Additive maps preserving Jordan zero-products on nest algebras

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Abstract

Let Alg N and Alg M be nest algebras associated with the nests N and M on Banach Spaces. Assume that N ∈ N and M ∈ M are complemented whenever N = N and M = M. Let Φ : Alg N → Alg M be a unital additive surjection. It is shown that Φ preserves Jordan zero-products in both directions, that is Φ(A)Φ(B) + Φ(B)Φ(A) = 0 ⇔ AB + BA = 0, if and only if Φ is either a ring isomorphism or a ring anti-isomorphism. Particularly, all unital additive surjective maps between Hilbert space nest algebras which preserves Jordan zero-products are characterized completely.

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1. Introduction

Recall that a Jordan ring 𝒜 is a non-associative commutative ring with product ◦ satisfying [(A ◦ A) ◦ B] ◦ A = (A ◦ A) ◦ (B ◦ A) for all A, B ∈ 𝒜. Let 𝒜 and 𝒜 be Jordan rings with Jordan product ◦. We say that a map Φ : 𝒜 → 𝒜 preserves Jordan zero-products (in both directions)
if, for $A, B \in \mathcal{A}$, $A \circ B = 0$ whenever (if and only if) $\Phi(A) \circ \Phi(B) = 0$. If $\mathcal{A}$ are associative ring, and if we define $A \circ B = AB + BA$, then $(\mathcal{A}, +, \circ)$ is a Jordan ring. Thus, if $\mathcal{A}$ and $\mathcal{B}$ are associative rings, we say that a map $\Phi : \mathcal{A} \to \mathcal{B}$ preserves Jordan zero-products (in both directions) if, for $A, B \in \mathcal{A}$, $\Phi(A)\Phi(B) + (\Phi(B))\Phi(A) = 0$ whenever (if and only if) $AB + BA = 0$. The question of characterizing additive maps preserving Jordan zero-products was recently discussed in [11]. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be an additive surjective map between some operator algebras $\mathcal{A}$ and $\mathcal{B}$. Under some mild conditions, it was shown in [11] that, if $\Phi$ preserves Jordan zero-products, then $\Phi$ is a Jordan homomorphism multiplied by a central element. Such operator algebras include von Neumann algebras, $C^*$-algebras and standard operator algebras, etc. Particularly, if $H$ and $K$ are infinite-dimensional (real or complex) Hilbert spaces and $\mathcal{A} = \mathcal{B}(H)$ and $\mathcal{B} = \mathcal{B}(K)$, then there exists a nonzero scalar $c$ and an invertible linear or conjugate-linear operator $A : H \to K$ such that either $\Phi(A) = cTA^*T^{-1}$ for all $A \in \mathcal{B}(H)$, or $\Phi(A) = cTA^*T^{-1}$ for all $A \in \mathcal{B}(H)$. Note that all operator algebras treated in [11] are self-adjoint or semi-simple and prime.

Nest algebras are important operator algebras that are neither self-adjoint nor semi-simple and prime. It is interesting to consider the question of characterizing the Jordan zero-product preserving additive maps on nest algebras over Banach spaces. The purpose of this paper is to discuss this question and give a characterization of such maps on nest algebras. We mention here that, in [16], it was shown that every weakly continuous unital surjective linear map preserving Jordan zero-products in both directions between nest algebras over Hilbert spaces is either an isomorphism or an anti-isomorphism.

Let $X$ and $Y$ be Banach spaces over field $F (= \mathbb{R}$ or $\mathbb{C}$, the field of real numbers or the field of complex numbers). $\mathcal{B}(X, Y)$ and $\mathcal{F}(X, Y)$ (simply $\mathcal{B}(X)$ and $\mathcal{F}(X)$ if $Y = X$) denote the Banach space of all bounded linear operators from $X$ into $Y$ and the subspace of all finite rank operators in $\mathcal{B}(X, Y)$, respectively. Recall that a nest in $X$ is a chain $\mathcal{N}$ of closed (under norm topology) subspaces of $X$ containing the trivial subspaces $0$ and $X$, which is closed under the formation of arbitrary closed linear span (denoted by $\bigvee$) and intersection (denoted by $\bigwedge$). $\text{Alg}_{\mathcal{N}}$ denotes the associated nest algebra, which is the set of all operators $T$ in $\mathcal{B}(X)$ such that $TN \subseteq N$ for every element $N \in \mathcal{N}$. When $\mathcal{N} \neq \{0, X\}$, we say that $\mathcal{N}$ is nontrivial. It is clear that if $\mathcal{N}$ is trivial, then $\text{Alg}_{\mathcal{N}} = \mathcal{B}(X)$. It is also obvious that for the case dim $X < \infty$, every nest algebra over $X$ is isomorphic to some upper-triangular block matrix algebra. Denote $\text{Alg}_{\mathcal{N}} =: \text{Alg}_{\mathcal{N}} = \mathcal{N} \cap \mathcal{F}(X)$, the set of all finite rank operators in $\text{Alg}_{\mathcal{N}}$. For $N \in \mathcal{N}$, let $N_- = \bigvee \{M \in \mathcal{N} | M \subseteq N\}$, $\mathcal{N}_+ = \bigwedge \{M \in \mathcal{N} | N \subseteq M\}$ and $N^\perp = (N_-)^\perp$, where $N^\perp = \{f \in X^* | N \subseteq \ker(f)\}$ and $X^*$ is the dual of $X$. We denote $0_- = 0$ and $X_+ = X$. It is well known that a rank one operator $x \otimes f$ belongs to $\text{Alg}_{\mathcal{N}}$ if and only if there is some $N \in \mathcal{N}$ such that $x \in N$ and $f \in N^\perp$; every operator in $\text{Alg}_{\mathcal{N}}$ is a finite sum of rank-1 operators in $\text{Alg}_{\mathcal{N}}$. Moreover, Erdos in [6] for Hilbert space case and Spandoudakis in [15] for general Banach space case proved that $\text{Alg}_{\mathcal{N}}$ is a dense subset of $\text{Alg}_{\mathcal{N}}$ under the strong operator topology. If $H$ is a Hilbert space and $\mathcal{N}$ is a nest in $H$, then there is a strongly closed totally ordered subset $\mathcal{P} \subseteq \mathcal{N}$ of projections such that $\{PH | P \in \mathcal{P}\} = \mathcal{N}$, and vice versa. This is not the case for nests in general Banach spaces since there exist subspaces that are not complemented, which is one of the main difficulties in the study of nest algebras over Banach spaces. For more information on nest algebras, we refer to [5].

Let $\mathcal{N}$ and $\mathcal{M}$ be nests on Banach spaces $X$ and $Y$, respectively. Assume that $N \in \mathcal{N}$ and $M \in \mathcal{M}$ are complemented whenever $N_- = N$ and $M_- = M$. There are many nests in Banach spaces satisfying this condition and containing elements that are not complemented. Let $\Phi : \text{Alg}_{\mathcal{N}} \to \text{Alg}_{\mathcal{M}}$ be a unital additive surjection. Our main result is to show that $\Phi$ preserves Jordan zero-products in both directions if and only if $\Phi$ is either a ring isomorphism or a ring anti-isomorphism (see Theorem 2.1 for infinite dimensional case and Theorem 2.3 for finite
dimensional case). Further more, if \( \dim X = \infty \), then \( \Phi \) must have one of the following forms: an algebraic isomorphism, a conjugate algebraic isomorphism, an algebraic anti-isomorphism and a conjugate algebraic anti-isomorphism. Particularly, all unital additive surjective map between Hilbert space nest algebras which preserves Jordan zero-products in both directions are characterized (see Corollary 2.2 and Theorem 2.3). Note that, in above results, no continuity of the maps is assumed.

The paper is organized as follows. In Section 2, we list the main results obtained in the present paper. In Section 3 we give some lemmas and general properties of Jordan zero-product preserving additive maps, which are needed both for proving our main results and for further study. Section 4 is devoted to proofs of the main results, mainly for infinite dimensional case (i.e., Theorem 2.1) since finite dimensional case is a immediate consequence of the lemmas in Section 3. By our approach, to prove Theorem 2.1, a key and difficult step is to show that the map in question preserves rank one operators. Finally, the question whether the unital assumption may be deleted is also discussed.

2. Main results

In this section, we state the main results obtained in this paper.

**Theorem 2.1.** Let \( \mathcal{N} \) and \( \mathcal{M} \) be nests on infinite dimensional Banach spaces \( X \) and \( Y \) over real or complex field \( \mathbb{F} \), respectively, with \( N \in \mathcal{N} \) and \( M \in \mathcal{M} \) complemented in \( X \) and \( Y \) respectively whenever \( N_\perp = N \) and \( M_\perp = M \). Let \( \Phi : \text{Alg}\mathcal{N} \to \text{Alg}\mathcal{M} \) be a unital additive surjective map. Then \( \Phi \) preserves Jordan zero-products in both directions if and only if either

(1) there exist a dimension preserving order isomorphism \( \theta : \mathcal{N} \to \mathcal{M} \) and an invertible bounded linear or conjugate-linear operator \( T : X \to Y \) such that \( T(N) = \theta(N) \) for every \( N \in \mathcal{N} \) and

\[ \Phi(A) = TAT^{-1} \quad \text{for all } A \in \text{Alg}\mathcal{N}; \]

or

(2) there exist a dimension preserving order isomorphism \( \theta : \mathcal{N}^{\perp} \to \mathcal{M} \) and an invertible bounded linear or conjugate-linear operator \( T : X^* \to Y \) such that \( T(N^{\perp}) = \theta(N^{\perp}) \) for every \( N \in \mathcal{N} \) and

\[ \Phi(A) = TA^*T^{-1} \quad \text{for all } A \in \text{Alg}\mathcal{N}. \]

We remark that, in the above result, the unital additive surjections which preserve Jordan zero-products in both directions between infinite dimensional Banach space nest algebras are automatically continuous in norm topology and in strong operator topology as well as in weak operator topology.

