CONFORMALLY INVARIANT METRICS AND UNIFORM STRUCTURES *). I

BY

I. S. GÁL

(Communicated by Prof. J. F. Koksma at the meeting of January 30, 1960)

1. Introduction

Here we shall show that on any Riemann surface \( X \) one can introduce two metrics \( d^i \) \((i = 1, 2)\) which are of considerable interest. First of all the definitions will be intrinsic and so these metrics are invariant under conformal mappings: If \( X \) and \( Y \) are realizations of the same Riemann surface and if \( f \) maps \( X \) conformally onto \( Y \) then the distance between any two points of \( X \) is the same as the distance between their images under \( f \) in \( Y \). Next, save for a few exceptional cases, the topology \( \mathcal{T}^i \) generated on the set \( X \) by the metric function \( d^i \) is identical with the topology \( \mathcal{T} \) of the manifold structure of \( X \). In other words, generally the metric \( d^i \) is compatible with the topology of the Riemann surface \( X \).

Finally, these metrics seem to stand up well under the crucial test of considering special cases: The first metric \( d^1 \) leads us back to Carathéodory’s prime ends and the second to the Riemannian metric introduced by Bergman. (See ref. 4 and 2, respectively).

The metrics \( d^1 \) will be defined by extending the notion of extremal length due to Ahlfors and Beurling. (See ref. 1, 14, 16 and 18). Here we shall need the notion of differential forms on Riemann surfaces, the modulus and exterior product of such forms and their integrals. However this paper can be read profitably without being acquainted with the theory of complex analytic manifolds or arbitrary Riemann surfaces. For several of the results concern schlichtartig Riemann surfaces and these, with the exception of the Riemann sphere, are conformally equivalent to domains of the finite plane. (See ref. 14 p. 155 and ref. 19 p. 224). Thus by restricting oneself to conformal types of connected open sets of the complex plane one obtains results which are somewhat more restricted but nevertheless interesting. In this specialized form the main result states that given any connected open set \( X \) in the complex plane it is possible to define two metric functions \( d^1 \) and \( d^2 \) on \( X \) such that if \( f \) maps \( X \) conformally onto the plane region \( f(X) \) then \( d^i(x_1, x_2) = d^i(f(x_1), f(x_2)) \) for any pair of points \( x_1, x_2 \in X \).

The only exceptional cases with respect to \( d^1 \) are the Riemann sphere,

*) This research was supported by the Office of Naval Research and the Office of Ordnance Research.
the finite plane and the unit disk. The sphere and the finite plane are exceptional also with respect to \( d^3 \) and these are the only exceptional surfaces. The definitions of the functions \( d^4 \) hold good in all exceptional cases but the metrics defined in this way are trivial in the sense that 
\[ d^3(x_1, x_2) = 0 \]
for every point pair \( x_1, x_2 \in X \). These surfaces will not be exceptional if we disregard the metrics themselves and concentrate only on the uniform structures generated by these metrics. Uniform structures will be used everywhere but save for Sections 6 and 9 it is sufficient to know the following: The uniform structure \( \mathcal{U} \) generated by the metric \( d \) on \( X \) is a family of subsets \( U \) of the product \( X \times X \): Precisely, we let 
\[ U \in \mathcal{U} \]
if and only if there is an \( \varepsilon > 0 \) such that \( (x_1, x_2) \in U \) for every pair \( x_1, x_2 \in X \) satisfying \( d(x_1, x_2) < \varepsilon \). The point is that two metrics \( d' \) and \( d'' \) generate the same uniform structure \( \mathcal{U} \) if and only if for every metric space \( Y \) the same functions \( f: X \to Y \) are uniformly continuous with respect to \( d' \) and \( d'' \).

All papers dealing with prime ends were directly or indirectly inspired by Osgood’s conjecture on the uniform continuity of conformal maps: (See ref. 13 p. 56).

If \( X \) and \( Y \) are simply connected open sets whose boundaries are Jordan curves and if \( f: X \to Y \) is a conformal map of \( X \) onto \( Y \) then \( f \) can be extended to a one-to-one continuous map between their closures \( \bar{X} \) and \( \bar{Y} \).

After proving this conjecture Carathéodory was lead to the following: Every simply connected domain \( X \) can be immersed as a dense subset in a compact topological space \( \bar{X} \) such that if \( X \) and \( Y \) are simply connected and \( f \) maps \( X \) conformally onto \( Y \) then \( f \) can be extended to a one-to-one continuous map between \( \bar{X} \) and \( \bar{Y} \). (See ref. 4). Carathéodory’s construction of \( \bar{X} \) is the first non-trivial compactification constructed by specifying the proper equivalence classes of Cauchy nets. (Or filters if we wish). The equivalence classes whose elements have no limit in \( X \), called prime ends, are the elements of the Carathéodory boundary \( \partial X \). We shall call \( \bar{X} \) the Carathéodory compactification of \( X \). If \( X \) is the disk \( [z: |z| < 1] \) then its compactification in the Carathéodory sense is \( [z: |z| \leq 1] \). Using the Riemann mapping theorem and Carathéodory’s main result on the extension of conformal maps to the boundaries we see that every \( \bar{X} \) is homeomorphic to the closed disk and every Carathéodory boundary is a circle.

