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# A Discrete Conservative Model for the Linear Vibrating String and Rod

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**Abstract**—In this paper, we shall suggest and study a conservative discrete model for the linear vibrating string and rod fixed at the end points. We shall prove that the difference systems involved in our models may be seen as second-order unconditionally stable finite difference schemes of the classical equations of the linear vibrating string and vibrating rod. If the forces acting on the string (or rod) are conservative the total energy of the discrete solutions of our models is conserved and we can prove that we have stability for every choice of the time step $\Delta t$. We have considered both hinged and clamped rod; the constrains are naturally included into the model and the conservation of energy is still proved by giving a suitable definition of potential energy. Some numerical examples are presented.

## 1. INTRODUCTION

In some previous papers (see, f.i., [1–3]), some discrete models for the vibrating string and rod have been considered following the approach suggested by Greenspan (see, f.i., [4,5]), who starts from a computer approach of the mechanics of particles in order to study a physical phenomenon. In the quoted references [1–3] it has been shown as the difference systems arising from the discrete models may be seen as particular second-order finite difference schemes of differential equations which model the motion of the continuous string (or rod). The stability of the considered difference systems have been studied by using standard techniques.

We remark now that, in the conservative cases, such discrete models are not conservative as the computed total energy is not constant in time.

In this paper, we shall suggest and study new conservative discrete models for the linear vibrating string and rod. In Section 2, some definitions and notations are introduced and a short review of the main previous results is given. In Section 3, the conservative model for the linear string is presented and theorems regarding energy conservation, discretization order, and stability are proved. In Section 4, similar results for the discrete rod are given. In Section 5, some numerical examples emphasizing the stability conditions, accuracy, and energy conservation are produced.

## 2. SHORT REVIEW OF PREVIOUS RESULTS

A discrete string (or rod) fixed at the end points, free to move in $xy$ plane vertically only (transversal vibrations) may be represented as a set of $n+1$ particles $P_0, P_1, \ldots, P_n$ whose centers

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX have coordinates $x_i = i\Delta x$ and $y_i$. $P_0$ and $P_n$ which mass is $m/2$, are not in motion and have coordinates (of the centers) $x_0 = 0$, $x_n = L$, and $y_0 = y_n = 0$, while $P_i$ $(i = 1, 2, \ldots, n-1)$, have mass $m$ and are free to move vertically only.

If $\Delta t$ is the finite time step, $y_{i,k}$, $(i = 0, 1, \ldots, n,\ k = 1, 2, \ldots)$ represents the position in the $xy$ plane of the particle $P_i$ at time $t_k = k\Delta t$. In the discrete model, given the initial position $y_{i,0}$ and speed $v_{i,0}$, the motion of the particle $P_i$ is determined for every $t_k$ by

$$F_{i,k} = ma_{i,k}. \tag{1}$$

In [1–3] position, speed, and acceleration are linked according to the *leap-frog* formulas

$$\begin{aligned} v_{i,1/2} &= v_{i,0} + \frac{\Delta t}{2} a_{i,0}, \\ v_{i,k+1/2} &= v_{i,k-1/2} + \Delta t a_{i,k}, \\ y_{i,k+1} &= y_{i,k} + \Delta t v_{i,k+1/2}. \end{aligned} \tag{2}$$

Once a proper structure for $F_{i,k}$ is given, then the motion of each particle will be determined recursively by (1) and (2).

In the case of the string, we consider as internal forces the tension only [1,2], in the case of the rod the tension and the bending moment, as external force the gravity [3].

In the linear case in [1], it was assumed the stress-strain law to be linear (Hooke's law)

$$F_{i,k} = T^+_{i,k} - T^-_{i,k} = KL\frac{\sqrt{\Delta x^2 + (y_{i+1,k} - y_{i,k})^2}}{\Delta x} - KL\frac{\sqrt{\Delta x^2 + (y_{i,k} - y_{i-1,k})^2}}{\Delta x}. \tag{3}$$

For the bending moment it was assumed that this vary linearly with the angular deformation (Euler's law). In order to simulate, in a one-dimensional model, the bending moment of the rod, in [3] the following bending force was defined:

$$F_{i,k} = F^-_{i,k} - F^{+-}_{i,k} + F^+_{i,k} - F^{-+}_{i,k}, \tag{4}$$

where

$$\begin{aligned} F^{-+}_{i,k} &= \frac{LC}{\Delta x}\frac{\theta_{i+1,k} - \theta_{i,k}}{\sqrt{\Delta x^2 + (y_{i+1,k} - y_{i,k})^2}}, \\ F^{+-}_{i,k} &= \frac{LC}{\Delta x}\frac{\theta_{i-1,k} - \theta_{i-2,k}}{\sqrt{\Delta x^2 + (y_{i,k} - y_{i-1,k})^2}}, \\ F^{+}_{i,k} &= \frac{LC}{\Delta x}\frac{\theta_{i,k} - \theta_{i-1,k}}{\sqrt{\Delta x^2 + (y_{i+1,k} - y_{i,k})^2}}, \\ F^{-}_{i,k} &= \frac{LC}{\Delta x}\frac{\theta_{i,k} - \theta_{i-1,k}}{\sqrt{\Delta x^2 + (y_{i,k} - y_{i-1,k})^2}}. \end{aligned} \tag{5}$$

For the vibrating string, from (1)–(3) we obtain, in the linear case (see [1]),

$$\frac{y_{i,k+1} - 2y_{i,k} + y_{i,k-1}}{\Delta t^2} = \frac{T_0 L}{M\Delta x^2}\left[y_{i+1,k} - 2y_{i,k} + y_{i-1,k}\right] - g, \tag{6}$$

where $L = n\Delta x$ and $M = nm$.