Since every linear subspace of a Hilbert space is complemented, the following corollary is immediate from Theorem 2.1 and gives a complete characterization of unital additive surjective maps preserving Jordan zero-products in both directions between nest algebras on infinite dimensional Hilbert spaces.

**Corollary 2.2.** Let \( \mathcal{N} \) and \( \mathcal{M} \) be nests in infinite dimensional Hilbert spaces \( H \) and \( K \) over (real or complex) field \( \mathbb{F} \), respectively. Let \( \Phi : \text{Alg}\mathcal{N} \to \text{Alg}\mathcal{M} \) be a unital additive surjective map. Then \( \Phi \) preserves Jordan zero-products in both directions if and only if either (1) or (2) in Theorem 2.1 holds.
Now we consider the finite dimensional case. It is clear that every nest algebra on finite dimensional spaces is isomorphic to an upper triangular block matrix algebras. So, without loss of the generality we may discuss the question for upper triangular block matrix algebras. Let $M_n = M_n(\mathbb{F})$ be the matrix algebra over field $\mathbb{F}$. Let $\mathcal{F} = \mathcal{F}(n_1, n_2, \ldots, n_k) \subseteq M_n$ be a upper triangular block matrix subalgebra, i.e., $n_1 + n_2 + \cdots + n_k = n$, $\mathcal{F} = \{A = (A_{ij})_{k \times k} | A_{ij} \in M_{n_i,n_j} \text{and} A_{ij} = 0 \text{ if } i > j \}$.

**Theorem 2.3.** Let $\mathbb{F}$ be the real or complex field, and $m$, $n$ be positive integers greater than 1. Let $\mathcal{F} = \mathcal{F}(n_1, n_2, \ldots, n_k) \subseteq M_n(\mathbb{F})$ and $\mathcal{S} = \mathcal{F}(m_1, m_2, \ldots, m_r) \subseteq M_m(\mathbb{F})$ be upper triangular block matrix algebras, and $\Phi : \mathcal{F} \to \mathcal{S}$ be an additive unital surjective map. Then $\Phi$ preserves Jordan zero-products in both directions if and only if either

1. $(n_1, n_2, \ldots, n_k) = (m_1, m_2, \ldots, m_r)$, there exist an automorphism $\tau : \mathbb{F} \to \mathbb{F}$ and an invertible matrix $T \in \mathcal{F}$ such that
   \[ \Phi(A) = T A \tau T^{-1} \quad \text{for all} \ A \in \mathcal{F}; \]

or

2. $(n_1, n_2, \ldots, n_k) = (m_r, m_{r-1}, \ldots, m_1)$, there exist an automorphism $\tau : \mathbb{F} \to \mathbb{F}$ and an invertible block matrix $T = (T_{ij})_{k \times k}$ with $T_{ij} \in M_{n_i,n_j}$ and $T_{ij} = 0$ whenever $i + j > k + 1$, such that
   \[ \Phi(A) = T A^{\tau^r} T^{-1} \quad \text{for all} \ A \in \mathcal{F} \]

where $A_\tau = (\tau(a_{ij}))_{n \times n}$ for $A = (a_{ij})_{n \times n} \in M_n(\mathbb{F})$ and $A^{\tau}$ is the transpose of $A$. If $\mathbb{F} = \mathbb{R}$, then $\tau = \text{id}(i.e., \tau(t) = t \text{ for all } t \in \mathbb{R})$.

To get a characterization of additive surjective maps between nest algebras that preserve Jordan zero-products in both directions, we conjecture that the assumption “$\Phi$ is unital” is not necessary. We are not able to solve this conjecture in the present paper. However, the following results suggest that this conjecture have a affirmative answer.

**Theorem 2.4.** Let $\mathbb{F}$ be the real or complex field, and $m$, $n$ be positive integers greater than 1. Let $\mathcal{F} = \mathcal{F}(n_1, n_2, \ldots, n_k) \subseteq M_n(\mathbb{F})$ and $\mathcal{S} = \mathcal{F}(m_1, m_2, \ldots, m_r) \subseteq M_m(\mathbb{F})$ be upper triangular block matrix algebras, and $\Phi : \mathcal{F} \to \mathcal{S}$ be a linear surjective map. Then $\Phi$ preserves Jordan zero-products in both directions if and only if either

1. $(n_1, n_2, \ldots, n_k) = (m_1, m_2, \ldots, m_r)$, there exist a nonzero scalar $c$ and an invertible matrix $T \in \mathcal{F}$ such that
   \[ \Phi(A) = c T A T^{-1} \quad \text{for all} \ A \in \mathcal{F}; \]

or

2. $(n_1, n_2, \ldots, n_k) = (m_r, m_{r-1}, \ldots, m_1)$, there exist a nonzero scalar $c$ and an invertible block matrix $T = (T_{ij})_{k \times k}$ with $T_{ij} \in M_{n_i,n_j}$ and $T_{ij} = 0$ whenever $i + j > k + 1$, such that
   \[ \Phi(A) = c T A^{\tau^r} T^{-1} \quad \text{for all} \ A \in \mathcal{F}. \]

For the infinite dimensional case, if we omit the assumption that $\Phi$ is unital in Theorem 2.1, we have to assume that $\Phi$ is linear and continuous in strong operator topology.

**Theorem 2.5.** Let $\mathcal{N}$ and $\mathcal{M}$ be nests in infinite dimensional Banach spaces $X$ and $Y$ over real or complex field $\mathbb{F}$, respectively, with $N \in \mathcal{N}$ and $M \in \mathcal{M}$ complemented in $X$ and $Y$, respectively.
respectively, whenever $N_\prec = N$ and $M_\prec = M$. Let $\Phi : \Alg N \to \Alg M$ be a strongly continuous linear surjective map. Then $\Phi$ preserves Jordan zero-products in both directions if and only if either

(1) there exist a dimension preserving order isomorphism $\theta : N \to M$, a nonzero constant $c$ and an invertible bounded linear operator $T : X \to Y$ such that $T(N) = \theta(N)$ for every $N \in N$ and

$$\Phi(A) = cT A T^{-1} \quad \text{for all } A \in \Alg N;$$

or

(2) there exist a dimension preserving order isomorphism $\theta : N_\perp \to M_\perp$, a nonzero constant $c$ and an invertible bounded linear operator $T : X^* \to Y$ such that $T(N_\perp) = \theta(N_\perp)$ for every $N \in N$ and

$$\Phi(A) = cT A^* T^{-1} \quad \text{for all } A \in \Alg N.$$

3. Lemmas

In this section, we give some lemmas and general structural features of additive surjective maps which preserve Jordan zero-products in both directions between nest algebras on Banach spaces which are needed to prove our main results listed in Section 2.

Lemma 3.1. Let $N$ and $M$ be nests on Banach spaces $X$ and $Y$ over (real or complex) field $F$, respectively. Let $\Phi : \Alg N \to \Alg M$ be a unital additive surjective map. Assume that $\Phi$ preserves Jordan zero-products in both directions. Then $\Phi$ is injective, preserves idempotents in both directions and preserves square-zero in both directions. Moreover,

$$\Phi(P A P) = \Phi(P) \Phi(A) \Phi(P), \quad (3.1)$$

$$\Phi(P A (I - P) + (I - P) A P) = \Phi(P) \Phi(A) (I - \Phi(P)) \quad (3.2)$$

$$\Phi(P A + A P) = \Phi(P) \Phi(A) + \Phi(A) \Phi(P) \quad (3.3)$$

and

$$\Phi(P) \Phi(P A (I - P)) \Phi(P) = \Phi(P) \Phi((I - P) A P) \Phi(P) = 0 \quad (3.4)$$

hold for all idempotent $P$ and all $A$ in $\Alg N$.

Proof. It is trivial to verify that $\Phi$ is injective. We will complete the proof by checking several claims.

Claim 1. $\Phi$ preserves idempotents and square-zero in both directions.

For every idempotent operator $P \in \Alg N$, since $P (I - P) + (I - P) P = 0$, we have $\Phi(P) \Phi(I - P) + \Phi(I - P) \Phi(P) = 0$. Therefore $\Phi(P)^2 = \Phi(P)$ and vice versa. The second assertion is obvious.

Claim 2. Assume that $P \in \Alg N$ is an idempotent operator. Denote

$$X_1 = P \Alg N P, \quad X_2 = P \Alg(M \setminus N)(I - P),$$

$$X_3 = (I - P) \Alg N P, \quad X_4 = (I - P) \Alg N (I - P);$$
and let
\[
Y_1 = \Phi(P)\mathcal{H}\Phi(P), \quad Y_2 = \Phi(P)\mathcal{H}(I - \Phi(P))
\]
\[
Y_3 = (I - \Phi(P))\mathcal{H}\Phi(P), \quad Y_4 = (I - \Phi(P))\mathcal{H}(I - \Phi(P)).
\]

Then \( \Phi(X_1) = Y_1, \Phi(X_4) = Y_4 \) and \( \Phi(X_2 + X_3) = Y_2 + Y_3 \).