One way of distinguishing between the topologically equivalent Carathéodory compactifications of various simply connected domains \( X \) is by describing how \( X \) is embedded in its compactification \( \bar{X} \). In other words the distinction is accomplished by the injection map \( f: X \to \bar{X} \) which maps \( x \in X \) into the same point but considered as a element of \( \bar{X} \). Two compactifications of an arbitrary topological space \( X \) are considered equivalent if and only if there is a homeomorphism of one onto the other which reduces to identity on \( X \). A great disadvantage of this method of distinction is that it involves the compactifications \( \bar{X} \).
An intrinsic method of distinction between the various compactifications \( \bar{X} \) can be obtained via a simple fact from the theory of uniform structures: If \( \bar{X} \) is a compact Hausdorff space then there exists exactly one uniform structure \( \mathcal{U} \) compatible with its topology. This means that there is a unique filter \( \mathcal{U} \) on \( \bar{X} \times \bar{X} \) such that for each \( \xi \in \bar{X} \) the neighborhoods of \( \xi \) are exactly the sets \( \bar{U}[\xi] = [\eta : (\xi, \eta) \in \bar{U}] \) where \( \bar{U} \) varies over \( \mathcal{U} \). Now let \( X \) be a dense subspace of \( \bar{X} \). The sets \( \bar{U} \cap X \times X \) form a uniform structure \( \mathcal{U} \) on \( X \) called the restriction of \( \mathcal{U} \) to \( X \). Although \( \mathcal{U} \) is always compatible with the topology of \( X \) it is not uniquely determined by \( \bar{X} \) but it depends on the way how \( X \) is embedded in \( \bar{X} \). It can be proved that two compactifications of \( X \) are equivalent if and only if their restricted structures are identical. (See the end of Section 6).

In order to give a simple and intrinsic description of the Carathéodory compactification \( X \) of a simply connected domain \( X \) it is thus sufficient to give a similar description of the uniform structure \( \mathcal{U} \) obtained by restricting the unique structure of \( \bar{X} \) to \( X \). This structure \( \mathcal{U} \) will be called the \textit{Carathéodory structure} of \( X \). We shall associate with each Riemann surface \( X \) a uniform structure \( \mathcal{V} \) compatible with its topology and such that if \( X \) is equivalent to the disk then \( \mathcal{V} \) is its Carathéodory structure. Therefore the completion of \( X \) with respect to \( \mathcal{V} \) is a natural generalization of the Carathéodory compactification. The elements of \( \bar{X} - X \) can be considered as the \textit{prime ends} of the Riemann surface \( X \). We shall see that if \( X \) is not simply connected then there is a conformally invariant metric on \( X \) which generates \( \mathcal{V} \). This is the metric \( d^1 \) discussed in the beginning.

I feel I should mention some of the important papers dealing with prime ends and compactifications which I found especially helpful while developing the theory which is presented here: The fact that the introduction of prime ends can be accomplished by completing the bounded simply connected plane domain \( X \) with respect to a suitable metric \( d \) was first formulated by MAZURKIEWICZ in [12]. The definition of the "prime end metric" \( d \) however involves Euclidean properties of \( X \) and is not invariant under conformal mappings. A conformally invariant definition of the prime ends of a bounded simply connected domain was first given by E. SCHLESINGER in [18]. Let \( X \) be an arbitrary Riemann surface and \( x_1, x_2, \ldots \) be a sequence of points of \( X \). A.HLIFORS gave a conformally invariant definition stating which sequences \( x_1, x_2, \ldots \) should define prime ends of \( X \) and which of these sequences should be considered equivalent. As far as I know these ideas were published only in a survey by ROYDEN. (See ref. 17). In [18] Schlesinger proves that for bounded simply connected domains the conformally invariant definitions and Carathéodory's original definitions are indeed equivalent. Although this is a valuable information it is not sufficient to conclude that conformal maps of such domains can be extended to the boundaries. Prime end were introduced and generalized in many ways but no one since Carathéodory
and Terasaka (see ref. 20) proved the extension property, which after all is the reason for the introduction of prime ends. At present due to the conformally invariant definition of the Carathéodory structure the extension property is almost immediate. (See Section 6). For the sake of completeness I would like to mention a paper by Ursell and L. C. Young and a survey by Piranian. (See ref. 21 and 15, respectively). These deal mainly with the special problem of comparing the Euclidean and the Carathéodory boundaries of bounded simply connected plane domains. A summary of the results presented here was published in [7]. In the last section a few words are said about the Bergman kernel function and the associated Riemannian metric. I plan to write a paper dealing exclusively with the metric \(d^2\) and the Riemannian metric which it generates. This will give an opportunity for a more detailed discussion.

2. Extremal length on Riemann surfaces

The purpose of this section is to define the notion of extremal length on arbitrary Riemann surfaces and collect the basic results on extremal length which will be needed in the sequel. We start with the simple case when \(X\) is a region in the finite plane: Let \(\Gamma\) be a collection of rectifiable curves \(\gamma\) in the region \(X\). For our purposes it is sufficient to consider simple closed curves and arcs. It can also be assumed that \(\gamma\) is piecewise smooth. We define what is meant by the extremal length \(\lambda = \lambda\{\Gamma\}\) of the family \(\Gamma\): Given a non-negative and square integrable function \(\sigma\) on \(X\) we let

\[
L(\sigma) = \text{glb} \int_{\gamma \in \Gamma} \sigma|dz|
\]

and

\[
A(\sigma) = \int_X \sigma^2 dxdy.
\]

Then letting \(\sigma\) vary over the family of these functions we define

\[
\lambda = \lambda\{\Gamma\} = \text{ lub} \frac{L(\sigma)^2}{A(\sigma)}.
\]

One can easily show that \(\lambda\) is invariant in the following sense: If \(Y\) is a region in the finite plane and if \(f\) maps \(X\) conformally onto \(Y\) then the extremal length \(\lambda\{f(\Gamma)\}\) of the family of image curves \(f(\gamma)\) is \(\lambda\{\Gamma\}\).

Let \(X\) be the rectangular region given by the inequalities \(a' < x < a''\) and \(-b < y < b\). Let \(\Gamma\) consist of the straight line segments connecting the points \((a', \eta)\) and \((a'', \eta)\) where \(-b < \eta < b\) or let \(\Gamma'\) be the family of those arcs \(\gamma\) whose end points are \((a', \eta)\) and \((a'', \eta)\) for some \(\eta\) in \((-b, b)\). A simple computation shows that in either case \(\lambda\{\Gamma'\} = (a'' - a')/2b\). (See for example ref. 9 p. 16–17 and ref. 16 p. 7–8).