For the vibrating rod, if $T_0 = 0$, from (1), (2), and (4) we obtain, in the linear case (that is, assuming $1 + ((y_{i,k} - y_{i,k-1})/\Delta x)^2 \simeq 1$), (see [3])

$$\frac{y_{i,k+1} - 2y_{i,k} + y_{i,k-1}}{\Delta t^2} = -\frac{L^2 C}{M\Delta x^4}\left[y_{i+2,k} - 4y_{i+1,k} + 6y_{i,k} - 4y_{i-1,k} + y_{i-2,k}\right] - g, \tag{7}$$

while, if $T_0 \neq 0$, from (1)–(5), we obtain [3]

$$\frac{1}{\Delta t^2}\left[y_{i,k+1} - 2y_{i,k} + y_{i,k-1}\right] = \frac{LT_0}{M\Delta x^2}\left[y_{i+1,k} - 2y_{i,k} + y_{i-1,k}\right] - \frac{L^2C}{M\Delta x^4}\left[y_{i+2,k} - 4y_{i+1,k} + 6y_{i,k} - 4y_{i-1,k} + y_{i-2,k}\right] - g, \quad (8)$$

where $C$ is a constant linking the bending action with the deformation of the rod.

The discrete schemes (6)–(8) are second-order finite difference schemes for the following classical differential equations:

$$\frac{\partial^2 y}{\partial t^2} = \frac{LT_0}{M}\frac{\partial^2 y}{\partial x^2} - g, \quad (9)$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{L^2C}{M}\frac{\partial^4 y}{\partial x^4} - g, \quad (10)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{LT_0}{M}\frac{\partial^2 y}{\partial x^2} - \frac{L^2C}{M}\frac{\partial^4 y}{\partial x^4} - g. \quad (11)$$

It is easy to verify that the previous discrete schemes are not conservative.

## 3. A CONSERVATIVE SCHEME FOR THE STRING

We now remark, that in all conservative schemes considered by Greenspan (see, f.i., [4,5]) the forces $F_{i,k}$, acting on each particle $P_i$ at time $t_k$ of the discrete model, depend on the positions of the particles $P_{i-1}$, $P_i$, and $P_{i+1}$ at times $t_k$ and $t_{k+1}$, while all the forces considered in the discrete scheme in Section 2 depend on the positions at time $t_k$ only.

We now redefine the tension force in (4) as

$$F_{i,k} = \frac{1}{2}\left(T^+_{i,k} + T^+_{i,k+1}\right) - \frac{1}{2}\left(T^-_{i,k} + T^-_{i,k+1}\right), \quad (12)$$

(which reduces to (3) if we use a clock with arbitrarily small time step) and we assume that position, speed, and acceleration of particle $P_i$ are linked (instead of (2)) by the formulas

$$\begin{aligned} \frac{1}{2}\left(v_{i,k+1} + v_{i,k}\right) &= \frac{1}{\Delta t}\left(y_{i,k+1} - y_{i,k}\right), \\ \frac{1}{\Delta t}\left(v_{i,k+1} - v_{i,k}\right) &= a_{i,k}. \end{aligned} \quad (13)$$

From (12), (13), and (1) we obtain, in the linear case, the following discrete scheme for $i = 1, \dots, n-1$:

$$\begin{aligned} \frac{T_0}{2\Delta x}\left[\left(y_{i+1,k+1} + y_{i+1,k}\right) - 2\left(y_{i,k+1} + y_{i,k}\right) + \left(y_{i-1,k+1} + y_{i-1,k}\right)\right] &= \frac{m}{\Delta t}\left(v_{i,k+1} - v_{i,k}\right), \\ \frac{1}{2}\left(v_{i,k+1} + v_{i,k}\right) &= \frac{1}{\Delta t}\left(y_{i,k+1} - y_{i,k}\right). \end{aligned} \quad (14)$$

### 3.1. Energy Conservation Theorem

Let

$$K_{i,k} = \frac{1}{2} m v_{i,k}^2 \quad (15)$$

be the kinetic energy of $P_i$ at time $t_k$, and

$$K_k = \frac{1}{2} m \sum_{i=0}^{n} v_{i,k}^2 \quad (16)$$

be the kinetic energy of the vibrating discrete string at time $t_k$, and let $W_m$ be the work done by the particle $P_i$ from the time $t_0$ to the time $t_m$. By using (12) and the techniques developed by Greenspan in [4,5], we can prove that

$$W_m = K_m - K_0. \tag{17}$$

Indeed, if $W_{i,m}$ is the work done by the particle $P_i$ from time $t_0$ to time $t_m$, we have

$$\begin{aligned} W_{i,m} &= \sum_{k=0}^{m-1} m a_{i,k} \left(y_{i,k+1} - y_{i,k}\right) \\ &= m \sum_{k=0}^{m-1} \frac{v_{i,k+1} - v_{i,k}}{\Delta t} \left(y_{i,k+1} - y_{i,k}\right) \\ &= m \sum_{k=0}^{m-1} \left(v_{i,k+1} - v_{i,k}\right) \frac{v_{i,k+1} + v_{i,k}}{2} \\ &= \frac{m}{2} \sum_{k=0}^{m-1} \left(v_{i,k+1}^2 - v_{i,k}^2\right) \\ &= \frac{m}{2} \left(v_{i,m}^2 - v_{i,0}^2\right). \end{aligned} \tag{18}$$

From (15) and (18), we have $W_{i,m} = K_{i,m} - K_{i,0}$, and by (16) defining $W_m = \sum_{i=0}^{n} W_{i,m}$, (17) follows.