If \( A \in X_2 \), then \( (P + A)^2 = P + A \), and we have \( \Phi(P + A)^2 = \Phi(P + A) \). Because \( \Phi \) preserves square zeros operators by Claim 1, it is easily checked that \( \Phi(P)\Phi(A)\Phi(P) + \Phi(P)\Phi(A) = \Phi(P)\Phi(A) \). This clearly yields \( \Phi(P)\Phi(A) + \Phi(A)\Phi(P) = \Phi(A) \), and therefore we get \( \Phi(A) = (I - \Phi(P))\Phi(A) \). So \( \Phi(A) \in Y_2 + Y_3 \), and then \( \Phi(X_2) \subseteq Y_2 + Y_3 \).

Next we want to prove \( \Phi(X_1) \subseteq Y_1, \Phi(X_4) \subseteq Y_4 \). Take arbitrarily \( A \in X_4 \). Since \( PA + AP = 0 \), we have \( \Phi(P)\Phi(A) + \Phi(A)\Phi(P) = 0 \). Multiplying the above equation from left and right by \( \Phi(P) \), respectively, we see that \( \Phi(P)\Phi(A) = \Phi(A)\Phi(P) = \Phi(P)\Phi(A)\Phi(P) = 0 \). So \( \Phi(A) \in Y_4 \), and consequently, \( \Phi(X_4) \subseteq Y_4 \). Similarly, we have \( \Phi(X_1) \subseteq Y_1 \). Now, since \( \mathcal{N} = X_1 + X_2 + X_3 + X_4 \) and \( \mathcal{M} = Y_1 + Y_2 + Y_3 + Y_4 \), the surjectivity of \( \Phi \) implies that the claim is true.

**Claim 3.** Eqs. (3.1)–(3.4) hold true.

Assume that \( P \in \text{Alg}\mathcal{N} \) is an idempotent operator. By Claim 2, for any \( A \in \text{Alg}\mathcal{N} \), we have
\[
\Phi(PAP) = \Phi(P)\Phi(PAP)\Phi(P)
\]
\[
= \Phi(P)(\Phi(PAP) + \Phi(PA(I - P)) + \Phi((I - P)AP)
\]
\[
+ \Phi((I - P)A(I - P))\Phi(P)
\]
\[
= \Phi(P)(\Phi(PAP + PA(I - P) + (I - P)AP + (I - P)A(I - P))\Phi(P)
\]
\[
= \Phi(P)\Phi(A)\Phi(P),
\]
that is, Eq. (3.1) is true.

Eq. (3.2) holds because
\[
\Phi(A) = \Phi((P + (I - P))A(P + (I - P)))
\]
\[
= \Phi(PAP) + \Phi(PA(I - P) + (I - P)AP) + \Phi((I - P)A(I - P))
\]
\[
= \Phi(P)\Phi(A)\Phi(P) + \Phi(PA(I - P) + (I - P)AP) + \Phi(I - P)\Phi(A)\Phi(I - P)
\]
and
\[
\Phi(A) = \Phi(P)\Phi(A)\Phi(P) + \Phi(P)\Phi(A)(I - \Phi(P))
\]
\[
+ (I - \Phi(P))\Phi(A)\Phi(P) + (I - \Phi(P))\Phi(A)(I - \Phi(P)).
\]

Now
\[
\Phi(PA + AP) = \Phi(2PAP + PA(I - P) + (I - P)AP)
\]
\[
= 2\Phi(PAP) + \Phi(PA(I - P) + (I - P)AP)
\]
\[
= 2\Phi(P)\Phi(A)\Phi(P) + \Phi(P)\Phi(A)(I - \Phi(P))
\]
\[
+ (I - \Phi(P))\Phi(A)\Phi(P)
\]
\[
= \Phi(P)\Phi(A) + \Phi(A)\Phi(P).
\]
Hence, we have respectively zero-products in both directions either Lemma 3.2. Hence, \( \Phi(P(A(I - P)) \in \Phi(P) \text{Alg.} \mathcal{M}(I - \Phi(P)) + (I - \Phi(P)) \text{Alg.} \Phi(P) \). Hence, \( \Phi(P) \Phi(P(A(I - P)) \Phi(P) = 0 \). The same reason implies that \( \Phi(P) \Phi((I - P)A) \Phi(P) = 0 \), too. This completes the proof of Eq. (3.4). □

**Lemma 3.2.** Let \( \mathcal{N} \) and \( \mathcal{M} \) be nests on Banach spaces \( X \) and \( Y \) over (real or complex) field \( \mathbb{F} \), respectively. Let \( \Phi : \text{Alg.} \mathcal{N} \to \text{Alg.} \mathcal{M} \) be a unital additive surjective map that preserves Jordan zero-products in both directions. If \( P \in \text{Alg.} \mathcal{N} \) is an idempotent with range \( \text{ran}(P) \in \mathcal{N} \), then, either

1. \( \Phi(EPA(I - P)) = \Phi(E) \Phi(PA(I - P)) \) and \( \Phi(PA(I - P)E) = \Phi(PA(I - P)) \Phi(E) \)
   hold for all \( A \in \text{Alg.} \mathcal{N} \) and all idempotent \( E \in \text{Alg.} \mathcal{N} \);
   or
2. \( \Phi(EPA(I - P)) = \Phi(PA(I - P)) \Phi(E) \) and \( \Phi(PA(I - P)E) = \Phi(E) \Phi(PA(I - P)) \)
   hold for all \( A \in \text{Alg.} \mathcal{N} \) and all idempotent \( E \in \text{Alg.} \mathcal{N} \).

**Proof.** We complete the lemma by several claims.

**Claim 1.** The equation

\[
\Phi(PA(I - P)) = \Phi(P)\Phi(A)(I - \Phi(P)) + (I - \Phi(P))\Phi(A)\Phi(P)
\]  \hspace{1cm} (3.5)

holds for all \( A \in \text{Alg.} \mathcal{N} \).

In fact, Eq. (3.5) follows directly from Eq. (3.2) in Lemma 3.1 since \( (I - P)AP = 0 \) due to \( \text{ran}P \in \mathcal{N} \).

**Claim 2.** The equations

\[
\Phi(P)\Phi(A)(I - \Phi(P))\Phi(B)\Phi(P) = 0
\] \hspace{1cm} (3.6)

and

\[
(I - \Phi(P))\Phi(A)\Phi(P)\Phi(B)(I - \Phi(P)) = 0
\] \hspace{1cm} (3.7)

hold for all \( A, B \in \text{Alg.} \mathcal{N} \).

It follows from Claim 1, i.e., the Eq. (3.5), for any \( A, B \in \text{Alg.} \mathcal{N} \),

\[
\Phi(P)\Phi(A)(I - \Phi(P))\Phi(B)\Phi(P) + (I - \Phi(P))\Phi(A)\Phi(P)\Phi(B)(I - \Phi(P)) + \Phi(P)\Phi(B)(I - \Phi(P))\Phi(A)\Phi(P) + (I - \Phi(P))\Phi(B)\Phi(P)\Phi(A)(I - \Phi(P))
\]
\[
= [\Phi(P)\Phi(A)(I - \Phi(P)) + (I - \Phi(P))\Phi(A)\Phi(P)]
\]
\[
\times [\Phi(P)\Phi(B)(I - \Phi(P)) + (I - \Phi(P))\Phi(B)\Phi(P)]
\]
\[
+ [\Phi(P)\Phi(B)(I - \Phi(P)) + (I - \Phi(P))\Phi(B)\Phi(P)]
\]
\[
\times [\Phi(P)\Phi(A)(I - \Phi(P)) + (I - \Phi(P))\Phi(A)\Phi(P)]
\]
\[
= \Phi(PA(I - P))\Phi(PB(I - P)) + \Phi(PB(I - P))\Phi(PA(I - P)) = 0.
\]

Hence, we have

\[
\Phi(P)\Phi(A)(I - \Phi(P))\Phi(B)\Phi(P) + \Phi(P)\Phi(B)(I - \Phi(P))\Phi(A)\Phi(P) = 0
\]
and

\[(I - \Phi(P))\Phi(A)\Phi(B)(I - \Phi(P)) + (I - \Phi(P))\Phi(B)\Phi(A)(I - \Phi(P)) = 0.\]

Since, by Lemma 3.1, \(\Phi\) is bijective and \(\Phi^{-1}\) preserves Jordan zero-products in both directions, it is obvious that

\[\Phi^{-1}(\Phi(P)\Phi(A)(I - \Phi(P)))\Phi^{-1}(\Phi(P)\Phi(B)(I - \Phi(P))) + \Phi^{-1}(\Phi(P)\Phi(B)(I - \Phi(P)))\Phi^{-1}(\Phi(P)\Phi(A)(I - \Phi(P))) = 0.\]

By Eq. (3.5) again, one gets

\[\Phi^{-1}(\Phi(P)\Phi(A)(I - \Phi(P)))\Phi^{-1}((I - \Phi(P))\Phi(B)(I - \Phi(P))) + \Phi^{-1}((I - \Phi(P))\Phi(B)(I - \Phi(P)))\Phi^{-1}(\Phi(P)\Phi(A)(I - \Phi(P))) = 0.\]

Note that the range of \(P\) is an element in the nest \(\mathcal{N}\), so we have

\[(I - P)\Phi^{-1}(\Phi(P)\Phi(A)(I - \Phi(P))) = 0\]

by Eq. (3.4) in Lemma 3.1. Similarly, one has

\[\Phi^{-1}(\Phi(P)\Phi(A)(I - \Phi(P)))P = 0.\]

Hence

\[\Phi^{-1}(\Phi(P)\Phi(A)(I - \Phi(P)))\Phi^{-1}((I - \Phi(P))\Phi(B)\Phi(P)) + \Phi^{-1}((I - \Phi(P))\Phi(B)\Phi(P))\Phi^{-1}(\Phi(P)\Phi(A)(I - \Phi(P))) = 0\]

and this implies that

\[\Phi(P)\Phi(A)(I - \Phi(P))\Phi(B)\Phi(P) + (I - \Phi(P))\Phi(B)\Phi(P)\Phi(A)(I - \Phi(P)) = 0.\]

Now it is clear that Claim 2 holds true.