Now we choose \(b = \pi\) and \(a' = \log r < a'' = \log R\) where \(0 < r < R < +\infty\). The map \(z \rightarrow e^z\) transforms this rectangle in the annulus formed by the circles \(|z| = r\) and \(|z| = R\) slit along \(r < z < R\). The first family \(\Gamma\) is transformed into the family of concentric circles with center \(z = 0\) and the
second into the family of simple closed curves separating \(|z|=r\) from \(|z|=R\). Using the conformal invariance of \(\lambda\) we obtain

**Lemma 2.1.** Let \(X\) be the annulus \(r<|z|<R\) and let \(\Gamma\) be either the family of concentric circles \(|z|=\varrho(r<\varrho<R)\) or the family of simple closed arcs separating the boundary components of \(X\). Then in both cases

\[
\lambda(\Gamma) = \frac{2\pi}{\log \frac{R}{r}}.
\]

**Note:** For the family of separating closed arcs the result is due to **AHLFORS** and **BEURLING**. (See ref. 1).

The extremal length of a family \(\Gamma\) of curves \(\gamma\) on a Riemann surface \(X\) will be defined by using differential forms instead of square integrable functions \(\varrho\). In what follows let \(\omega\) denote a first order form on \(X\) so that \(\omega\) associates with each \(x \in X\) a vector \(\omega(x)\) of the space of tangent vectors at \(x\). Locally \(\omega\) is representable by a form \(pdx+qdy\) where \(p\) and \(q\) are complex valued. We assume that \(\omega\) is pure so that \(*\omega+i\omega=0\) or locally \(pdx+qdy=pdz\). Then it is meaningful to speak about the integral of the modulus \(|\omega|\) along the curves of \(\Gamma\). (See for example ref. 19 p. 171 where the special case of pure holomorphic \(\omega\)'s is considered. This additional condition is superfluous. For the invariance of

\[
\frac{1}{0} |p(x, y)| (\dot{x}^2 + \dot{y}^2)^{1/2} \, dt
\]

under a change of coordinates \(u=u(x, y), v=v(x, y)\) is guaranteed by the Cauchy–Riemann equations of \(u\) and \(v\). We let

\[
L(\omega) = \text{glb} \int_{\gamma \in \Gamma} |\omega|
\]

and

\[
A(\omega) = \frac{i}{2} \int_{X} \omega \wedge \bar{\omega}
\]

where \(\bar{\omega}\) is the conjugate of \(\omega\) and \(\omega \wedge \bar{\omega}\) is the exterior product of \(\omega\) and \(\bar{\omega}\). We can now define the extremal length of the family \(\Gamma\) as

\[
\lambda = \lambda(\Gamma) = \text{lub} \frac{L(\omega)^2}{A(\omega)}
\]

where the least upper bound is taken with respect to all pure first order forms \(\omega\) satisfying \(A(\omega)<+\infty\). (Square integrable forms.)

The following lemma states an important monotonicity property of the extremal length:

**Lemma 2.2:** Let \(X\) be an open set of the Riemann surface \(Y\), let \(\Gamma_Y\) be a family of curves in \(Y\) and let \(\Gamma_X\) be a subfamily such that its elements belong entirely to \(X\). Then

\[
\lambda_X(\Gamma_X) \geq \lambda_Y(\Gamma_Y)
\]

where \(\lambda_X\) and \(\lambda_Y\) denote extremal lengths with respect to the surfaces \(X\) and \(Y\), respectively.
Proof: Given $\omega$ on $Y$ and $\varepsilon > 0$ we can find a $\gamma \in \Gamma_X$ such that
\[
\left( \int_{\gamma} |\omega| \right)^2 < (1 + \varepsilon) L_X(\omega)^2.
\]
Since $\lambda_X\{\Gamma_X\}$ is a least upper bound we see that
\[
\left( \int_{\gamma} |\omega| \right)^2 < (1 + \varepsilon) \lambda_X\{\Gamma_X\} A_X(\omega) < (1 + \varepsilon) \lambda_X\{\Gamma_X\} A_Y(\omega).
\]
Using $\Gamma_X \subseteq \Gamma_Y$ we obtain
\[
L_Y(\omega)^2 < (1 + \varepsilon) \lambda_X\{\Gamma_X\} A_Y(\omega)
\]
and so $\lambda_Y\{\Gamma_Y\} < (1 + \varepsilon) \lambda_X\{\Gamma_X\}$ where $\varepsilon > 0$ is arbitrary.

We shall also need the following simple

Lemma 2.3. Let $X$ and $Y$ be domains in the finite plane and let $\Gamma_Y$ be a family of curves in $Y$ such that every $\gamma_x = \gamma_y \cap X$ is a curve in $X$. If $\Gamma_X$ denotes the family of these curves $\gamma_x$ then $\lambda_X\{\Gamma_X\} \leq \lambda_Y\{\Gamma_Y\}$.

Proof: Given an admissible $\varrho$ on $X$ we extend it to an admissible function $\tilde{\varrho}$ on $X \cup Y$ by letting $\tilde{\varrho}(y) = 0$ in $Y - X$. By definition we have
\[
\text{glb} \left( \int_{\gamma_x \in \Gamma_X} |\tilde{\varrho}|^2 \right) < \lambda_Y\{\Gamma_Y\} A_Y(\tilde{\varrho}).
\]
Since $\Gamma_X$ contains the arcs $\gamma_y \cap X$ we obtain
\[
\text{glb} \left( \int_{\gamma_x \in \Gamma_X} |\varrho|^2 \right) < \lambda_Y\{\Gamma_Y\} A_Y(\varrho) = \lambda_Y\{\Gamma_Y\} A_X(\varrho).
\]
Therefore
\[
L_X(\varrho)^2 < (1 + \varepsilon) \lambda_Y\{\Gamma_Y\} A_X(\varrho)
\]
for every $\varrho$ on $X$ and so $\lambda_X\{\Gamma_X\} \leq \lambda_Y\{\Gamma_Y\}$.

