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Triviality of φ_4^4 theory in a finite volume scheme adapted to the broken phase

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ABSTRACT

We study the standard one-component φ^4-theory in four dimensions. A renormalized coupling is defined in a finite size renormalization scheme which becomes the standard scheme of the broken phase for large volumes. Numerical simulations are reported using the worm algorithm in the limit of infinite bare coupling. The cutoff dependence of the renormalized coupling closely follows the perturbative Callan Symanzik equation and the triviality scenario is hence further supported.

1. Introduction

The Z(2) symmetric quantum field theory of a single scalar field with φ^4 interaction is the number one textbook prototype model for all kinds of field theoretic methods. At the same time, with its possibility of spontaneous symmetry breaking, it may be viewed as the crudest caricature of the Higgs sector of the Standard Model. In this context the strongly conjectured triviality of the model in four space time dimensions is of physical interest as it leads to upper bounds[1] on the Higgs mass [2]. This is so because triviality means that the cutoff cannot be removed from the regularized theory without ending in a free Gaussian theory. The model is then interpreted as an effective theory in which an upper limit on tolerable unphysical cutoff effects implies an upper bound on the interaction strength which in turn is responsible for mass generation by the Higgs field.

Unfortunately, in the four-dimensional case we still have to rely on numerical methods to demonstrate triviality beyond the perturbative regime. Such studies in the lattice regularization have been strongly boosted by a series of papers by Lüscher and Weisz (LW), of which the first two have dealt with the one component model in the symmetric [3] and in the broken [4] phase. Here control over the lattice theory was gained by combining large orders in the hopping parameter expansion with careful perturbative renormalization group evolution [5]. These studies were in addition corroborated by some early Monte Carlo simulations as for example [6,7]. In these cases the Ising model was considered as the limit of φ^4 theory at infinite bare coupling. Barring a complicated non-monotonic relation between bare and (natural) renormalized couplings on the lattice, this limit is the most interesting case for questions concerning triviality.

In recent years one of the authors has taken up the subject again after some progress had been made in Monte Carlo methods which allow to achieve a new level of precision in this context with only moderate investments in computing power. The main new ingredients are, on the one hand, the use of so-called worm algorithms [8,9] to simulate arbitrary order contributions of a hopping parameter expansion for observables on finite lattices instead of generating field configurations. The second ingredient is the use of finite volume renormalization schemes as in [10]. As triviality is an ultraviolet renormalization effect, more computing power can be devoted in this way to closely approaching the continuum limit as the thermodynamic limit does not have to be taken. In other words, the manageable ratios L/a between lattice size and spacing is used to achieve a significant range of small a and not for large L in physical units. In [11–13] such a strategy has been explored for the symmetric phase of the model. In this publication we now offer a finite size scaling study on the other side of the critical line.

In Section 2 we define our renormalization scheme, followed by basic definitions of φ^4 theory. In Section 4 the numerical method and achieved results are described followed by a brief summary. This work is based on the master thesis of the first author at Humboldt University, Berlin 2013.

2. Broken phase finite volume scheme

At first glance the title of this subsection might look paradoxical as there is no symmetry breaking in a finite volume. If we define

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[1] We have to remark here that such bounds are not universal but depend on the cutoff in use. Different lattice discretizations yield different bounds, see [1] for example.

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however an order parameter v_0 by the large distance behavior of the Z(2) symmetric fundamental two point correlation (ξ is the correlation length),

$$\langle\varphi(x)\varphi(0)\rangle \cong v_0^2 \quad \text{for } |x| \gg \xi, \tag{1}$$

then this definition has a smooth thermodynamic limit. To define definite renormalization conditions we employ the Fourier transform

$$G(p) = a^4 \sum_x \mathrm{e}^{-ipx}\langle\varphi(x)\varphi(0)\rangle \tag{2}$$

and extract v_0^2 from

$$G(p) = L^4 \delta_{p,0} v_0^2 + G_c(p), \qquad G_c(0) = 0, \tag{3}$$

where we have assumed a torus of extent L in each direction and G_c is the varying part of the correlation ('connected', although we here avoid the one-point function).