**Claim 3.** Either (1) \((I - \Phi(P))\Phi(A)\Phi(P) = 0\) holds for all \(A \in \text{Alg}\mathcal{N}\); or (2) \((I - \Phi(P))\Phi(A)(I - \Phi(P)) = 0\) holds for all \(A \in \text{Alg}\mathcal{N}\).

Notice that, for \(A, B \in \text{Alg}\mathcal{N}, A(\text{Alg}\mathcal{N})B = 0\) if and only if there exists \(N \in \mathcal{N}\) such that \(\text{ran}(B) \subseteq N \subseteq \ker(A)\). To see this, assume that \(A(\text{Alg}\mathcal{N})B = 0\) and consider the linear manifold \(N_0 = \text{span}\{Sx | S \in \text{Alg}\mathcal{N}\ \text{and} \ x \in \text{ran}(B)\}\). It is clear that \(N\) is invariant under every operator in \(\text{Alg}\mathcal{N}\) and hence its closure \(N = \overline{N_0} \in \mathcal{N}\) and \(\text{ran}(B) \subseteq N \subseteq \ker(A)\). The converse is obvious. So, for a fixed \(A\), the surjectivity of \(\Phi\) and Claim 2 together imply that there exist \(M_1, M_2 \in \mathcal{M}\) such that

\[\text{ran}(\Phi(P)) \subseteq M_1 \subseteq \ker(\Phi(P)\Phi(A)(I - \Phi(P))).\]
Since $M_1 + M_2 = M_1$ or $M_2$ and $\text{ran}(\Phi(P)) + \text{ran}(I - \Phi(P)) = Y$, we see that either $M_1 = Y$ or $M_2 = Y$. In the former case we have $\Phi(P)\Phi(A)(I - \Phi(P)) = 0$ and in the last case we have $(I - \Phi(P))\Phi(A)\Phi(P) = 0$.

Take nonzero $B \in P(\text{Alg} \mathcal{A})(I - P)$; then by what we have proved, we have $\Phi(P)\Phi(B)(I - \Phi(P)) = 0$ or $(I - \Phi(P))(B)\Phi(P) = 0$. Assume that $(I - \Phi(P))(B)\Phi(P) = 0$, we have to show that $\Phi(P)\Phi(A)(I - \Phi(P)) = 0$ for all $A$. If, on the contrary, there exists $A$ such that $\Phi(P)\Phi(A)(I - \Phi(P)) \neq 0$, then we must have $(I - \Phi(P))\Phi(A)\Phi(P) = 0$. Consider $A + B$. Since $\Phi(P)\Phi(A + B)(I - \Phi(P)) \neq 0$, we have $(I - \Phi(P))(A + B)\Phi(P) = 0$, that is, $(I - \Phi(P))\Phi(B)\Phi(P) = 0$. It follows that

$$\Phi(B) = \Phi(PB(I - P)) = (I - \Phi(P))(A)(I - \Phi(P))$$

contradicting to the injectivity of $\Phi$.

A similar argument shows that $(I - \Phi(P))(B)\Phi(P) = 0$ will imply that $(I - \Phi(P))(A)\Phi(P) = 0$ for all $A$, completing the proof of Claim 3.

**Claim 4.** The assertion of Lemma 3.2 holds true.

Assume that the case (1) in Claim 3 occur. Then, for any $A \in \text{Alg} \mathcal{A}$, we have $(I - \Phi(P))\Phi(A)\Phi(P) = 0$ and $\Phi(PA(I - P)) = \Phi(P)\Phi(A)(I - \Phi(P))$. Thus

$$\Phi(AP) = \Phi(P)\Phi(A)\Phi(P)$$

$$= \Phi(P)\Phi(A)\Phi(P) + (I - \Phi(P))\Phi(A)\Phi(P) \quad (3.8)$$

$$= \Phi(A)\Phi(P)$$

and similarly

$$\Phi((I - P)A) = (I - \Phi(P))\Phi(A). \quad (3.9)$$

Let $E$ be an arbitrary idempotent operator in $\text{Alg} \mathcal{A}$. Note that, $EP = PEP$. Hence,

$$\Phi(EP\Phi(I - P)) = \Phi(EP\Phi(I - P)) = \Phi(P)\Phi(EP\Phi(I - P))(I - \Phi(P))$$

$$= \Phi(P)\Phi(EP\Phi(I - P))(I - \Phi(P)) + \Phi(P)\Phi(EP\Phi(I - P))\Phi(P)$$

$$= \Phi(P)\Phi(EP\Phi(I - P)). \quad (3.10)$$

In the same way, one has

$$\Phi(EP\Phi(I - P)) = \Phi(EP\Phi(I - P))(I - \Phi(P)). \quad (3.11)$$

Since $EP + EP\Phi(I - P)$ is idempotent, Eqs. (3.8)–(3.11) give that

$$\Phi(EP\Phi(I - P)) = \Phi(E)\Phi(EP\Phi(I - P)) + \Phi(EP\Phi(I - P))\Phi(E)$$

$$= \Phi(E)\Phi(EP\Phi(I - P))$$

$$+ \Phi(EP\Phi(I - P))(I - \Phi(P))\Phi(E)\Phi(P)$$

$$= \Phi(E)\Phi(EP\Phi(I - P)).$$
Replacing $E$ by $I - E$ in above equation, we also have
\[
\Phi((I - E)PA(I - P)) = (I - \Phi(E))\Phi((I - E)PA(I - P)).
\]
It follows that
\[
\Phi(E)\Phi((I - E)PA(I - P)) = 0
\]
and therefore,
\[
\Phi(PEA(I - P)) = \Phi(E)[\Phi(PEA(I - P)) + \Phi((I - E)PA(I - P))] = \Phi(E)\Phi(PEA(I - P)).
\]
A similar argument for the idempotent $(I - P)E + PA(I - P)E$ shows that
\[
\Phi(PA(I - P)E) = \Phi(PA(I - P))\Phi(E).
\]
So the case (1) of the Lemma 3.2 occurs.

Similarly, if the case (2) of Claim 3 occurs, then we have $\Phi(PEA(I - P)) = \Phi(PEA(I - P))\Phi(E)$ and $\Phi(PA(I - P)E) = \Phi(E)\Phi(PA(I - P))$ hold for all $A$ and all idempotent $E$, that is, the case (2) of Lemma 3.2 occurs.

The proof is finished. □

To prove Lemma 3.4 and 3.5, we need the following lemma which appeared in [4].

**Lemma 3.3.** Let $\mathcal{N}$ be a nest on a (real or complex) Banach space $X$. If $N \in \mathcal{N}$ is complemented in $X$ whenever $N_\perp = N$, then the ideal $\text{Alg}_\mathcal{N}$ of finite rank operators of $\text{Alg}\mathcal{N}$ is contained in the linear span of the idempotents in $\text{Alg}\mathcal{N}$. Moreover, for every rank one nilpotent operator $F$ in $\text{Alg}\mathcal{N}$, there exist idempotent operators $P, Q$ in the nest algebra $\text{Alg}\mathcal{N}$ such that $F = P - Q$.

**Lemma 3.4.** Let $\mathcal{N}$ and $\mathcal{M}$ be nests on Banach spaces $X$ and $Y$ (over real or complex) field $\mathbb{F}$, respectively, with $N \in \mathcal{N}$ and $M \in \mathcal{M}$ complemented in $X$ and $Y$, respectively, whenever $N_\perp = N$ and $M_\perp = M$. Let $\Phi : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{M}$ be a unital additive surjective map. If $\Phi$ preserves Jordan zero-products in both directions, then $\Phi$ preserves rank one idempotents in both directions, and, there exists an additive bijective function $\tau : \mathbb{F} \rightarrow \mathbb{F}$ such that, for any idempotent $P \in \text{Alg}\mathcal{N}$ and any $\alpha \in \mathbb{F}$, we have
\[
\Phi(\alpha P) = \tau(\alpha)\Phi(P).
\]