If we combine the last two lemmas we see that the extremal length $\lambda\{\Gamma\}$ is uniquely determined by the support of $\Gamma$ and is independent of the domain containing $\Gamma$. This is a well known property of the extremal length and it can be easily established directly for arbitrary Riemann surfaces. In some instances the extremal length $\lambda$ can be obtained by varying $\omega$ over a proper subspace of the space of all square integrable first order differentials. (See ref. 16 pp. 8–9 and pp. 13–14). We can also modify the definition of extremal length by letting $\omega$ vary over an intrinsically defined proper subclass of the class of square integrable differentials. For instance we may require that $\omega$ be a holomorphic or harmonic differential.

3. Conformally invariant metrics

Using the notion of extremal length we shall here introduce on any Riemann surface $X$ two pseudo-metrics $d^i(i = 1, 2)$. The definitions will be intrinsic and so these pseudo-metrics will be invariant under conformal mappings. The topology $\mathcal{T}^i$ induced on $X$ by $d^i(i = 1, 2)$ will in general be the topology $\mathcal{T}$ of the manifold $X$. The exceptional cases are the following: If $X$ is conformally equivalent to the Riemann sphere or the
finite plane then $d^i(x, y) = 0$ for every $x, y \in X$ and so the topologies $T_i$ are anti-discrete. If $X$ is equivalent to the unit disk then $T_1$ is anti-discrete while $T_2$ is the topology of $X$. For every $X$ we have $T_1 < T_2 < T$.

We define the pseudo-metrics $d^i(i = 1, 2)$ as follows: Let $x, y \in X$ be fixed and consider simply connected domains $S$ in $X$ such that $x, y \in S$ and the boundary of $S$ is a Jordan curve or a Jordan arc $\gamma$. Let $\Gamma^1_{xy}$ denote the family of all possible boundary curves $\gamma$ and let $\Gamma^2_{xy}$ be the subfamily of Jordan curves. We shall prove that the extremal length $\Gamma^1_{xy}$ of the family $\Gamma^2_{xy}$ is always finite and so we may define $d^i(x, y) = (\lambda^i_{xy})^i$.

We start by proving the triangle inequality in the following restricted form:

**Lemma 3.1.** If $x_1, x_2, x_3 \in X$ and if both $d^i(x_1, x_2)$ and $d^i(x_2, x_3)$ are finite then $d^i(x_1, x_3)$ is finite and

$$d^i(x_1, x_3) < d^i(x_1, x_2) + d^i(x_2, x_3).$$

**Proof:** For the sake of simplicity let $d^i(x_1, x_2), ...$ be abbreviated by $d^i_1, ...$. Let $\varepsilon > 0$ and a form $\omega$ be given. By the definition of the extremal length there exist simply connected domains $S_r(r = 1, 3)$ with boundary curves $\gamma_r(r = 1, 3)$ such that

$$\int_{\gamma_r} |\omega| < (1 + \varepsilon) A(\omega)^{1/d^i}, \quad (r = 1, 3).$$

We may assume that $\gamma_1 \cap \gamma_3$ is a countable set of isolated points: In fact $\gamma_1 \cup \gamma_3$ can be covered by a locally finite system consisting of at most denumerably many coordinate neighborhoods $(N_k, \varrho_k)$ where $N_j \cap N_k = \emptyset$ for $j \neq k$ and the paths $\varrho_k(N_k \cap \gamma_r)$ can be replaced by suitable polygonal lines $\gamma_r'$ such that the integral of $|\omega|$ along $N_k \cap \gamma_r'$ is sufficiently close to its integral along $N_k \cap \gamma_r$. By (3.1) we have

$$\int_{\gamma_2 \cap \gamma_3} |\omega| < (1 + \varepsilon) A(\omega)^{1/d^i_1 + d^i_3}.$$  

We find now a simply connected domain $S$ containing both $x_1$ and $x_3$ and bounded by a curve $\gamma$ which is a subset of $\gamma_1 \cup \gamma_3$. The domain $S$ will then be modified so that the boundary $\gamma_2$ of the new domain $S_2$ will be a Jordan arc or a Jordan curve. In particular if both $\gamma_1$ and $\gamma_3$ are Jordan curves then so is $\gamma_2$. Hence for $i = 2$ the boundary $\gamma_2$ will always be a simple closed curve. If $x_1 \in S_1$ or if $x_3 \in S_3$ we can choose $S = S_1 = S_2$ or $S = S_3 = S_2$. If $x_1 \notin S_1$ and $x_3 \notin S_3$ then the points $x_1, x_2, x_3$ belong to distinct components of $S_1 \cup S_3 - \gamma_1 - \gamma_3$. These components $C_1, C_2, C_3$ are simply connected domains whose boundaries consist of Jordan arcs. The common boundary of $C_1$ and $C_2$ contains a Jordan arc whose possible end points are boundary points of $S_1 \cup S_2$. We join this open arc to the set $C_1 \cup C_2 \cup C_3$. By adding also an open arc of the common boundary of $C_2$ and $C_3$ we obtain a simply connected domain $S$ containing $x_1, x_2, x_3$ and bounded by a curve $\gamma$ contained in $\gamma_1 \cup \gamma_3$.

In order to modify $S$ we enclose each of the at most denumerably
many points \( p \in \gamma_1 \cap \gamma_3 \) into a small simple closed curve \( \gamma_p \) which cuts a Jordan arc off both \( \gamma_1 \) and \( \gamma_3 \). We replace each pair of these arcs by two arcs of \( \gamma_p \) and obtain a Jordan curve \( \gamma_2 \) whose interior \( S_2 \) contains \( x_1, x_2, x_3 \). By choosing sufficiently small curves \( \gamma_p \) the integral of \( |w| \) along \( \gamma_2 \) can be made arbitrarily close to its integral along \( \gamma \). Hence for the family \( \Gamma_{x_2} \) we have

\[
L(w) < (1 + \varepsilon)A(w) \frac{d_1^t + d_3^t}{d_2^t}
\]

and so \( d_2^t < d_1^t + d_3^t \). Our next object is to show

**Lemma 3.2.** If \( y \) belongs to a sufficiently small neighborhood of \( x \in X \) then \( \lambda_{xy}^t \) is finite.