If we now define the potential energy of particle $P_i$ at time $t_k$ as

$$\begin{aligned} V_{i,k} &= -\frac{T_0}{2\Delta x} y_{i,k} \left(y_{i+1,k} - 2y_{i,k} + y_{i-1,k}\right) \\ &= \frac{T_0}{2\Delta x} \left[y_{i,k} \left(y_{i,k} - y_{i+1,k}\right) + y_{i,k} \left(y_{i,k} - y_{i-1,k}\right)\right] \end{aligned} \tag{19}$$

and $V_k = \sum_{i=0}^{n} V_{i,k}$, the potential energy of the discrete string at time $t_k$, then

$$\begin{aligned} W_{i,m} &= \sum_{k=0}^{m-1} F_{i,k} \left(y_{i,k+1} - y_{i,k}\right) \\ &= \frac{T_0}{2\Delta x} \sum_{k=0}^{m-1} \left[\left(y_{i+1,k+1} + y_{i+1,k}\right) - 2\left(y_{i,k+1} + y_{i,k}\right)\right. \\ &\quad \left. + \left(y_{i-1,k+1} + y_{i-1,k}\right)\right] \left(y_{i,k+1} - y_{i,k}\right) \end{aligned} \tag{20}$$

and

$$W_m = V_0 - V_m. \tag{21}$$

From (17) and (21) it follows $K_m - K_0 = V_0 - V_m$, that is,

$$K_m + V_m = K_0 + V_0, \tag{22}$$

and the following conservation theorem is proved.

THEOREM 3.1. *The discrete model (14), with the potential energy given by (19) and the tension force given by (12) is conservative.*

REMARK 3.1. We remark that, defining $\Delta x T_0/m = V^2$ and neglecting the external force due to gravity ($g = 0$), it is easy to prove that (14) is a second-order finite difference scheme for the differential system

$$\begin{aligned} \frac{dy}{dt} &= v, \\ \frac{dv}{dt} &= V^2 \frac{\partial^2 y}{\partial x^2}. \end{aligned} \tag{23}$$

Indeed from (23), we have

$$\begin{aligned} y\,(x,t+\Delta\,t) &= y(x,t) + \int_t^{t+\Delta\,t} v(x,\tau)\,d\tau, \\ v\,(x,t+\Delta\,t) &= v(x,t) + V^2 \int_t^{t+\Delta\,t} \frac{\partial^2 y(x,\tau)}{\partial x^2}\,d\tau, \end{aligned} \tag{24}$$

and to discretize with (14) is equivalent to using the trapezoidal rule to approximate the integrals in (24), where in the second integral the second-order approximation $\partial^2 y(x,\tau)/\partial x^2 = (y(x-\Delta\,x,\tau) - 2y(x,\tau) + y(x+\Delta\,x,\tau))/\Delta\,x^2$ has been used.

As (23) is equivalent to the D' Alembert equation

$$\frac{\partial^2 y}{\partial t^2} = V^2 \frac{\partial^2 y}{\partial x^2}, \tag{25}$$

the scheme (14) is a second-order finite difference method for the D' Alembert equation (24) which solution, for every choice of $\Delta\,t$ and $\Delta\,x$, verifies the energy conservation law.

REMARK 3.2. It is obvious that if we consider also the external force due to gravity, being this a conservative one, the conservation energy theorem is still verified by the discrete model (14).

### 3.2. Stability Theorem

In order to study the stability of the difference system (14) we use the following lemma.

LEMMA 3.1. *Let $C_n$ be a symmetric definite positive matrix of order $n$, and $A$ and $B$ two matrices of order $2n$ defined by*

$$A = \begin{bmatrix} -C_n & -\alpha I_n \\ -\beta I_n & \gamma I_n \end{bmatrix}, \qquad B = \begin{bmatrix} C_n & -\alpha I_n \\ -\beta I_n & -\gamma I_n \end{bmatrix}, \tag{26}$$

*with $\alpha, \beta \in R^+$, $\gamma$ nonnegative, and $I_n$ the identity matrix of order $n$, then the eigenvalues of the pencil problem*

$$A\underline{x} = \lambda B\underline{x} \tag{27}$$

*are all complex and on the unit circle.*

PROOF. Let $\underline{x} = [\underline{x}_1, \underline{x}_2]^\top$ with $\underline{x}_1$, $\underline{x}_2$ vectors of order $n$. From (27) because of (26), we have

$$\begin{aligned} -(1+\lambda)\,C_n\underline{x}_1 - \alpha(1-\lambda)\underline{x}_2 &= 0, \\ -\beta(1-\lambda)\underline{x}_1 + \gamma(1+\lambda)\underline{x}_2 &= 0. \end{aligned} \tag{28}$$

The determinant of the matrix of the homogeneous linear system (28) must vanish and this implies, by using standard results of linear algebra [6]

$$\det\left(-\gamma(1+\lambda)^2 C_n - \alpha\beta(1-\lambda)^2 I_n\right) = 0. \tag{29}$$

Because $C_n = Q^\top \Omega Q$ with $Q$ orthogonal and $\Omega = \mathrm{diag}(\omega_1, \omega_2, \dots, \omega_n)$, $\omega_i > 0\ \forall i$, from (29) we have

$$\begin{aligned} &\det\left(-\gamma(1+\lambda)^2 Q^\top \Omega Q - \alpha\beta(1-\lambda)^2 Q^\top Q\right) \\ &= \det\left[Q^\top\left(-\gamma(1+\lambda)^2\Omega - \alpha\beta(1-\lambda)^2 I_n\right)Q\right] \\ &= \det\left(-\gamma(1+\lambda)^2\Omega - \alpha\beta(1-\lambda)^2 I_n\right) \\ &= \prod_{i=1}^{n}\left(-\gamma(1+\lambda)^2\omega_i - \alpha\beta(1-\lambda)^2\right) \\ &= \prod_{i=1}^{n}\left(\lambda^2(-\gamma\omega_i - \alpha\beta) + 2\lambda(-\gamma\omega_i + \alpha\beta) + (-\gamma\omega_i - \alpha\beta)\right) = 0. \end{aligned} \tag{30}$$

This implies, by solving the quadratic equation in $\lambda$ and remembering the positivity of $\alpha$, $\beta$, $\gamma$, and $\omega_i$, that the $2n$ roots $\lambda_i$ of (29) are complex and $|\lambda_i| = 1$. ∎

REMARK 3.3. With the same technique used in the proof of Lemma 3.1 it may be proved that $\det(A) \neq 0$, and so the values $\lambda_i$ are the inverse of the eigenvalues of the matrix $A^{-1}B$.