**Proof.** Taking arbitrary $\alpha, \beta \in \mathbb{F}$, since $(\alpha I - \alpha P)(\beta P) + (\beta P)(\alpha I - \alpha P) = 0$,
\[
\Phi(\alpha I)\Phi(\beta P) + \Phi(\beta P)\Phi(\alpha I) = \Phi(\alpha P)\Phi(\beta P) + \Phi(\beta P)\Phi(\alpha P). \tag{3.12}
\]
Letting $\alpha = 1$ in Eq. (3.12), we get
\[
2\Phi(\beta P) = \Phi(P)\Phi(\beta P) + \Phi(\beta P)\Phi(P). \tag{3.13}
\]
Multiplying Eq. (3.13) from left and right by $\Phi(P)$, respectively, we see that
\[
\Phi(P)\Phi(\beta P) = \Phi(\beta P)\Phi(P).
\]
On the other hand, taking $\beta = 1$ in Eq. (3.12), we obtain
\[
\Phi(\alpha I)\Phi(P) + \Phi(P)\Phi(\alpha I) = \Phi(\alpha P)\Phi(P) + \Phi(P)\Phi(\alpha P). \tag{3.14}
\]
Multiplying Eq. (3.14) from left and right by \( \Phi(P) \) and applying Eq. (3.13), we get
\[
\Phi(\alpha I) \Phi(P) = \Phi(P) \Phi(\alpha I).
\]
Since \( \Phi \) is surjective and preserves idempotents in both directions, it follows from Lemma 3.3 and the fact that the set of finite rank operators is strongly dense in the nest algebra \( Alg N \) that \( \Phi(\alpha I) = \tau(\alpha) I \) for some scalar \( \tau(\alpha) \). It is clear that \( \tau : \mathbb{F} \to \mathbb{F} \) is additive as \( \Phi \) is. According to the Eq. (3.14), it is easily seen that
\[
\Phi(P) \Phi(\alpha P) \Phi(P) = \Phi(\alpha P) \Phi(P) = \Phi(\alpha P) = \tau(\alpha) \Phi(P).
\]
Now, from \( \alpha P = P(\alpha P) P \) and Eq. (3.1), it follows that
\[
\Phi(\alpha P) = \tau(\alpha) \Phi(P).
\]
Let \( P \in Alg N \) be a rank one idempotent. Then \( P(Alg N) P = \mathbb{F}P \). By Claim 1 and Lemma 3.1, we see that \( \Phi(P)(Alg N) \Phi(P) = \Phi(P(Alg N) P) = \mathbb{F} \Phi(P) \). Therefore, \( \Phi(P)(Alg N) \Phi(P) \) is an one dimensional subalgebra of \( Alg N \). This forces that \( \Phi(P) \) is a rank one idempotent, i.e., \( \Phi \) preserves rank one idempotent. It is also clear from \( \Phi(\mathbb{F}P) = \mathbb{F} \Phi(P) \) that the additive function \( \tau : \mathbb{F} \to \mathbb{F} \) is bijective. By considering \( \Phi^{-1} \) one sees that \( \Phi \) preserves rank one idempotents in both directions.

The proof of the lemma is completed. \( \Box \)

**Lemma 3.5.** Let \( N \) and \( M \) be nests on Banach spaces \( X \) and \( Y \) (over real or complex) field \( \mathbb{F} \), respectively, with \( N \subset N \) and \( M \subset M \) complemented in \( X \) and \( Y \), respectively, whenever \( N_0 = N \) and \( M_0 = M \). Let \( \Phi : Alg N \to Alg M \) be a unital additive surjective map. If \( \Phi \) preserves Jordan zero-products in both directions, then there exists an automorphism \( \tau : \mathbb{F} \to \mathbb{F} \) such that \( \Phi(\lambda P) = \tau(\lambda) \Phi(P) \) and \( \Phi(\lambda F) = \tau(\lambda) \Phi(F) \) hold for all idempotent operator \( P \), finite rank operator \( F \) and scalar \( \lambda \in \mathbb{F} \); furthermore, for any \( F, G \in Alg N \) and \( A \in Alg N \), we have
\[
\Phi(FA + AF) = \Phi(F) \Phi(A) + \Phi(A) \Phi(F),
\]
and
\[
\Phi(FAF) = \Phi(F) \Phi(A) \Phi(F).
\]

**Proof.** By Lemma 3.4, there is an additive bijective function \( \tau : \mathbb{F} \to \mathbb{F} \) such that \( \Phi(\lambda P) = \tau(\lambda) \Phi(P) \) holds for all scalar \( \lambda \) and idempotent \( P \in Alg N \). Next we claim that \( \tau \) is in fact an automorphism of \( \mathbb{F} \) and \( \Phi(\lambda x \otimes f) = \tau(\lambda) \Phi(x \otimes f) \) holds for every rank one operator \( x \otimes f \in Alg N \) and every \( \lambda \in \mathbb{F} \).

First assume that \( x \otimes f \in Alg N \) is a rank one nilpotent. Then, by Lemma 3.3, there exist two idempotent operators \( Q_1, Q_2 \in Alg N \) such that \( x \otimes f = Q_1 - Q_2 \). Therefore, by Lemma 3.4, for any scalar \( \lambda \), we have \( \Phi(\lambda x \otimes f) = \Phi(\lambda Q_1 - \lambda Q_2) = \tau(\lambda) (\Phi(Q_1) - \Phi(Q_2)) = \tau(\lambda) \Phi(x \otimes f) \). Let \( \lambda \) and \( \mu \) be any nonzero scalars. Since \( \mu x \otimes f \) is still nilpotent, we have
\[
\tau(\lambda \mu) \Phi(x \otimes f) = \Phi(\mu \lambda x \otimes f) = \tau(\lambda) \Phi(\mu x \otimes f) = \tau(\lambda) \tau(\mu) \Phi(x \otimes f).
\]
Hence
\[
\tau(\lambda \mu) = \tau(\lambda) \tau(\mu),
\]
that is, \( \tau \) is an automorphism.
If \( x \otimes f \) is not a nilpotent, then \( \langle x, f \rangle \neq 0 \) and \( \frac{x}{\langle x, f \rangle} \otimes f \) is a rank one idempotent. Therefore, 
\[
\Phi(x \otimes f) = \Phi\left( \langle x, f \rangle \frac{x}{\langle x, f \rangle} \otimes f \right) = \tau(\langle x, f \rangle) \Phi\left( \frac{x}{\langle x, f \rangle} \otimes f \right).
\]
It follows that 
\[
\Phi(\lambda x \otimes f) = \Phi\left( \lambda \langle x, f \rangle \frac{x}{\langle x, f \rangle} \otimes f \right) = \tau(\lambda \langle x, f \rangle) \Phi\left( \frac{x}{\langle x, f \rangle} \otimes f \right) = \tau(\lambda) \tau(\langle x, f \rangle) \Phi\left( \frac{x}{\langle x, f \rangle} \otimes f \right).
\]
Hence \( \Phi(\lambda F) = \tau(\lambda) \Phi(F) \) holds for all \( F \in \text{Alg}_{\mathcal{N}} \) and \( \lambda \in \mathbb{F} \).

Now let us show that \( \Phi(FG + GF) = \Phi(F)\Phi(G) + \Phi(G)\Phi(F) \) for all \( F, G \in \text{Alg}_{\mathcal{N}} \). By virtue Eq. (3.3) in Lemma 3.1, we know that \( \Phi(FP + PF) = \Phi(F)\Phi(P) + \Phi(P)\Phi(F) \) holds for all \( F \in \text{Alg}_{\mathcal{N}} \) and all idempotent \( P \in \text{Alg}_{\mathcal{N}} \). Since \( FP + PF \) is of finite rank, for every \( \lambda \in \mathbb{F} \), we have \( \Phi(\lambda(FP + PF)) = \tau(\lambda)\Phi(FP + PF) \). It follows that 
\[
\Phi(F\lambda P + \lambda PF) = \tau(\lambda)\Phi(F)\Phi(P) + \Phi(P)\Phi(F)) = \Phi(F)\Phi(\lambda P) + \Phi(\lambda P)\Phi(F).
\]
This, together with Lemma 3.3, implies that 
\[
\Phi(FG + GF) = \Phi(F)\Phi(G) + \Phi(G)\Phi(F) \quad (3.15)
\]
holds for all finite rank operator \( F \) and all rank one operator \( G \) in the domain nest algebra, and consequently, holds for all \( F, G \in \text{Alg}_{\mathcal{N}} \).

Next we show that, for any rank one idempotent \( P \), any \( \lambda \in \mathbb{F} \), and any \( A \in \text{Alg}_{\mathcal{N}} \), we have \( \Phi(\lambda PA + A(\lambda P)) = \Phi(\lambda P)\Phi(A) + \Phi(A)\Phi(\lambda P) \).

Let \( B = (I - P)A(I - P) \). Then \( \Phi(\lambda PB + B(\lambda P)) = \Phi(B)\Phi(\lambda P) + \Phi(\lambda P)\Phi(B) = 0, A \) is of finite rank and hence, by Eq. (3.15),
\[
\Phi(\lambda PA + A(\lambda P)) = \Phi(\lambda P)(A - B) + (A - B)(\lambda P)) = \Phi(\lambda P)\Phi(A) - \Phi(A)\Phi(\lambda P)
\]
Thus, by Lemma 3.3, it is easily seen that 
\[
\Phi(FA + AF) = \Phi(F)\Phi(A) + \Phi(A)\Phi(F) \quad (3.16)
\]
holds for all finite rank operator \( F \) and all operator \( A \) in \( \text{Alg}_{\mathcal{N}} \).