**Proof:** We choose a coordinate neighborhood \( (O, \varphi) \) covering the given point \( x \). Since \( \varphi(x) \) is in the open plane set \( \varphi(O) \) we can choose \( R > 0 \) so that the disk with center \( \varphi(x) \) and radius \( R \) belongs to \( \varphi(O) \). Let \( \Gamma \) be the family of those simple closed curves \( \gamma \) in the plane which separate the two boundary components of the annulus \( A \) having center \( \varphi(x) \) and radii \( r < R \). If \( y \in X \) is chosen such that \( |\varphi(x) - \varphi(y)| < r \) then the inverse image of every \( \gamma \in \Gamma \) belongs to the family \( \Gamma_{x_2}^t \). Hence by Lemmas 2.1 and 2.2 we have

\[
\lambda_{xy}^t < \frac{2\pi}{\log \frac{R}{r}}
\]

Therefore if \( y \) belongs to the open neighborhood

\[
O_x = \{ y : |\varphi(x) - \varphi(y)| < r \}
\]

then \( \lambda_{xy}^t \) is finite.

Since \( r > 0 \) can be arbitrarily small the last inequality also shows that for every \( \varepsilon > 0 \) there is an open neighborhood \( O_x \) of \( x \) such that \( d^t(x, y) < \varepsilon \) for every \( y \in O_x \). Therefore we proved:

The topology induced on \( X \) by the pseudo-metric \( d^t \) is not stronger than the topology of \( X \).

If \( y = x \) then \( y \in O_x \) for every \( r > 0 \) and so \( d^t(x, x) = 0 \) for every \( x \in X \). It is now easy to show

**Lemma 3.4.** The distance \( d^t(x, y) \) is finite for every pair of points \( x, y \in X \).

**Proof:** Since \( X \) is arcwise connected we can choose a simple arc \( \gamma \) with end points \( x, y \in \gamma \). We associate with each \( p \in \gamma \) an open set \( O_p \) having the property that \( d^t(p, q) \) is finite for every \( q \in O_p \). Since \( \gamma \) is compact there are finitely many points \( p_1, \ldots, p_n \in \gamma \) such that the union of the corresponding \( O_p \)'s covers \( \gamma \). We may assume that \( p_1 = x \) and \( p_n = y \). Using the triangle inequality at most \( n \) times in succession we see that \( d^t(p_1, p_n) = d^t(x, y) \) is finite.

The symmetry of the functions \( d^t(i = 1, 2) \) is obvious from the definition
of the families \( I^n_{xy} \). Hence we proved that \( d^1 \) and \( d^2 \) are indeed pseudometrics on the set \( X \). Let \( f \) map the Riemann surface \( X \) conformally onto \( Y \) and let \( y_i = f(x_i) (i = 1, 2) \). Then \( f \) establishes a one-to-one correspondence between the simply connected domains \( S_{x, x} \) and \( S_{y, y} \) so that each \( \gamma_{x, x} \) is mapped onto a \( \gamma_{y, y} \) and conversely each \( \gamma_{y, y} \) is the image of some \( \gamma_{x, x} \). The extremal length being a conformal invariant it follows that \( l^1_{x, x} = l^2_{y, y} \) and so \( d^1(x_1, x_2) = d^2(y_1, y_2) \). In other words the pseudometrics \( d^1(i = 1, 2) \) are conformally invariant.

4. The compatibility of the metrics

The purpose of this section is to prove that if the Riemann surface \( X \) is multiply connected then the uniform topology \( \mathcal{F}^t \) generated on \( X \) by the pseudo-metric \( d^t(i = 1, 2) \) is the original topology \( \mathcal{T} \) of the manifold \( X \). First we shall prove this result in the special case when the Riemann surface \( X \) is conformally equivalent to a region on the Riemann sphere \( P_1 \). We shall interpret \( P_1 \) as the one point compactification of the finite plane. The topology of \( P_1 \) will also be denoted by \( \mathcal{T} \).

Let us first discuss the exceptional cases:

Lemma 4.1. If \( X \) is the Riemann sphere or if \( X \) is the finite plane then \( \mathcal{F}^1 \) and \( \mathcal{F}^2 \) are trivial i.e. \( d^1(p, q) = d^2(p, q) = 0 \) for every \( p, q \in X \).

Proof: If \( Y \) is the Riemann sphere then \( I_{pq}^1 = I_{pq}^2 \). The extremal length \( \lambda_{pq}^1 = \lambda_{pq}^2 \) can be estimated from above by using Lemma 2.2 and choosing \( X \) to be an annulus \( A \) such that both \( p \) and \( q \) belong to the unbounded component of \( P_1 - A \). We keep the outer radius \( R \) fixed and let the inner radius \( r \to 0 \). By Lemmas 2.1 and 2.2 we obtain \( \lambda_{pq}^1 = \lambda_{pq}^2 = 0 \). If \( Y \) is the finite plane \( \lambda_{pq}^2 \) can be estimated by using the same lemmas and choosing \( X \) to be an annulus \( A \) having center \( p \), a fixed inner radius \( r > |p - q| \) and outer radius \( R \to \infty \). Since \( \lambda_{pq}^1 < \lambda_{pq}^2 \) we obtain \( \lambda_{pq}^1 = \lambda_{pq}^2 = 0 \).

Lemma 4.2. If \( X \) is the unit disk then \( \mathcal{F}^1 \) is trivial.

Proof: We can again apply Lemmas 2.1 and 2.2: We let \( A \) be an annulus with center on the boundary of \( X \), let \( R \) be so small that both \( p \) and \( q \) are outside of \( A \) and let the inner radius \( r \to 0 \).

