfactors and abelian subalgebras, Pacific J. Math. 219 (2) (2005) 365–390.
[17] S. Szarek, Metric entropy of homogeneous spaces, in: Quantum Probability, in: Banach Center Publ., vol. 43, Polish Acad. Sci., Warsaw, 1998, pp. 395–410.
[18] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory II, Invent. Math. 118 (1994) 411–440.
[19] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory III: The absence of Cartan subalgebras, Geom. Funct. Anal. 6 (1996) 172–199.
[20] D. Voiculescu, Free entropy dimension $\leqslant 1$ for some generators of property T factors of type II_1, J. Reine Angew. Math. 514 (1999) 113–118.