Replacing \( A \) by \( FA + AF \) in Eq. (3.16), we get 
\[
\Phi(F(FA + AF) + (FA + AF)F) = \Phi(F)\Phi(FA + AF) + \Phi(FA + AF)\Phi(F)
\]
and 
\[
\Phi(F(FA + AF) + (FA + AF)F) = \Phi(F)\Phi(FA) + 2\Phi(FA)\Phi(A) \Phi(F) + (A)\Phi(F)^2
\]
Comparing above two equations yields 
\[
\Phi(FAF) = \Phi(F)\Phi(A)\Phi(F). \quad (3.17)
\]
Replacing \( F \) by \( F + G \) in Eq. (3.17), one gets 
\[
\Phi(FAG + GAF) = \Phi(F)\Phi(A)\Phi(G) + \Phi(G)\Phi(A)\Phi(F)
\]
holds for all finite rank operators \( F, G \) and all operators \( A \) in \( \text{Alg}_{\mathcal{N}} \). \( \square \)
The following lemma is a characterization of rank one operators in nest algebras and was proved in [10, Lemma 2.1].

**Lemma 3.6.** Let \( \mathcal{A} \) be a standard subalgebra in a nest algebra \( \text{Alg} \mathcal{N} \). Then the following statements are equivalent.

1. \( A \in \mathcal{A} \) is of rank one.
2. For any \( B, C \in \mathcal{A} \), \( BAC = 0 \) will imply either \( BA = 0 \) or \( AC = 0 \).
3. For any rank one operators \( B, C \in \mathcal{A} \), \( BAC = 0 \) will imply either \( BA = 0 \) or \( AC = 0 \).
4. For any nilpotent rank one operators \( B, C \in \mathcal{A} \), \( BAC = 0 \) will imply either \( BA = 0 \) or \( AC = 0 \).

**4. Proofs of the main results**

Now let us give proofs of the results listed in Section 2.

**Proof of Theorem 2.1.** Obviously, if the condition (1) or (2) holds, then \( \Phi \) is a Jordan ring isomorphism and hence preserves Jordan zero-products in both directions.

Assume that \( \Phi \) preserves Jordan zero-products in both directions. We have to show that either (1) or (2) is true. We will complete the proof by checking several Claims.

The following claim is the key claim for our proof.

**Claim 1.** \( \Phi \) preserves rank one operators in both directions.

For every rank one operator \( x \otimes f \in \text{Alg} \mathcal{N} \), let \( N_x = \cap \{ N | x \in N \in \mathcal{N} \} \). Then \( x \in N_x \) and \( f \in (N_x)^\perp \). \( x \otimes f \) is either a multiple of an idempotent operator or a square zero operator.

**Case 1.** \( \langle x, f \rangle = \lambda \neq 0 \). Then \( x \otimes f = \lambda P \), where \( P \) is some rank one idempotent operator in \( \text{Alg} \mathcal{N} \). By Lemma 3.4, \( \Phi \) preserves rank one idempotents in both directions and there exists an automorphism \( \tau : \mathbb{F} \to \mathbb{F} \) such that \( \Phi(x \otimes f) = \Phi(\lambda P) = \tau(\lambda) \Phi(P) \). Therefore, \( \Phi(x \otimes f) \) is of rank one.

**Case 2.** \( \langle x, f \rangle = 0 \) but \( f \not\in (N_x)^\perp \). Then we can take \( x_0 \in N_x \) such that \( \langle x_0, f \rangle = 1 \). Thus \( x_0 \otimes f \in \text{Alg} \mathcal{N} \) is a rank one idempotent. Denote \( \Phi(x \otimes f) = T \) and \( \Phi(x_0 \otimes f) = y \otimes g \). By Lemma 3.5,

\[
0 = \Phi(x \otimes f \cdot x_0 \otimes f \cdot x \otimes f) \\
= \Phi(x \otimes f) \Phi(x_0 \otimes f) \Phi(x \otimes f) \\
= Ty \otimes T^* g.
\]

So one of the \( Ty \) and \( T^* g \) is zero. By Lemma 3.5 again,

\[
T = \Phi(x \otimes f) = \Phi(x \otimes f \cdot x_0 \otimes f + x_0 \otimes f \cdot x \otimes f) \\
= \Phi(x \otimes f) \Phi(x_0 \otimes f) + \Phi(x \otimes f) \Phi(x \otimes f) \\
= Ty \otimes g + y \otimes T^* g,
\]

and therefore, \( T \) is of rank one.
Case 3. \((x, f) = 0, f \in (N_x)_{-}^\perp\) and \((N_x)_{-} \neq N_x\). In this case, take \(f_0 \in (N_x)_{-}^\perp \setminus N_x^\perp\). By the Case 1 and 2, we see that \(\Phi(x \otimes f_0)\) is of rank one. Denote \(\Phi(x \otimes f_0) = y \otimes g\). Choose \(x_1 \in N_x\) such that \((x_1, f_0) = 1\) and let \(S = \Phi(x_1 \otimes f)\). Then
\[
0 = \Phi(x_1 \otimes f \cdot x \otimes f_0 \cdot x_1 \otimes f) \\
= \Phi(x_1 \otimes f) \Phi(x \otimes f_0) \Phi(x_1 \otimes f) \\
= S y \otimes S^* g.
\]
So either \(Sy = 0\) or \(S^* g = 0\). Since, by Lemma 3.5,
\[
\Phi(x \otimes f) = \Phi((x \otimes f_0)(x_1 \otimes f) + (x_1 \otimes f)(x \otimes f_0)) \\
= y \otimes g S + S y \otimes g \\
= y \otimes S^* g + S y \otimes g,
\]
we see that \(\Phi(x \otimes f)\) is a rank one operator.

Case 4. \((x, f) = 0, f \in (N_x)_{-}^\perp\) and \((N_x)_{-} = N_x\).

In this case, by the hypothesis on \(N\), \(N_x\) is complemented. Thus there exists a bounded idempotent operator \(P\) with range \(N_x\). Then \(P \in \text{Alg} N\) and \(PAP = AP\) for all \(A\) in the nest algebra \(\text{Alg} N\). Hence, Lemma 3.2 is applicable.

Assume that the case (1) of Lemma 3.2 holds true. Note that \(x \otimes f = P(x \otimes f)(I - P)\). By Lemma 3.5, it is obvious that, to show that \(\Phi(x \otimes f)\) is of rank one, we need only to check that, for any rank one nilpotent operators \(T \in \text{Alg} \mathcal{M}\) and \(S \in \text{Alg} \mathcal{M}\), \(T \Phi(x \otimes f) S = 0\) implies either \(T \Phi(x \otimes f) = 0\) or \(\Phi(x \otimes f) S = 0\). To do this, assume that \(T\) and \(S\) be such rank one nilpotents so that \(T \Phi(x \otimes f) S = 0\). Then, by Lemma 3.3, there exist idempotent operators \(Q_1, Q_2 \in \text{Alg} \mathcal{M}\) and \(R_1, R_2 \in \text{Alg} \mathcal{M}\) such that \(T = Q_1 - Q_2\) and \(S = R_1 - R_2\), respectively. Furthermore, by Lemma 3.1, there are idempotents \(E_1, E_2 \in \text{Alg} N\) and \(F_1, F_2 \in \text{Alg} N\), so that \(\Phi(E_i) = Q_i\) and \(\Phi(F_i) = R_i, i = 1, 2\). Hence, \(T \Phi(x \otimes f) S = 0\) and Lemma 3.2 together imply that
\[
0 = (Q_1 - Q_2) \Phi(x \otimes f)(R_1 - R_2) \\
= \Phi(E_1) \Phi(P(x \otimes f)(I - P)) \Phi(F_1) - \Phi(E_2) \Phi(P(x \otimes f)(I - P)) \Phi(F_2) \\
- \Phi(E_2) \Phi(P(x \otimes f)(I - P)) \Phi(F_1) + \Phi(E_1) \Phi(P(x \otimes f)(I - P)) \Phi(F_2) \\
= \Phi(P E_1 P(x \otimes f)(I - P)) \Phi(F_1) - \Phi(P E_1 P(x \otimes f)(I - P)) \Phi(F_2) \\
- \Phi(P E_2 P(x \otimes f)(I - P)) \Phi(F_1) + \Phi(P E_2 P(x \otimes f)(I - P)) \Phi(F_2) \\
= \Phi(P E_1 P(x \otimes f)(I - P) F_1) - \Phi(P E_1 P(x \otimes f)(I - P) F_2) \\
- \Phi(P E_2 P(x \otimes f)(I - P) F_1) + \Phi(P E_2 P(x \otimes f)(I - P) F_2) \\
= \Phi((E_1 - E_2)(x \otimes f)(F_1 - F_2)).
\]
It follows that \((E_1 - E_2)(x \otimes f)(F_1 - F_2) = 0\) and hence either \((E_1 - E_2)(x \otimes f) = 0\) or \((x \otimes f)(F_1 - F_2) = 0\). Applying Lemma 3.2 again, either
\[
T \Phi(x \otimes f) = (\Phi(E_1) - \Phi(E_2)) \Phi(x \otimes f) = \Phi((E_1 - E_2)(x \otimes f)) = 0
\]
or
\[
\Phi(x \otimes f) S = \Phi(x \otimes f)(\Phi(F_1) - \Phi(F_2)) = \Phi((x \otimes f)(F_1 - F_2)) = 0.
\]
Therefore, \(\Phi(x \otimes f)\) is a rank one operator.
If the case (2) of Lemma 3.2 happens, one can similarly check that $\Phi(x \otimes f)$ is of rank one, completing the proof of Claim 1.