We now complete our renormalization scheme by singling out two small torus momenta

$$p_* = \frac{2\pi}{L}(1,0,0,0), \qquad p_{**} = \frac{2\pi}{L}(1,1,0,0) \tag{4}$$

beside zero momentum. We match $G(p)$ to the form

$$G(p) = Z\left\{L^4\delta_{p,0}v^2 + \frac{1}{\hat{p}^2 + m^2}\right\} \quad \text{at } p \in \{0, p_*, p_{**}\} \tag{5}$$

which simultaneously fixes the wave function renormalization factor Z, a renormalized expectation value v and the renormalized mass m. By solving these conditions we obtain

$$z^2 = (mL)^2 = \frac{G(p_{**})\hat{p}_{**}^2 L^2 - G(p_*)\hat{p}_*^2 L^2}{G(p_*) - G(p_{**})} \tag{6}$$

and

$$w^2 = (vL)^2 = \frac{G(0)}{G(p_*)}\frac{1}{L^2\hat{p}_*^2 + z^2} - z^{-2}, \tag{7}$$

where we have introduced the dimensionless finite size scaling quantities z and w and the usual lattice momentum

$$\hat{p}_\mu = \frac{2}{a}\sin(ap_\mu/2). \tag{8}$$

It is not difficult to see that in the thermodynamic limit $z \to \infty$ our definitions of m and v approach those of m_R and v_R in [4]. Apart from this limit however, each fixed value of z defines a different renormalization scheme and the perturbative coefficients of the continuum perturbative Callan Symanzik β function, for instance, will depend on z beyond the scheme independent one and two loop terms.

As usual in the spontaneously broken theory we define the renormalized coupling constant in terms of v by setting

$$g = \frac{3m^2}{v^2} = \frac{3z^2}{w^2}. \tag{9}$$

3. Some basic φ^4 formulae

The action in the lattice form is given by

$$S = \sum_x\left[\varphi(x)^2 + \lambda\left(\varphi(x)^2 - 1\right)^2\right] - 2\kappa\sum_{x\mu}\varphi(x)\varphi(x+\hat{\mu}) \tag{10}$$

with all dimensionless quantities. This is equivalent to the field theoretic form

$$S = a^4\sum_x\left\{\frac{1}{2}(\partial_\mu\phi)^2 + \frac{\mu_0^2}{2}\phi^2 + \frac{g_0}{4!}\phi^4\right\} \tag{11}$$

with mass dimension one field[2] ϕ if we match

$$a\phi = \sqrt{2\kappa}\varphi \tag{12}$$

$$a^2\mu_0^2 = \frac{1-2\lambda}{\kappa} - 8 \tag{13}$$

$$g_0 = \frac{6\lambda}{\kappa^2} = 6\lambda\left(\frac{a^2\mu_0^2 + 8}{1-2\lambda}\right)^2. \tag{14}$$

Classically, the symmetric phase arises for $a^2\mu_0^2 > 0$ where φ, ϕ fluctuate around zero with a bare mass given by

$$m_0 = \mu_0 \quad \text{(symmetric phase)}. \tag{15}$$

For $a^2\mu_0^2 < 0$ the field fluctuates around one of two equivalent nonzero values $\pm\bar{\varphi}$. The quadratic fluctuations around either constant field are now controlled by the bare mass

$$m_0 = \sqrt{-2\mu_0^2} \quad \text{(brokenphase)}. \tag{16}$$

Note that in LW μ_0 does not appear, as in [3] it is replaced by m_0 while in [4] the action in terms of ϕ is not written and only the m_0 for the broken phase appears. Consequently the relations between m_0 and κ, λ differ in the two papers as emphasized in a footnote in [4].

Following LW we explore the plane of bare parameters by approaching the critical line (continuum limit) on trajectories at fixed λ and define the β-function

$$\beta(am, g) = \left.\frac{\partial g}{\partial \ln(am)}\right|_\lambda. \tag{17}$$

This definition entails the following tree level lattice artefact contributions

$$\beta(am, g) = \frac{4a^2m^2}{8 + a^2m^2}g + \mathrm{O}(g^2) \quad \text{(symmetric phase)} \tag{18}$$

and $(m^2 \to -m^2/2)$

$$\beta(am, g) = -\frac{4a^2m^2}{16 - a^2m^2}g + \mathrm{O}(g^2) \quad \text{(broken phase)}. \tag{19}$$

Although these artefacts are small they may be avoided by switching to modified couplings

$$\tilde{g} = g \times \begin{cases} (1 + a^2m^2/8)^{-2} & \text{symmetric,} \\ (1 - a^2m^2/16)^{-2} & \text{broken.} \end{cases} \tag{20}$$

The perturbative continuum β function for the coupling g of the previous subsection – and also $\tilde{g}$ formed from it – is given by

$$\beta(0, g) = b_1 g^2 + b_2 g^3 + b_{3,z} g^4 + \mathrm{O}(g^5),$$
$$b_1 = \frac{3}{(4\pi)^2}, \qquad b_2 = -\frac{17}{3(4\pi)^4}. \tag{21}$$

While the first two coefficients are scheme independent, the three loop term $b_{2,z}$ is at the moment not known for our present scheme at finite z. The infinite volume case is found in [4],

$$b_{3,\infty} = \frac{14.715616}{(4\pi)^6}. \tag{22}$$

[2] Our ϕ corresponds to φ_0 in the LW papers.