We may now prove the following.

THEOREM 3.2. *The difference system (14) is unconditionally stable.*

PROOF. The difference system (14) may be written in the matrix form

$$A\underline{z}_{k+1} = B\underline{z}_k, \tag{31}$$

where

$$\underline{z}_k = [\underline{y}_k, \underline{v}_k]^{\mathrm{T}}, \tag{32}$$

and $A$ and $B$ are of the form (26) with

$$\alpha = \frac{m}{\Delta t}, \qquad \beta = \frac{1}{\Delta t}, \qquad \gamma = \frac{1}{2}, \tag{33}$$

and

$$\frac{2\Delta x}{T_0} C_n = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots \\ -1 & 2 & -1 & \cdots & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -1 & 2 \end{bmatrix}. \tag{34}$$

Because $C_n$ is $\forall n$ a symmetric positive definite matrix, the proof follows from Lemma 3.1. ∎

## 4. THE CONSERVATIVE SCHEME FOR THE ROD

In order to obtain a conservative scheme as a model for the linear rod, we now redefine the force $F_{i,k}$ in (4) as

$$F_{i,k} = \frac{1}{2}\left(F^{-}_{i,k+1} + F^{-}_{i,k}\right) - \frac{1}{2}\left(F^{+-}_{i,k+1} + F^{+-}_{i,k}\right) + \frac{1}{2}\left(F^{+}_{i,k+1} + F^{+}_{i,k}\right) - \frac{1}{2}\left(F^{-+}_{i,k+1} + F^{-+}_{i,k}\right) \tag{35}$$

(which reduces to (4) if we use a clock with arbitrarily small time step).

If we neglect the tension and we assume that position, velocity, and acceleration are linked by (13), from (35) and (1) for $i = 1, 2, \dots, n-1$, we obtain the following relations:

$$\begin{aligned} &\frac{LC}{2\Delta x^3}\left[\left(-y_{i+2,k+1} + 4y_{i+1,k+1} - 6y_{i,k+1} + 4y_{i-1,k+1} - y_{i-2,k+1}\right)\right. \\ &\quad \left. + \left(-y_{i+2,k} + 4y_{i+1,k} - 6y_{i,k} + 4y_{i-1,k} - y_{i-2,k}\right)\right] \\ &= \frac{m}{\Delta t}\left(v_{i,k+1} - v_{i,k}\right), \\ \frac{1}{2}\left(v_{i,k+1} + v_{i,k}\right) &= \frac{1}{\Delta t}\left(y_{i,k+1} - y_{i,k}\right). \end{aligned} \tag{36}$$

In (36) we assume $y_{-1} = y_0 = y_n = y_{n+1} = 0$.

### 4.1. Energy Conservation Theorem

As done in the case of the string, we have that the work done by the kinetic energy of the rod from time $t_0$ to time $t_m$ is

$$W_m = K_m - K_0. \tag{37}$$

Now we define the potential energy of particle $P_i$, $i = 1, \dots, n-1$ at time $t_k$ as

$$V_{i,k} = \frac{LC}{2\Delta x^3}\left(y_{i+2,k}y_{i,k} - 4y_{i+1,k}y_{i,k} + 6y^2_{i,k} - 4y_{i-1,k}y_{i,k} + y_{i-2,k}y_{i,k}\right). \tag{38}$$

Then

$$W_{i,m} = \sum_{k=0}^{m-1} F_{i,k}\left(y_{i,k+1} - y_{i,k}\right) = \frac{LC}{2\Delta x^3} \sum_{k=0}^{m-1} \left[-\left(y_{i+2,k+1} + y_{i+2,k}\right)\right.$$
$$+ 4\left(y_{i+1,k+1} + y_{i+1,k}\right) - 6\left(y_{i,k+1} + y_{i,k}\right) + 4\left(y_{i-1,k+1} + y_{i-1,k}\right)$$
$$\left. - \left(y_{i-2,k+1} + y_{i-2,k}\right)\right]\left(y_{i,k+1} - y_{i,k}\right). \tag{39}$$

Being $V_k = \sum_{i=0}^{n-1} V_{i,k}$, we have

$$W_m = V_0 - V_m. \tag{40}$$

From (37) and (40) we have

$$K_m - K_0 = V_0 - V_m,$$

that is, $V_0 + K_0 = V_m + K_m$, and the following conservation theorem is proved.

THEOREM 4.1. *The discrete model (36), with the potential energy given by (38) is conservative.*

REMARK 4.1. We remark that, being $\epsilon = LC\Delta x/m$, it is easy to prove that (36) is a second-order difference scheme of the differential system

$$\frac{\partial y}{\partial t} = v,$$
$$\frac{\partial v}{\partial t} = -\epsilon \frac{\partial^4 y}{\partial x^4}. \tag{41}$$

Indeed from (41) we have

$$y\left(x, t + \Delta t\right) = y(x,t) + \int_t^{t+\Delta t} v(x,\tau)\, d\tau,$$
$$v\left(x, t + \Delta t\right) = v(x,t) - \epsilon \int_t^{t+\Delta t} \frac{\partial^4 y(x,\tau)}{\partial x^4}\, d\tau, \tag{42}$$

and to discretize with (36) is equivalent to using the trapezoidal rule to approximate the integrals in (42), where in the second integral the second-order approximation

$$\frac{\partial^4 y(x,\tau)}{\partial x^4} = \frac{y(x - 2\Delta x, \tau) - 4y\left(x - \Delta x, \tau\right) + 6y(x,\tau) - 4y\left(x + \Delta x, \tau\right) + y\left(x + 2\Delta x, \tau\right)}{\Delta x^4}$$

has been used.

As (41) is equivalent to the classical vibrating rod equation

$$\frac{\partial^2 y}{\partial t^2} = -\epsilon \frac{\partial^4 y}{\partial x^4}, \tag{43}$$

the scheme (36) is a second-order finite difference method for the equation (43) whose solutions, for every choice of $\Delta t$ and $\Delta x$, verify the energy conservation law.