In the sequel we’ll denote $\mathcal{E}_1(\mathcal{N}) = \bigcup\{N \in \mathcal{N} | \dim N^\perp > 1\}$, $\mathcal{E}_2(\mathcal{N}) = \bigcup\{N^\perp | N \in \mathcal{N}, \dim N > 1\}$, $\mathcal{D}_1(\mathcal{N}) = \bigcup\{N \in \mathcal{N} | N \neq \mathcal{X}\}$, $\mathcal{D}_2(\mathcal{N}) = \bigcup\{N^\perp | N \in \mathcal{N} \text{ and } N \neq 0\}$. Note that $\mathcal{D}_1(\mathcal{N})$ is dense in $\mathcal{X}$ and $\mathcal{D}_2(\mathcal{N})$ is $\mathcal{W}^*$-dense in $\mathcal{X}^*$, and actually norm-dense whenever $\mathcal{X}$ is reflexive.

Claim 2. Either

(1) there exist a dimension preserving order isomorphism $\theta : \mathcal{N} \to \mathcal{M}$, an automorphism $\tau : \mathcal{F} \to \mathcal{F}$ and $\tau$-linear bijective maps $T : D_1(\mathcal{N}) \to D_1(\mathcal{M})$ and $S : D_2(\mathcal{N}) \to D_2(\mathcal{M})$ such that $T(N) = \theta(N)$ for $N \neq \mathcal{X}$, $S(N^\perp) = \theta(N^\perp)$ for $N \neq 0$, and

$$\Phi(x \otimes f) = Tx \otimes Sf$$

for all $x \otimes f \in \text{Alg.} \mathcal{N}$; or

(2) There exist a dimension preserving order isomorphism $\theta : \mathcal{N}^\perp \to \mathcal{M}$, an automorphism $\tau : \mathcal{F} \to \mathcal{F}$ and $\tau$-linear bijective maps $T : D_2(\mathcal{N}) \to D_1(\mathcal{M})$ and $S : D_1(\mathcal{N}) \to D_2(\mathcal{M})$ such that $T(N^\perp) = \theta(N^\perp)$ for $N \neq 0$, $S(N) = \theta(N^\perp)$ for $N \neq \mathcal{X}$, and

$$\Phi(x \otimes f) = Tf \otimes Sx$$

for all $x \otimes f \in \text{Alg.} \mathcal{N}$.

By Claim 1 and a result concerning characterization of additive maps between nest algebras that preserve rank one operators due to Bai and Hou (Ref. [2]), the following assertion is true.

Either

(i) there exist a dimension preserving order isomorphism $\theta : \mathcal{N} \to \mathcal{M}$ and additive bijective maps $T : D_1(\mathcal{N}) \to D_1(\mathcal{M})$ and $S : D_2(\mathcal{N}) \to D_2(\mathcal{M})$ such that for each $N \in \mathcal{N}$ and $x \otimes f \in \text{Alg.} \mathcal{N}$, $T|_{\mathcal{E}_1(\mathcal{N})} : \mathcal{E}_1(\mathcal{N}) \to \mathcal{E}_1(\mathcal{M})$ and $S|_{\mathcal{E}_2(\mathcal{N})} : \mathcal{E}_2(\mathcal{N}) \to \mathcal{E}_2(\mathcal{M})$ are $\tau$-linear bijective, $T(N) = \theta(N)$ and $S(N^\perp) = \theta(N^\perp)$,

$$\Phi(x \otimes f) = \begin{cases} Tx \otimes Sf & \text{if } x \in \mathcal{E}_1(\mathcal{N}) \text{ and } f \in \mathcal{E}_2(\mathcal{N}), \\ Te_0 \otimes S(F(x_0)f) & \text{if } x \in \mathcal{E}_1(\mathcal{N}) \text{ and } f \notin \mathcal{E}_2(\mathcal{N}), \\ T\left[f(x)\right] \otimes Sf_0 & \text{if } x \notin \mathcal{E}_1(\mathcal{N}) \text{ and } f \notin \mathcal{E}_2(\mathcal{M}); \end{cases}$$

or

(ii) There exist a dimension preserving order isomorphism $\theta : \mathcal{N}^\perp \to \mathcal{M}$ and additive bijective maps $T : D_2(\mathcal{N}) \to D_1(\mathcal{M})$ and $S : D_1(\mathcal{N}) \to D_2(\mathcal{M})$ such that for every $N \in \mathcal{N}$ and $x \otimes f \in \text{Alg.} \mathcal{N}$, $T|_{\mathcal{E}_2(\mathcal{N})} : \mathcal{E}_2(\mathcal{N}) \to \mathcal{E}_1(\mathcal{M})$ and $S|_{\mathcal{E}_1(\mathcal{N})} : \mathcal{E}_1(\mathcal{N}) \to \mathcal{E}_2(\mathcal{M})$ are $\tau$-linear bijective, $T(N^\perp) = \theta(N^\perp)$ and $S(N) = \theta(N^\perp)$,

$$\Phi(x \otimes f) = \begin{cases} Tf \otimes Sx & \text{if } x \in \mathcal{E}_1(\mathcal{N}) \text{ and } f \in \mathcal{E}_2(\mathcal{N}), \\ Tf_0 \otimes S(F(x_0)f) & \text{if } x \in \mathcal{E}_1(\mathcal{N}) \text{ and } f \notin \mathcal{E}_2(\mathcal{M}), \\ Te_0 \otimes S\left[f(x)\right] & \text{if } x \notin \mathcal{E}_1(\mathcal{N}) \text{ and } f \notin \mathcal{E}_2(\mathcal{M}). \end{cases}$$

The pathological cases occur if and only if $\dim(0_+) = 1$ with $e_0 \in 0_+$ or $\dim(X^\perp) = 1$ with $f_0 \in X^\perp$.

Thus we need check further that both $T$ and $S$ are $\tau$-linear, and either $\Phi(x \otimes f) = Tx \otimes Sf$ for all $x \otimes f \in \text{Alg.} \mathcal{N}$ or $\Phi(x \otimes f) = Tf \otimes Sx$ for all $x \otimes f \in \text{Alg.} \mathcal{N}$.
If (i) holds, we will prove \( T : \mathcal{D}_1(N) \to \mathcal{D}_1(M) \) and \( S : \mathcal{D}_2(N) \to \mathcal{D}_2(M) \) are \( \tau \)-linear. By using Lemma 3.5, there is an automorphism \( \tau : \mathbb{F} \to \mathbb{F} \) such that \( \Phi(\lambda F) = \tau(\lambda)\Phi(F) \) for every \( F \in \text{Alg}_x(N) \) and every scalar \( \lambda \), i.e., \( \Phi \) is \( \tau \)-linear on \( \text{Alg}_x(N) \). From this it is easily seen that both \( T \) and \( S \) are \( \tau \)-linear. In fact, if there exist \( x \in \delta_1(N) \), \( f \notin \mathcal{E}_2(N) \) such that \( x \otimes f \in \text{Alg}_\tau(N) \), then \( 0_+ = [e_0] \) and \( x = (x, f) e_0 \). For every \( \alpha \in \mathbb{F} \), \( T e_0 \otimes S \alpha f = \Phi(\alpha e_0 \otimes f) = \tau(\alpha) \Phi(e_0 \otimes f) = \tau(\alpha) T e_0 \otimes S \alpha f \). Thus we have \( S(\alpha f) = \tau(\alpha) S f \). This implies that \( S \) is \( \tau \)-linear on \( \mathcal{D}_2(N) \). Similarly, one can check that \( T(\alpha x) = \tau(\alpha) Ax \) for all \( x \in \mathcal{D}_1(N) \). Now it is clear that \( \Phi(x \otimes f) = TX \otimes SF \) for all \( x \otimes f \in \text{Alg}_x(N) \).

Similarly, the case (ii) holds will imply that (2) holds.

**Claim 3.** If (1) in Claim 2 holds, then \( T \) is bijective \( \tau \)-linear and \( \Phi(A) = TAT^{-1} \) on \( \mathcal{D}_1(N) \) for all \( A \in \text{Alg}_\tau(N) \).

By Lemma 3.5 and Claim 2 above, for any \( x \otimes f \), \( A \in \text{Alg}_\tau(N) \), we have

\[
\Phi(A)TX \otimes SF + TX \otimes \Phi(A)^*SF = \Phi(A(x \otimes f) + (x \otimes f)A) = TAx \otimes SF + TX \otimes SA^*f.
\]

It follows that

\[
\Phi(A)TX \in [TX, TAx] \quad \forall x \in \mathcal{D}_1(N),
\]

that is, \( T^{-1} \Phi(A)T \) is a locally linear combination of \( I \) and \( A \) on \( \mathcal{D}_1(N) \). Here \( y, z \) denotes the linear span of \( y \) and \( z \). Since \( \text{dim } X = \infty \), \( I \) is not of finite rank. By virtue of a result in [12] (also, ref. [9]), we see that \( I, A \) is algebraic reflexive. So \( T^{-1} \Phi(A)T \in [I, A] \), i.e.,

\[
T^{-1} \Phi(A)T = (\alpha A + \beta A I)|_{\mathcal{D}_1(N)}
\]

for some scalars \( \alpha \) and \( \beta \). So, \( \Phi(A)|_{\mathcal{D}_1(N)} = \alpha A T A T^{-1} + \beta A I \) for every \( A \in \text{Alg}_\tau(N) \). By the additivity of \( \Phi \) it is easily seen that there exists a constant \( \alpha \) such that \( \alpha A = \alpha \) for all \( A \). It is also clear that \( \beta I = 0 \) whenever \( F \) is finite rank since \( \Phi \) maps finite rank operators to finite rank operators. Note that \( \Phi \) maps idempotents to idempotents, so \( \alpha = \pm 1 \). Then, applying Lemma 3.5 again, for any \( A \) and any finite rank operator \( F \) in \( \text{Alg}_\tau(N) \), we have

\[
\alpha(T FAT^{-1} + T AFT^{-1}) = \Phi(FA + AF) = \Phi(F)\Phi(A) + \Phi(A)\Phi(F) = \alpha^2(T FAT^{-1} + T AFT^{-1}) + 2\alpha \beta \alpha T FT^{-1} = T FAT^{-1} + T AFT^{-1} + 2\alpha \beta \alpha T FT^{-1},
\]

this enforces that \( \alpha = 1 \) and \( \beta = 0 \) for all \( A \), as desired.