4. Worm simulations

4.1. Brief summary of the method

The renormalization scheme of Section 2 is defined entirely in terms of the two point correlation. Worm simulations are ideally suited for its numerical computation in the Ising limit $\lambda = \infty$. The worm ensemble is given by the partition function

$$\mathcal{Z} = \sum_{u,v} Z(u,v) = \sum_{u,v,k} t^{\sum_{x,\mu} k(x,\mu)} \delta\big[\partial^*_\mu k_\mu - q_{u,v}\big]. \tag{23}$$

In this formula we sum over link variables $k(x,\mu) \equiv k_\mu(x) = 0, 1$ and $\delta[\ldots]$ enforces the constraint that is most easily described in words: each site except u, v must be surrounded by an even number of $k = 1$ links while at u, v (unless $u = v$) this number must be odd. The fugacity is $t = \tanh(2\kappa)$. The k configurations are in one-to-one correspondence with strong coupling graphs with lines drawn on links with $k(x,\mu) = 1$. At the same time we have the connection with the spin formulation

$$Z(u,v) = \mathcal{N} \sum_{\varphi} \mathrm{e}^{2\kappa \sum_{x,\mu} \varphi(x)\varphi(x+\hat{\mu})} \varphi(u)\varphi(v) \tag{24}$$

where for the Ising limit the sum is over $\varphi(x) = \pm 1$ and $\mathcal{N}$ is a normalization factor. In [9] a lot more details about this reformulation and the efficient simulation of (23) can be found. It is obvious now that the two point function is given by

$$\big\langle \varphi(x)\varphi(0) \big\rangle = \frac{\langle\langle \delta_{x,u-v} \rangle\rangle}{\langle\langle \delta_{u,v} \rangle\rangle} \tag{25}$$

with the double angles referring to expectation values with respect to (23). The required Fourier transforms can be directly accumulated from

$$G(p) = \frac{\langle\langle \mathrm{e}^{-ip(u-v)} \rangle\rangle}{\langle\langle \delta_{u,v} \rangle\rangle} = \frac{\langle\langle \prod_\mu \cos(p_\mu (u-v)_\mu) \rangle\rangle}{\langle\langle \delta_{u,v} \rangle\rangle}. \tag{26}$$

For the last step we have used the invariance under individual reflections along each direction. Note that with the small momenta of interest we do not expect very rapid oscillations. As only ratios of G, where the wave function renormalization cancels, are of interest, the denominator $\langle\langle \delta_{u,v} \rangle\rangle = G(0)^{-1}$ (inverse susceptibility) is never really needed here.

4.2. Numerical results

At first we have explored how z depends on the hopping parameter for lattice sizes $L/a = 8, 16, 32$. The results are shown in Fig. 1. We are here just above the infinite volume critical point which is known [14] to occur close to $2\kappa \approx 0.149$. Each data point in the plot corresponds to 10^6 iterations, where an iteration [9] consists of one worm move per site. From these results we have decided to adopt in the following the target value $z^2 = 10$ for our study. Our results are summarized in Table 1.

Each line corresponds to a statistics of 8×10^7 iterations. By some tuning we found values of κ that lead to $z^2 = 10$ within errors. The directly measured couplings (9) are given in the fourth column while the rightmost column differs by two tiny corrections. By the first order reweighting technique described in [11] the value is adjusted to $z^2 = 10$ exactly and then the cutoff correction (20) is applied. The first correction is clearly only a change within the error bars, but, although to a much lesser degree than in [11], it in addition lowers the statistical error slightly.