### 4.2. Stability Theorem

As done for the vibrating string we can now prove the following.

THEOREM 4.2. *The difference system (36) is unconditionally stable.*

PROOF. The difference system (36) may be written in the matrix form

$$A\underline{z}_{k+1} = B\underline{z}_k, \tag{44}$$

where $A$ and $B$ are of the form (26), $\underline{z}_k$ is defined as in (32), $\alpha$, $\beta$, $\gamma$ are defined by (33) and

$$\frac{2\Delta x^3}{LC} C_n = \begin{bmatrix} 6 & -4 & 1 & 0 & \cdots & \cdots & 0 \\ -4 & 6 & -4 & 1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & -4 & 6 & -4 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 1 & -4 & 6 & -4 \\ 0 & \cdots & \cdots & \cdots & 1 & -4 & 6 \end{bmatrix}. \tag{45}$$

Being $C_n$ related to the square of the matrix defined in (34), it is $\forall n$, a symmetric definite positive matrix, and the proof follows again from Lemma 3.1.

### 4.3. Hinged and Clamped Rod

In Section 4, we have arbitrarily assumed the fictitious particles $P_{-1}$ and $P_{n+1}$ to not be in motion. In order to study the two classical problems of a hinged rod and of a clamped one, we must modify in (36) the first and the last equation in the system formed by the first $n-1$ equations according to the boundary conditions.

HINGED ROD. The hinged rod may be simulated by assuming [7]

$$y_{-1,k} = -y_{1,k}, \qquad y_{n-1,k} = -y_{n+1,k}, \tag{46}$$

which allows us to substitute in (36) for $i=1$ and $i=n-1$ the following equations:

$$\begin{aligned}
&\frac{LC}{2\Delta x^3}\left[(-5y_{1,k+1}+4y_{2,k+1}-y_{3,k+1})+(-5y_{1,k}+4y_{2,k}-y_{3,k})\right] \\
&\quad = \frac{m}{\Delta t}\left(v_{1,k+1}-v_{1,k}\right), \\
&\frac{LC}{2\Delta x^3}\left[(-y_{n-3,k+1}+4y_{n-2,k+1}-5y_{n-1,k+1})+(-y_{n-3,k}+4y_{n-2,k}-5y_{n-1,k})\right] \\
&\quad = \frac{m}{\Delta t}\left(v_{n-1,k+1}-v_{n-1,k}\right).
\end{aligned} \tag{47}$$

CLAMPED ROD. The clamped rod may be simulated by assuming [7] that

$$y_{-1,k} = y_{1,k}, \qquad y_{n-1,k} = y_{n+1,k}, \tag{48}$$

which allows us to substitute in (36) for $i=1$ and $i=n-1$ the following equations:

$$\begin{aligned}
&\frac{LC}{2\Delta x^3}\left[(-7y_{1,k+1}+4y_{2,k+1}-y_{3,k+1})+(-7y_{1,k}+4y_{2,k}-y_{3,k})\right] \\
&\quad = \frac{m}{\Delta t}\left(v_{1,k+1}-v_{1,k}\right), \\
&\frac{LC}{2\Delta x^3}\left[(-y_{n-3,k+1}+4y_{n-2,k+1}-7y_{n-1,k+1})+(-y_{n-3,k}+4y_{n-2,k}-7y_{n-1,k})\right] \\
&\quad = \frac{m}{\Delta t}\left(v_{n-1,k+1}-v_{n-1,k}\right).
\end{aligned} \tag{49}$$

After some calculations, it is easy to see that the new models obtained by modifying (36) according to (47) and (49), respectively, are still conservative when we define the potential energy as

$$V_k^h = V_k - \frac{LC}{2\Delta x^3}\left(y_{1,k}^2 + y_{n-1,k}^2\right) \tag{50}$$

for the hinged rod, and

$$V_k^c = V_k + \frac{LC}{2\Delta x^3}\left(y_{1,k}^2 + y_{n-1,k}^2\right) \tag{51}$$

for the clamped rod.

Also, for the stability conditions it is easy to show, in virtue of Lemma 3.1, that the schemes are still unconditionally stable. Indeed, the new matrices differ from the one defined in (45) only in position $(1,1)$ and $(n,n)$, where we have instead of a 6, a 5, or a 7, respectively. Now we remark that, if we consider a rod in which we take into account the tension force (defined as in (3) and (12)) and the bending force (defined as in (4), (5), and (35)), as internal forces, and gravity, as an external force, we still have a conservative scheme whose finite difference system is unconditionally stable. This system is a second-order finite difference scheme for the equation

$$\frac{\partial^2 y}{\partial t^2} = V^2\frac{\partial^2 y}{\partial x^2} - \epsilon\frac{\partial^4 y}{\partial x^4} - g. \tag{52}$$

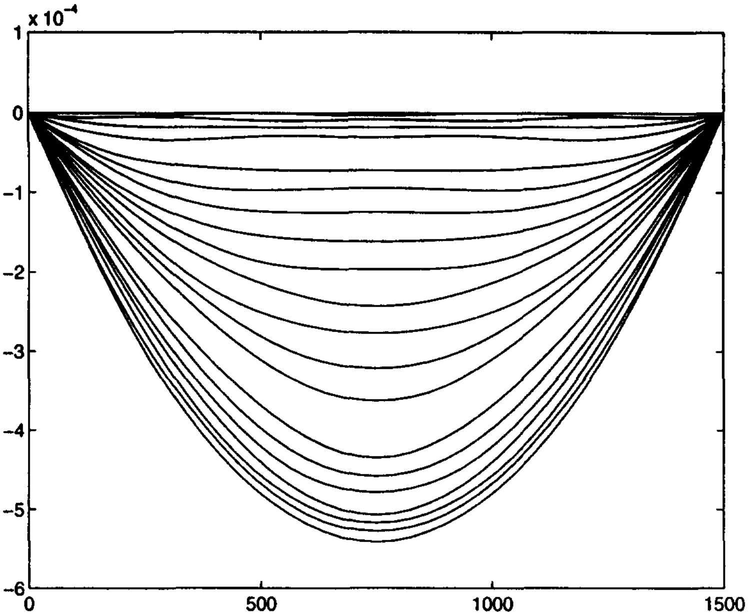


Figure 1. Position.