Similarly, one can check that

**Claim 4.** If (2) in Claim 2 holds, then \( T \) is bijective \( \tau \)-linear and \( \Phi(T) = T A^* T^{-1} \) on \( \mathcal{D}_2(N) \) for all \( A \in \text{Alg}_\tau(N) \).

**Claim 5.** The \( \tau \)-linear operator \( A \) in Claim 3 (res. in Claim 4) is linear or conjugate linear, and can be extended to a bounded linear or conjugate linear invertible operator from \( X \) onto \( Y \) (res. from \( X^* \) onto \( Y \)).

Claims 3 and 4 together assert that \( \Phi \) is either a ring isomorphism or a ring anti-isomorphism from \( \text{Alg}_\tau(N) \) onto \( \text{Alg}_\tau(M) \). Then, Claim 5 is an immediate consequence of a result in [10].

The proof of Theorem 2.1 is finished. □
Proof of Theorem 2.3. In fact, Theorem 2.3 is an immediate consequence of Lemma 3.5. Assume \( \Phi: \mathcal{T}(n_1, n_2, \ldots, n_k) \to \mathcal{T}(m_1, m_2, \ldots, m_r) \) is a unital additive surjective map and preserves Jordan zero-products in both directions. Then, by Lemma 3.5, \( \Phi(FG + GF) = \Phi(F)\Phi(G) + \Phi(G)\Phi(F) \) holds for all \( F, G \in \mathcal{T}(n_1, n_2, \ldots, n_k) \), that is, \( \Phi \) is a Jordan ring isomorphism. Since every Jordan ring isomorphism between upper triangular block matrix algebras is either a ring isomorphism or a ring anti-isomorphism, we see that either (1) or (2) holds (Ref. for example [1]).

The converse is obvious. \( \square \)

Proof of Theorem 2.4. Obviously, we only need to check the “only if” part. Assume that \( \Phi: \mathcal{T}(n_1, n_2, \ldots, n_k) \to \mathcal{T}(m_1, m_2, \ldots, m_r) \) is linear and preserves Jordan zero-products in both directions. Our approach is to show that \( \Phi(I) = cI \) for some nonzero scalar \( c \), and then apply Theorem 2.3 to \( c^{-1}\Phi \). Since the argument is similar to the proof of Theorem 2.5, we omit it here. \( \square \)

Proof of Theorem 2.5. It is clear that \( \Phi \) is bijective. We’ll show that \( \Phi(I) = cI \) for some nonzero scalar \( c \). Let \( P \in \text{Alg} \mathcal{N} \) with \( P^2 = P \). Since \( P(I - P) + (I - P)P = 0 \), we have \( \Phi(I)\Phi(P) + \Phi(P)\Phi(I) = 2\Phi(P)^2 \). Multiplying the above equation from left and right by \( \Phi(P) \), respectively, we get
\[
\Phi(I)\Phi(P)^2 = \Phi(P)^2\Phi(I).
\]
Similarly, it follows from \( \Phi(P)^2\Phi(I) + \Phi(I)\Phi(P)\Phi(I) = 2\Phi(I)\Phi(P)^2 \) and \( \Phi(P)\Phi(I)^2 + \Phi(I)\Phi(P)\Phi(I) = 2\Phi(P)^2\Phi(I) \) that \( \Phi(P)\Phi(I)^2 = \Phi(I)^2\Phi(P) \).

Hence, by Lemma 3.3, for each finite rank operator \( F \in \text{Alg} \mathcal{N} \), we have
\[
\Phi(F)\Phi(I)^2 = \Phi(I)^2\Phi(F). \quad (4.1)
\]
Let \( A, B \in \text{Alg} \mathcal{N} \) with \( BA = 0 \). For every idempotent \( P \), it follows from \( AP(I - P)B + (I - P)BAP = 0 \) that \( \Phi(AP)\Phi((I - P)B) + \Phi((I - P)B)\Phi(AP) = 0 \). Thus for every idempotent \( P \) we have
\[
\Phi(AP)\Phi(B) + \Phi(B)\Phi(AP) = \Phi(AP)\Phi(PB) + \Phi(PB)\Phi(AP). \quad (4.2)
\]
On the other hand, \( A(I - P)PB + PBA(I - P) = 0 \) implies that \( \Phi(A(I - P))\Phi(PB) + \Phi(PB)\Phi(A(I - P)) = 0 \), and hence for every idempotent \( P \)
\[
\Phi(A)\Phi(PB) + \Phi(PB)\Phi(A) = \Phi(AP)\Phi(PB) + \Phi(PB)\Phi(AP). \quad (4.3)
\]
Combining (4.2) and (4.3), we get \( \Phi(AP)\Phi(B) + \Phi(B)\Phi(AP) = \Phi(A)\Phi(PB) + \Phi(PB)\Phi(A) \) for every idempotent \( P \). Hence for every finite rank operator \( F \in \text{Alg} \mathcal{N} \),
\[
\Phi(AF)\Phi(B) + \Phi(B)\Phi(AF) = \Phi(A)\Phi(FB) + \Phi(FB)\Phi(A). \quad (4.4)
\]
Take \( A = Q \) and \( B = I - Q \) for some idempotent operator \( Q \in \text{Alg} \mathcal{N} \), then \( BA = 0 \). It follows from (4.4) that \( \Phi(QF)^2\Phi(I - Q) + \Phi(I - Q)\Phi(QF) = \Phi(Q)\Phi(F(I - Q)) + \Phi(F(I - Q))\Phi(Q) \). Thus we see that
\[
\Phi(QF)^2\Phi(I) + \Phi(I)\Phi(QF) - \Phi(Q)\Phi(F) - \Phi(F)\Phi(Q)
\]
\[
= \Phi(QF)^2\Phi(Q) + \Phi(Q)\Phi(QF) - \Phi(Q)\Phi(FQ) - \Phi(FQ)\Phi(Q).
\]
On the other hand, taking \( A = I - Q \) and \( B = Q \), we obtain from (4.4) another equation
\[
\Phi(I)\Phi(QF) + \Phi(QF)\Phi(I) - \Phi(F)\Phi(Q) - \Phi(Q)\Phi(F)
\]
\[
= \Phi(Q)\Phi(FQ) + \Phi(FQ)\Phi(Q) - \Phi(QF)\Phi(Q) - \Phi(Q)\Phi(QF).
\]
Hence \( \Phi(QF + FQ)\Phi(I) + \Phi(I)\Phi(QF + FQ) = 2(\Phi(Q)\Phi(F) + \Phi(F)\Phi(Q)) \) holds for every idempotent \( Q \). This further implies that

\[
\Phi(FE + EF)\Phi(I) + \Phi(I)\Phi(FE + EF) = 2(\Phi(F)\Phi(E) + \Phi(E)\Phi(F))
\]

(4.5) holds for every finite rank operator \( E \in \mathbb{Alg}_N \). Multiplying (4.5) from left and right by \( \Phi(I) \), respectively, we see that

\[
\Phi(I)^2\Phi(FE + EF) + \Phi(I)\Phi(FE + EF)\Phi(I) = 2\Phi(I)(\Phi(F)(\Phi(E) + \Phi(E)\Phi(F))
\]

and

\[
\Phi(I)\Phi(FE + EF)\Phi(I) + \Phi(FE + EF)\Phi(I)^2 = 2(\Phi(F)\Phi(E) + \Phi(E)\Phi(F))\Phi(I).
\]

These two equations, together with Eq. (4.1), entail that

\[
\Phi(I)(\Phi(F)\Phi(E) + \Phi(E)\Phi(F)) = (\Phi(F)\Phi(E) + \Phi(E)\Phi(F))\Phi(I).
\]

(4.6)

By the linearity and the strong continuous property of \( \Phi \), the Eq. (4.6) holds for all operators \( F, E \in \mathbb{Alg}_N \). Let \( F = E = A \) in (4.6), we get

\[
\Phi(I)\Phi(A)^2 = \Phi(A)^2\Phi(I)
\]

(4.7)

holds for all \( A \in \mathbb{Alg}_N \). By the surjectivity and the strong continuity of \( \Phi \), the Eq. (4.7) implies that \( \Phi(I) \) commutes with all idempotent operators and hence there must exist a nonzero scalar \( c \) such that \( \Phi(I) = cI \). Let \( \Psi(\cdot) = \frac{1}{c}\Phi(\cdot) \), then \( \Psi : \mathbb{Alg}_N \rightarrow \mathbb{Alg}_N \) is a unital linear surjection preserving Jordan zero-product in both directions. Now applying Theorem 2.1 to complete the proof of the Theorem 2.5. □

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