These data are plotted in Fig. 2. The dotted (blue), dashed (red) and solid (black) curves derive from integrating the Callan Symanzik equation

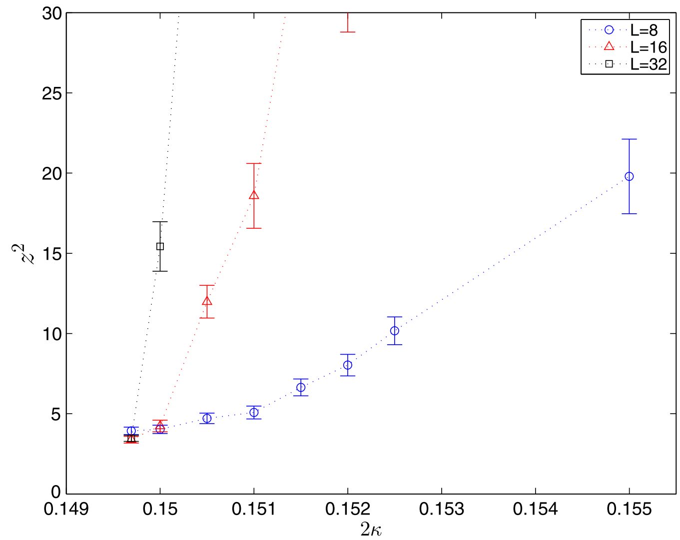

Fig. 1. Finite size mass against hopping parameter for $L/a = 8, 16, 32$.

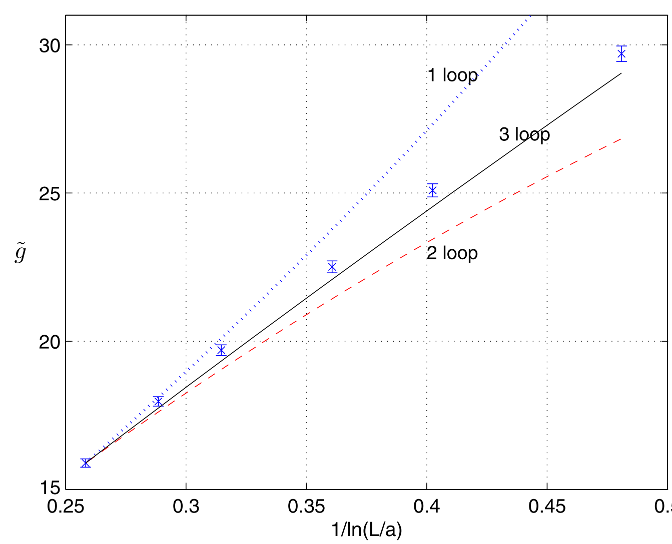

Fig. 2. Coupling $\tilde{g}$ at $z^2 = 10$ as a function of the cutoff. The curves stem from integrations of the renormalization group equation at various loop order truncations. The leftmost point is taken as initial value.

Table 1
Simulation results to determine the renormalized coupling in the continuum limit (growing L/a) for fixed $z^2 = 10$.

L	2κ	z^2	g	$\tilde{g}\vert_{z^2=10}$
8	0.152460	10.024(96)	29.13(30)	29.70(26)
12	0.150992	10.008(98)	24.88(26)	25.09(22)
16	0.150450	9.964(99)	22.39(24)	22.51(20)
24	0.150046	9.974(98)	19.65(21)	19.70(18)
32	0.149899	9.980(97)	17.95(19)	17.97(16)
48	0.149790	10.065(96)	15.90(17)	15.89(14)

$$\frac{d\tilde{g}}{d\ln(L/a)} = -\beta(0, g) \tag{27}$$

with the continuum β-function at 1, 2, 3 loop perturbative precision. Beside the universal coefficients (21) we here use the infinite volume value (22) for the three loop coefficient. As discussed before this is only indicative with the presently unknown coefficient for $z^2 = 10$ certainly being slightly different. The experience in the symmetric phase has been, however, that at this size the difference may not be very sizeable.

5. Summary

We have defined a finite size renormalization scheme for φ^4 theory, which in the infinite volume limit goes over into the one that is standard in the broken phase of the model. In the Ising limit of infinite bare coupling, we have numerically generated values of the renormalized coupling as a function of the lattice cutoff. Using novel simulation techniques we computed precise values which turn out to closely follow the perturbative renormalization group. The data points are nicely sandwiched between the one and two loop trajectories. The three loop curve falls in between and is only about two sigma (2%) away from our data, although the three loop coefficient is taken for $z^2 = \infty$ rather than $z^2 = 10$ studied here. If we conclude agreement with perturbation theory in the range studied then this should be even better justified for larger L/a and $\tilde{g}$ tends to zero in the continuum limit at a logarithmic rate. This supports the triviality scenario once more by combining numerical and perturbative methods.

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