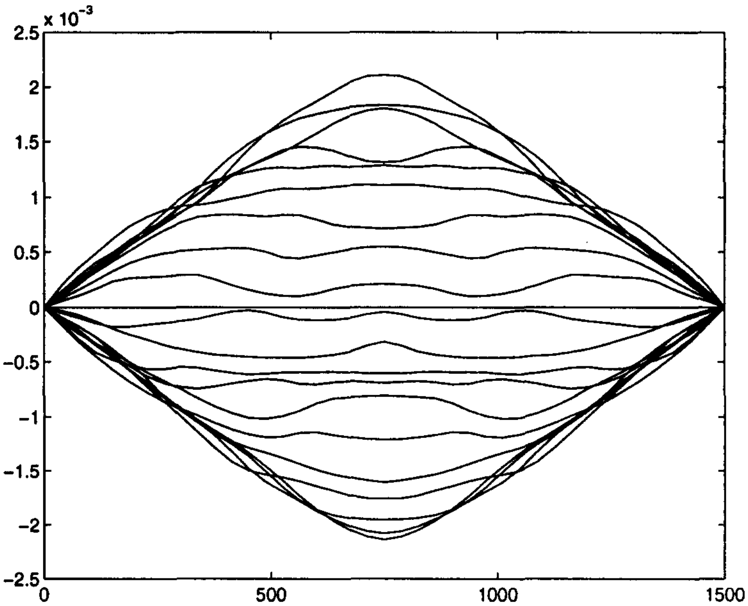


Figure 2. Velocity.

Table 1. Position.

| $P_i$ | $\Delta t = 1$ | $\Delta t = 0.1$ | $\Delta t = 0.01$ | $\Delta t = 0.001$ | $\Delta t = 0.0005$ |
|---|---|---|---|---|---|
| 2 | −3.7752e −05 | −6.1221− e06 | −2.7002− e05 | −2.7624− e05 | −2.7666− e05 |
| 11 | −3.3443e −04 | −4.4121− e05 | −2.1496 −e04 | −2.1333− e04 | −2.1332− e04 |
| 21 | −4.6724e −04 | −6.2382− e05 | −2.7698− e04 | −2.8173− e04 | −2.8177− e04 |
| 31 | −3.3443e −04 | −4.4121− e05 | −2.1496− e04 | −2.1333− e04 | −2.1332− e04 |
| 40 | −3.7752e −05 | −6.1221− e06 | −2.7002− e05 | −2.7624− e05 | −2.7666− e05 |

## 5. NUMERICAL RESULTS

We now give some numerical results obtained by considering a vibrating string and a vibrating rod. From these it may be seen that, for different choices of the time step $\Delta t$, the energy is

Table 2. Velocity.

| $P_i$ | $\Delta t = 1$ | $\Delta t = 0.1$ | $\Delta t = 0.01$ | $\Delta t = 0.001$ | $\Delta t = 0.0005$ |
|---|---|---|---|---|---|
| 2 | 1.7344e −04 | −8.3407e −05 | −8.7885e −05 | −1.4350e −04 | −1.4164e −04 |
| 11 | 1.0386e −03 | −8.1030e −04 | −1.1739e −03 | −1.1204e −03 | −1.1194e −03 |
| 21 | 1.2385e −03 | −1.2254e −03 | −2.0771e −03 | −2.0977e −03 | −2.1024e −03 |
| 31 | 1.0386e −03 | −8.1030e −04 | −1.1739e −03 | −1.1204e −03 | −1.1194e −03 |
| 40 | 1.7344e −04 | −8.3407e −05 | −8.7885e −05 | −1.4350e −04 | −1.4164e −04 |

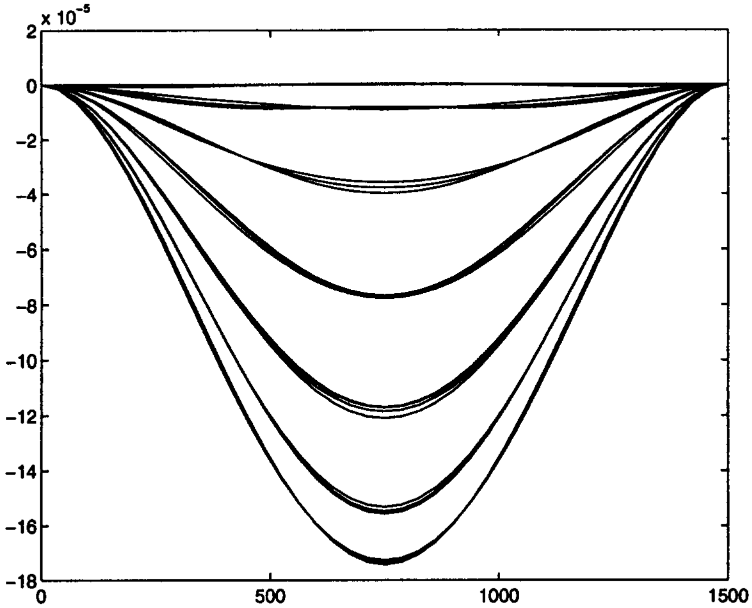


Figure 3. Position.