The pseudo-metrics \( d^1 \) and \( d^2 \) behave very differently. Later it will be proved for instance that if \( X \) is conformally equivalent to the unit disk then \( d^2 \) is compatible with the topology of \( X \) i.e. \( \mathcal{F}^2 = \mathcal{T} \). For the time being we shall study only the metric \( d^1 \). Let us first restrict ourselves to schlichtartig Riemann surfaces, so that \( X \) can be considered as a region of the Riemann sphere. For these we have:

Lemma 4.3: If \( X \) has a compact boundary component then \( \mathcal{F}^1 \) and \( \mathcal{F}^2 \) are proper metrics and \( \mathcal{F}^1 = \mathcal{F}^2 = \mathcal{T} \).

Proof: We have \( \mathcal{F}^1 < \mathcal{F}^2 < \mathcal{T} \) and so it is sufficient to prove that given \( p \in X \) and \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( d^1(p, q) < \delta \) only if
\[|p - q| < \varepsilon.\] Since \(d^1\) is conformally invariant we can restrict ourselves to the case \(p \neq \infty\). We choose a compact boundary component \(C\) and a bounded open set \(O\) in the finite plane containing \(C\) and the point \(p\). We may assume that \(O\) is simply connected and its boundary is a Jordan curve contained in \(X\). We define \(\omega(x) = 1\) for \(|x - p| < r\) where \(r\) is so large that \(O\) is covered by the disk \(|x - p| < r\). If \(|x - p| > r\) we let \(\omega(x) = r|x - p|^{-1}\).

Let \(\delta > 0\) be so small that \(A(\omega)^4\) is less than

(i) the Euclidean distance of \(C\) from the boundary of \(O\), and also less than

(ii) the length of any crosscut of \(O\) separating \(p\) from \(C\).

Now let \(|p - q| > \varepsilon\) and let \(S\) be a simply connected domain in \(X\) such that \(p, q \in S\). If the boundary \(\gamma\) of \(S\) intersects the boundary of \(O\) then \(\gamma \cap O\) contains a crosscut of \(O\) which separates \(p\) from \(C\) or \(\gamma \cap O\) contains an arc connecting \(cO\) with \(C\). In either case

\[\int_{\gamma} |\omega| = \int_{\gamma} |dz| > \delta A(\omega)^4.\]

If \(\gamma \subset O\) then \(\gamma\) is a simple closed curve and so using \(p, q \in S\) we obtain

\[\int_{\gamma} |\omega| = \int_{\gamma} |dz| > 2|p - q| > 2\varepsilon.\]

Hence if \(\delta\) is chosen such that \(\delta A(\omega)^4 < 2\varepsilon\) then for the family \(I_p^1\) we have \(L(\omega) > \delta A(\omega)^4\). We proved that if \(|p - q| > \varepsilon\) then \(d^1(p, q) > \delta\). If \(p = \infty \in X\) we apply a conformal map e.g. we let \(z \rightarrow -z^{-1}\). Hence in any case \(T^1 = T^2 = T\).

**Lemma 4.4.** If \(X\) has no compact boundary components and if \(X \subset P_1\) then \(X\) is simply connected.

This can be proved for instance by using the following lemma: (See ref. 11 Vol. 2 p. 380).

**If the region \(X\) is properly contained in the sphere \(S^2\) and if \(S^2 - X\) is connected then \(X\) is a simply connected domain.**

As usual a region means a connected open set in \(S^2\). In the reference given it is required that \(S^2 - X\) be a semi-continuum, which means that any two points of \(S^2 - X\) can be joined by a continuum. The present condition is obtained by using the compactness of \(S^2 - X\) and the fact that the closure of a connected set is itself connected.

**Proof:** To prove that \(X\) is simply connected let \(C\) be a component of \(S^2 - X\). By \(\partial C\) and \(\partial X\) we denote the boundaries of \(C\) and \(X\) in \(S^2\). Since \(S^2 - X\) is closed we have \(\partial C \subset S^2 - X\) and \(\partial X \subset S^2 - X\). Every point \(p\) in the interior of \(S^2 - X\) has connected neighborhoods \(O_p \subset S^2 - X\) and so if \(p \in C\) then \(O_p \subset C\). Therefore \(\partial C \subset \partial X\). Since \(S^2\) is connected \(\partial C\) is not void. By hypothesis every component of \(\partial X\) contains \(\infty\), and so \(\partial X\) is connected in \(S^2 - X\). Since \(C\) is a maximal connected set and \(S^2 - X \supset \partial X \supset \partial C \neq \phi\) we see that \(C \supset \partial X\). Therefore each component of
$S^2 - X$ contains $\infty$, and so $S^2 - X$ consists of only one component. By the above lemma $X$ is simply connected.

The results can be summarized as follows:

**Theorem 4.1.** The conformally invariant pseudo-metric $d^1$ of a schlichtartig Riemann surface $X$ is compatible with the topology of $X$ unless $X$ is simply connected when $d^1$ is identically zero.

The methods developed in the proof of this theorem can be used to prove

**Theorem 4.2.** If the Riemann surface $X$ is multiply connected then the conformally invariant metrics $d^1$ and $d^2$ are compatible with the topology of $X$.

**Proof:** Since in general $T^1 < T^2 < T$ it is sufficient to prove that $T^1 > T$. Let $p \in X$ be fixed. We want to prove that given any neighborhood $N_p$ of $p$ there is a $\delta > 0$ such that $d^1(p, q) < \delta$ only if $q \in N_p$. By hypothesis there is a simple closed curve $c$ which is not homotopic to 0. We can choose $c$ such that $p \notin c$. There is an arc $\alpha$ connecting $c$ with $p$. We may assume that $\alpha$ is closed i.e. homeomorphic to $[0, 1]$ and its end points are $p$ and a point on $c$. Then $c \cup \alpha$ is compact and so there are finitely many coordinate neighborhoods $(O_k, \varphi_k) (k = 1, \ldots, n)$ such that $O = \cup O_k$ covers $c \cup \alpha$. These coordinate neighborhoods can be chosen such that $O$ is homeomorphic to an annulus.