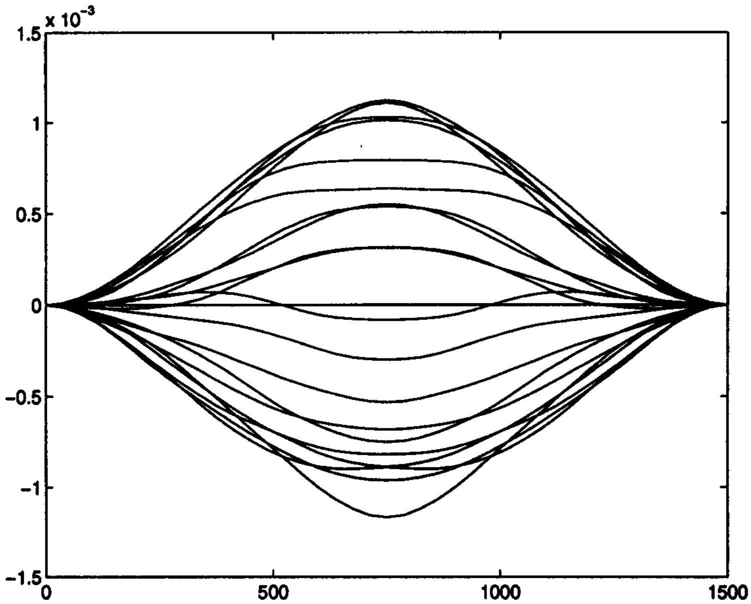


Figure 4. Velocity.

always the same at every time $t_k$, while the position and the velocity at time $t_k$ depend on the choice of $\Delta t$.

**EXAMPLE 1.** We have considered a harmonic steel string with $T_0 = 62.5\,\text{N}$, $L = 1.5\,\text{m}$, circular section with diameter $1\,\text{mm}$ and density $7.8\,\text{Kg/dm}^3$. We have taken $n = 41$, and so we have

Table 3. Position.

| $P_i$ | $\Delta t = 1$ | $\Delta t = 0.1$ | $\Delta t = 0.01$ | $\Delta t = 0.001$ | $\Delta t = 0.0005$ |
|---|---|---|---|---|---|
| 2 | −5.7947e −07 | −1.7415e −07 | −1.1197e −06 | −1.1544e −06 | −1.1510e −06 |
| 11 | −3.5887e −05 | −6.3006e −06 | −6.6968e −05 | −6.5555e −05 | −6.5667e −05 |
| 21 | −6.5848e −05 | −9.8317e −06 | −1.2303e −04 | −1.1693e −04 | −1.1672e −04 |
| 31 | −3.5887e −05 | −6.3006e −06 | −6.6968e −05 | −6.5555e −05 | −6.5667e −05 |
| 40 | −5.7947e −07 | −1.7415e −07 | −1.1197e −06 | −1.1544e −06 | −1.1510e −06 |

Table 4. Velocity.

| $P_i$ | $\Delta t = 1$ | $\Delta t = 0.1$ | $\Delta t = 0.01$ | $\Delta t = 0.001$ | $\Delta t = 0.0005$ |
|---|---|---|---|---|---|
| 2 | 1.0559e −05 | 1.6574e −06 | 5.2761e −06 | 4.9634e −06 | 6.2362e −06 |
| 11 | 5.8056e −04 | 2.1255e −04 | 5.8064e −04 | 4.7244e −04 | 4.7578e −04 |
| 21 | 1.0179e −03 | 5.6661e −04 | 9.4202e −04 | 1.1095e −03 | 1.1094e −03 |
| 31 | 5.8056e −04 | 2.1255e −04 | 5.8064e −04 | 4.7244e −04 | 4.7578e −04 |
| 40 | 1.0559e −05 | 1.6574e −06 | 5.2761e −06 | 4.9634e −06 | 6.2362e −06 |

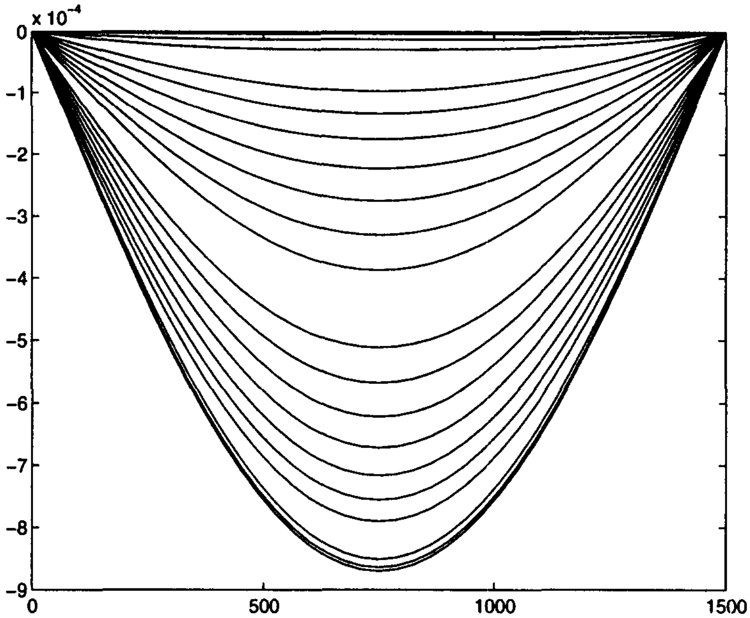


Figure 5. Position.

supposed the mass $m$ of $P_i$ to be $m \simeq 0.023\,\text{g}$. We have assumed as external forces only the gravity and the string to be at time $t = 0$, in a horizontal position at 1 m from the floor with null velocity.

Figures 1 and 2 show the positions and the velocities in the first 4 s for times $t_k = k \times 0.2\,\text{s}$ with $k = 0, 1, \dots, 20$. In Tables 1 and 2, we can see the positions and the velocities at time $t = 4\,\text{s}$ of particles $P_2$, $P_{11}$, $P_{21}$, $P_{31}$, $P_{40}$ obtained with $\Delta t = 1\,\text{s}$, $\Delta t = 0.1\,\text{s}$, $\Delta t = 0.01\,\text{s}$, $\Delta t = 0.001\,\text{s}$, and $\Delta t = 0.0005\,\text{s}$. For all $\Delta t$ and for every $t_k$, the computed total energy is always $9.015\text{e}-2\,\text{Kgm}^2/\text{s}^2$.