We define a form $\omega$ on $X$ as follows: The open set $O$ is a schlichtartig Riemann surface and so it has a global coordinate system $(O, \varphi)$. (See for example ref. 19 p. 224). We choose a smaller annulus $Q$ containing $c \cup \alpha$ so that $\varphi(Q)$ is a bounded region in the finite plane. Using this coordinate system we may define $\omega(z) = 1$ for every $z \in \varphi(Q)$ and $\omega(z) = 0$ for every $z \in \varphi(Q)$. Now we can determine a suitable $\delta > 0$: It is sufficient to choose $\delta$ so small that $\delta A(\omega)^{\frac{1}{n}}$ is smaller than

(i) the infimum of $|\varphi(q) - \varphi(p)|$ as $q$ varies over $\varphi(Q - N_p)$;

(ii) the length of any arc connecting the two boundary components of $\varphi(Q)$; and

(iii) the length of any crosscut of $\varphi(Q)$ which bounds a simply connected region containing $\varphi(p)$.

Now if $q \in N_p$ then for any $\gamma \in \Gamma^4_p$ we have

$$\int_\gamma |\omega| > \int_{\varphi(\gamma)} |dz| > \delta A(\omega)^{\frac{1}{n}}.$$ 

Therefore $L(\omega) > \delta A(\omega)^{\frac{1}{n}}$ and so

$$d^1(p, q) > L(\omega) A(\omega)^{-\frac{1}{n}} > \delta.$$ 

Theorem 4.2. is proved.

5. A uniform structure for schlichtartig Riemann surfaces

Here we shall study the pseudo-metric $d^1$ of schlichtartig Riemann surfaces. We shall be especially interested in the conformally invariant
uniform structure $\mathcal{U}$ generated by $d^1$. The Riemann sphere $P_1$ being compact there is a unique uniform structure $\mathcal{U}$ compatible with its topology. The uniform structure induced on a set $X \subseteq P_1$ will be denoted also by $\mathcal{U}$. Unless $X$ is compact there are other uniform structures besides $\mathcal{U}$ having the property that the uniform topology generated on $X$ by $\mathcal{U}$ is $\mathcal{T}$. (See ref. 6 and 8). If $X$ is a compact set in the finite plane then by the uniqueness principle its $\mathcal{U}$ is the same as the uniform structure generated on the plane by the Euclidean metric $|p-q|$. This is often called the "usual structure" of the plane or of the set $X$.

A region $X \subseteq P_1$ whose boundary components are finitely many simple closed curves will be called a Jordan domain. Our main object is to prove

**Theorem 5.1.** If $X$ is a multiply connected Jordan domain then $\mathcal{U}^1 = \mathcal{U}$.

In particular if $X$ is a bounded Jordan domain in the finite plane then $\mathcal{U}^1$ is the usual structure of $X$. Since $\mathcal{U}^1$ is invariant under conformal mappings we shall obtain the following corollary:

**Theorem 5.2.** If $X$ and $Y$ are bounded Jordan domains in the finite plane and if $f$ maps $X$ conformally onto $Y$ then $f$ is uniformly continuous with respect to the usual structure of the plane.

The uniform structures $\mathcal{U}$ and $\mathcal{U}^1$ are in general distinct structures. For instance let $X$ be the region obtained from the unit disk by omitting the segment $[0, 1)$ and, in order to make $X$ multiply connected, also the point $-\frac{1}{2}$. Then it is easy to see that $\{\frac{1}{2} + (i/n)\} (n=1, 2, \ldots)$ and $\{\frac{1}{2} - (i/n)\} (n=1, 2, \ldots)$ are non-equivalent Cauchy sequences with respect to $\mathcal{U}^1$. In general $\mathcal{U}^1$ can be weaker or stronger than $\mathcal{U}$ and $\mathcal{U}^1$ can also be incomparable with $\mathcal{U}$.

We start the proofs by first establishing the following

**Lemma 5.1.** If the bounded plane region $X$ is not simply connected and if its boundary consists of finitely many isolated points and simple closed curves then $\mathcal{U} \neq \mathcal{U}^1$.

**Proof:** Given $\delta > 0$ we wish to determine an $\varepsilon > 0$ such that $d^1(x, y) < \varepsilon$ implies $|x-y| < \delta$. The existence of a suitable $\varepsilon$ will be proved by contradiction: We suppose that there is a sequence of point-pairs $x_n, y_n \in X$ such that $d^1(x_n, y_n) \rightarrow 0$ but $|x_n-y_n| > \delta$ for every $n = 1, 2, \ldots$. We choose $\omega = 1$ and determine the simply connected domains $S_n$ with boundary curves $\gamma_n$ such that

\begin{equation}
\int_{\gamma_n} |\omega| < (1+\delta) A(\omega) d^1(x_n, y_n).
\end{equation}

If $X$ has more than two boundary components then $S_n$ being simply connected $\gamma_n$ intersects at most one boundary component. If $X$ has only two boundary components $C_1, C_2$ then there are at most finitely many
indices \( n \) with the property that \( \gamma_n \) connects \( C_1 \) with \( C_2 \). In fact for these \( \int |\omega| = \int |dz| \) is greater than or equal to the Euclidean distance of \( C_1 \) and \( C_2 \). We omit these finitely many terms of the sequence \( \{(x_n, y_n)\} \). If \( \tilde{\gamma}_n \) does not intersect any boundary component then \( \gamma_n \) is a simple closed curve enclosing \( x_n \) and \( y_n \). Hence

\[
\int_{\gamma_n} |\omega| = \int_{\gamma_n} |dz| > 2|x_n - y_n| > 2\delta.
\]

Using (5.1) and \( d^1(x_n, y_n) \to 0 \) we see that all but finitely many \( \tilde{\gamma}_n \) intersect exactly one boundary component. Since there are only finitely many boundary components we can select a subsequence of \( \{(x_n, y_n)\} \) and a boundary component \( C \) such that each \( \tilde{\gamma}_n \) intersects \( C \) and no other boundary components of \( X \).