EXAMPLE 2. We have considered a clamped rod of the same material of the string with $T_0 = 0$ (no tension), $L = 1.5\,\text{m}$, circular section with diameter 30 mm and density $7.8\,\text{Kg/dm}^3$. We have taken $n = 41$, and so we have supposed the total mass $M$ to be $M \simeq 8.27\,\text{Kg}$. We have assumed as external forces only the gravity and the rod to be at time $t = 0$, in a horizontal position at 1 m from the floor with null velocity.

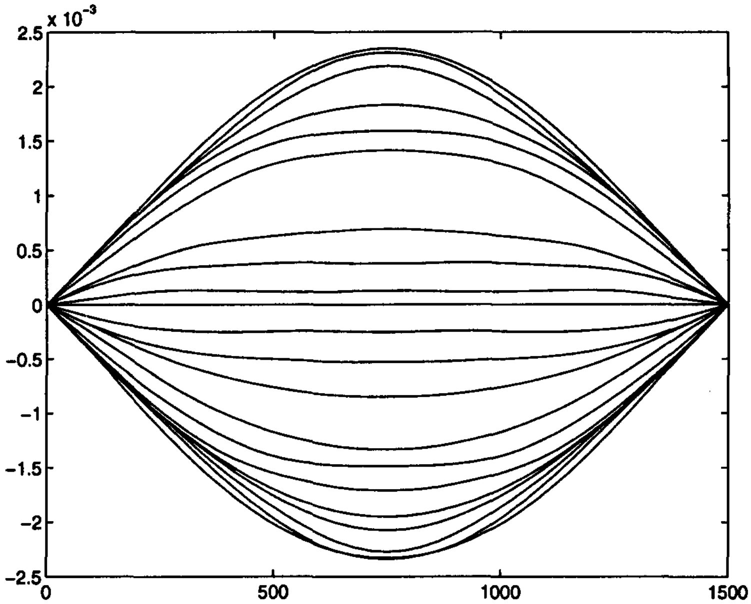


Figure 6. Velocity.

Table 5. Position.

| $P_i$ | $\Delta t = 1$ | $\Delta t = 0.1$ | $\Delta t = 0.01$ | $\Delta t = 0.001$ | $\Delta t = 0.0005$ |
|---|---|---|---|---|---|
| 2 | −6.7324e −05 | −4.9764e −05 | −6.2324e −05 | −6.3175e −05 | −6.3174e −05 |
| 11 | −6.0661e −04 | −4.4822e −04 | −5.6048e −04 | −5.6390e −04 | −5.6392e −04 |
| 21 | −8.5768e −04 | −6.3382e −04 | −7.9255e −04 | −7.9033e −04 | −7.9029e −04 |
| 31 | −6.0661e −04 | −4.4822e −04 | −5.6048e −04 | −5.6390e −04 | −5.6392e −04 |
| 40 | −6.7324e −05 | −4.9764e −05 | −6.2324e −05 | −6.3175e −05 | −6.3174e −05 |

Table 6. Velocity.

| $P_i$ | $\Delta t = 1$ | $\Delta t = 0.1$ | $\Delta t = 0.01$ | $\Delta t = 0.001$ | $\Delta t = 0.0005$ |
|---|---|---|---|---|---|
| 2 | 5.7907e −05 | −1.6089e −04 | −1.0477e −04 | −9.6037e −05 | −1.0651e −04 |
| 11 | 4.6927e −04 | −1.4564e −03 | −9.1843e −04 | −9.7206e −04 | −9.5839e −04 |
| 21 | 6.1140e −04 | −2.1179e −03 | −1.4101e −03 | −1.3331e −03 | −1.3391e −03 |
| 31 | 4.6927e −04 | −1.4564e −03 | −9.1843e −04 | −9.7206e −04 | −9.5839e −04 |
| 40 | 5.7907e −05 | −1.6089e −04 | −1.0477e −04 | −9.6037e −05 | −1.0651e −04 |

Figures 3 and 4 show the positions and the velocities in the first 4 s for times $t_k = k \times 0.2\,\mathrm{s}$ with $k = 0, 1, \dots, 20$. In Tables 3 and 4, we can see the positions and the velocities at time $t = 4\,\mathrm{s}$, of particles $P_2$, $P_{11}$, $P_{21}$, $P_{31}$, $P_{40}$ obtained with $\Delta t = 1\,\mathrm{s}$, $\Delta t = 0.1\,\mathrm{s}$, $\Delta t = 0.01\,\mathrm{s}$, $\Delta t = 0.001\,\mathrm{s}$, and $\Delta t = 0.0005\,\mathrm{s}$. For all $\Delta t$ and for every $t_k$, the computed total energy is always $81.13\,\mathrm{Kgm}^2/\mathrm{s}^2$.

EXAMPLE 3. With the same assumptions of Example 2, we have considered a hinged rod. Figures 5 and 6 show again the positions and the velocities in the first 4 s for times $t_k = k \times 0.2\,\mathrm{s}$ with $k = 0, 1, \dots, 20$, observe the difference in the boundary conditions (see Figures 3 and 4). In Tables 5 and 6, we can see the positions and the velocities at time $t = 4\,\mathrm{s}$ of particles $P_2$, $P_{11}$, $P_{21}$, $P_{31}$, $P_{40}$ obtained with $\Delta t = 1\,\mathrm{s}$, $\Delta t = 0.1\,\mathrm{s}$, $\Delta t = 0.01\,\mathrm{s}$, $\Delta t = 0.001\,\mathrm{s}$, and $\Delta t = 0.0005\,\mathrm{s}$. For all $\Delta t$ and for every $t_k$, the computed total energy is still always $81.13\,\mathrm{Kgm}^2/\mathrm{s}^2$.

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