The curve \( \gamma_n \) has finite arc length and so it has one or two accumulation points on \( C \). Therefore the simply connected domain \( S_n \) is the plane domain bounded by \( \gamma_n \) and an arc \( p_n q_n \) of the Jordan curve \( C \). Both the arc and \( C \) might degenerate to a single point. Using a compactness argument we select a subsequence of the sequence \( \{(x_n, y_n)\} \) such that \( p_n \to p \) and \( q_n \to q \) in the Euclidean sense.

The case \( p = q \) is not possible. For \( C \) being a simple closed curve or the point \( p \) itself we can choose a disk \( N_p \) with radius \( r < \delta/4 \) and center \( p \) such that \( N_p \cap C \) is an arc or the point \( p \). Now if \( n \) is so large that \( p_n, q_n \in N_p \) then the boundary of \( S_n \) consists of \( \gamma_n \) and the arc \( p_n q_n \) lying entirely in \( N_p \). Therefore \( S_n \) is contained in the disk with radius \( r + \frac{1}{2} \int_{\gamma_n} |\omega| \) and center \( p \). Since \( x_n, y_n \in S_n \) and \( |x_n - y_n| > \delta \) using (5.1) we obtain

\[
\frac{\delta}{2} < (1 + \delta) A(\omega)^\frac{1}{2} d^1(x_n, y_n)
\]

in contradiction to \( d^1(x_n, y_n) \to 0 \). Now let us suppose that \( p \neq q \) so that \( |p - q| > 0 \). If \( n \) is so large that \( |p_n - p| < \frac{1}{4} |p - q| \) and \( |q_n - q| < \frac{1}{4} |p - q| \) then

\[
\int_{\gamma_n} |\omega| = \int_{\gamma_n} |dz| > |p_n - q_n| > \frac{1}{2} |p - q|.
\]

In view of (5.1) this contradicts \( d^1(x_n, y_n) \to 0 \). Lemma 5.1 is proved.

**Lemma 5.2.** If \( X \) is a Jordan domain then \( \mathcal{U} \subset \mathcal{U} \).

**Proof:** We may again restrict ourselves to bounded Jordan domains in the finite plane. Given \( \varepsilon > 0 \) our object is to determine a \( \delta > 0 \) such that if \( x, y \in X \) and \( |x - y| < \delta \) then \( d^1(x, y) < \varepsilon \). On the contrary let us suppose that there is an \( \varepsilon > 0 \) and a sequence \( \{(x_n, y_n)\} \) of point-pairs \( x_n, y_n \in X \) such that \( |x_n - y_n| \to 0 \) but \( d^1(x_n, y_n) > \varepsilon \) for every \( n = 1, 2, \ldots \).

Let \( \overline{X} \) denote the closure of \( X \) in the Euclidean plane. Since \( \overline{X} \) is compact we can select a subsequence of \( \{(x_n, y_n)\} \) such that for some \( p \) in \( \overline{X} \) we have \( x_n \to p \) and \( y_n \to p \) in the Euclidean sense.

First suppose that \( p \in X \). Then \( R > 0 \) can be chosen such that the disk \( |x - p| < R \) belongs entirely to \( X \). We apply Lemma 2.2 with an annulus \( A \)
centered around $p$ having sufficiently small inner radius $r > 0$ and outer radius $R$. For every sufficiently large $n$ the points $x_n, y_n$ belong to the bounded component of $cA$. Hence if $r$ is small then for these points we have $d^1(x_n, y_n) < \varepsilon$ in contradiction to the hypothesis. Now let $p$ belong to one of the boundary components $C$ of $X$. We determine a family $\Gamma$ of curves $\gamma$ such that its extremal length $\lambda < \varepsilon^2$. A contradiction will be obtained by showing that $\Gamma \subseteq \Gamma_{x_n y_n}$ for every sufficiently high index $n$.

We choose $R > 0$ so small that $|q - p| > R$ for some $q \in X$. Since $C$ is a Jordan curve we can connect $q$ with $p$ by a simple arc $\pi \subset X$. We let $r > 0$ be so small that the extremal length of the family of circles $|x - p| = q (r < q < R)$ is less than $\varepsilon^2$. In addition $r$ can be chosen such that the intersection of $C$ with the disk $|x - p| < r$ is a simple arc. Every circle $|x - p| = q$ intersects $\pi$ in at least one point $x_q$. Since $x_q \in X$ and $X$ is open there is an arc $\gamma \subset X$ on the circle $|x - p| = q$ which contains $x_q$ and whose end points $p_\gamma, q_\gamma \in C$. Let $\Gamma$ be the family of all possible arcs $\gamma$ and let $\lambda$ be its extremal length. Each $\gamma \in \Gamma$ is an arc of one of the circles $|x - p| = q$ and on each circle $|x - p| = q$ lies at least one arc $\gamma \in \Gamma$. Applying Lemma 2.3 we see that $\lambda < \varepsilon^2$.

Given $\gamma \in \Gamma$ with end points $p_\gamma, q_\gamma \in C$ let $\gamma^*$ be that arc of $C$ which contains $p$ and whose end points are $p_\gamma, q_\gamma$. By the Jordan curve theorem $\gamma \cup \gamma^* \cup \{p_\gamma, q_\gamma\}$ is the boundary of a simply connected domain $S_\gamma \subset X$. Therefore $\gamma$ is the boundary with respect to $X$ of the simply connected domain $S_\gamma$. If $n$ is sufficiently large then $|x_n - p| < r$ and $|y_n - p| < r$. Since $|x - p| = r$ intersects $C$ in a simple arc $x_n, y_n \in S_\gamma$ and so $\gamma \in \Gamma_{x_n y_n}$. Therefore $d^1(x_n, y_n) < \varepsilon$ which is a contradiction. Lemma 5.2. and Theorem 5.1. are proved.

(To be continued)