Canonical bases for quantum generalized Kac–Moody algebras

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Abstract

We construct canonical bases for quantum generalized Kac–Moody algebras using semisimple perverse sheaves.

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0. Introduction

The quantum groups arise as certain deformations of the universal enveloping algebras of Kac–Moody algebras. One of the main achievements in the theory of quantum groups is the construction of canonical bases (or global bases) for integrable highest weight modules over a quantum group $U_v (\mathfrak{g})$. The canonical bases enjoy many remarkable properties such as positivity with respect to the $U_v (\mathfrak{g})$-action, and they all come from...
a single canonical basis for the negative part $U^-_v(g)$ of the quantum group via the action on the highest weight vector. These bases were constructed independently by Kashiwara (in a purely algebraic way) and Lusztig (in a geometric way via intersection cohomology) [10,12], and they are called the global bases and canonical bases, respectively. In [7], it was shown that these two bases coincide with each other. Lusztig’s construction was inspired by Ringel’s realization of $U^-_v(g)$ in terms of the Hall algebra of the underlying Dynkin diagram.

The generalized Kac–Moody algebras were introduced by Borcherds in his study of vertex algebras and Monstrous Moonshine [3]. In particular, the Monster Lie algebra, which is a special example of generalized Kac–Moody algebras, played a crucial role in the proof of the Moonshine conjecture [4]. Recently, Jeong, Kang and Kashiwara developed the crystal basis theory for quantum generalized Kac–Moody algebras, and constructed the global bases for $U_v(g)$-modules in the category $O_{int}$ and for the subalgebra $U^-_v(g)$ [8].

In this paper, we provide the geometric construction of canonical bases for quantum generalized Kac–Moody algebras. We first extend Ringel’s Hall algebra approach to the case of quantum generalized Kac–Moody algebras. In order to account for simple imaginary roots, we consider quivers in which edge loops are allowed, and add some nilpotency relations at each such edge loop. Next, we generalize to this context Lusztig’s construction of canonical bases in terms of perverse sheaves on varieties of representations of quivers.

An important difference between our construction and the classical one is that our canonical bases are made of semi-simple perverse sheaves rather than simple perverse sheaves. That this is necessary is already clear in the case of a quiver consisting of a single loop, where the corresponding generalized Kac–Moody algebra is the Heisenberg algebra and the space of representations is the nilpotent cone. In particular, the canonical basis is not (quasi)-orthonormal anymore but only (quasi)-orthogonal. This makes the extension of Lusztig’s construction more delicate. We conjecture that the canonical bases thus obtained coincide with the global bases constructed in [8]. We prove this conjecture in the case of generalized Kac–Moody algebras with no isotropic simple roots, which includes many interesting generalized Kac–Moody algebras such as the Monster Lie algebra. We expect our work will lead to a variety of combinatorial and geometric developments in the study of generalized Kac–Moody algebras and their representations.

1. Quantum generalized Kac–Moody algebras

1.1. Generalized root datum

Let $I$ be a countable index set. In this paper, a generalized root datum is a matrix $A = (a_{ij})_{i,j \in I}$ satisfying the following conditions:

(i) $a_{ij} \in \{2, 0, -2, -4, \ldots\}$,
(ii) $a_{ij} = a_{ji} \in \mathbb{Z}_{\leq 0}$.
Such a matrix is a special case of Borcherds–Cartan matrix (see [3]). Let $I^{re} = \{i \in I \mid a_{ii} = 2\}$ and $I^{im} = I \setminus I^{re}$. We also assume that we are given a collection of positive integers (the charge of $A$) $\mathbf{m} = (m_i)_{i \in I}$ with $m_i = 1$ whenever $i \in I^{re}$.

1.2. Definition

The **quantum generalized Kac–Moody algebra** associated with $(A, \mathbf{m})$ is the (unital) $\mathbb{C}(v)$-algebra $U_v(\mathfrak{g}_{A, \mathbf{m}})$ (or simply $U_v(\mathfrak{g})$) generated by the elements $K_i, K_i^{-1}, E_{i,k}, F_{i,k}$ for $i \in I, k = 1, \ldots, m_i$ subject to the following set of relations:

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, K_i K_j = K_j K_i, \quad (1.1)
\]

\[
K_i E_{jk} K_i^{-1} = v^{a_{ij}} E_{jk}, \quad K_i F_{jk} K_i^{-1} = v^{-a_{ij}} F_{jk}, \quad (1.2)
\]

\[
E_{ik} F_{jl} - F_{jl} E_{ik} = \delta_{lk} \delta_{ij} \frac{K_i - K_i^{-1}}{v - v^{-1}}, \quad (1.3)
\]

\[
\sum_{n=0}^{1-a_{ij}} (-1)^n \left[ \frac{1 - a_{ij}}{n} \right] E_{ik}^{1-a_{ij}-n} E_{jl} E_{ik}^n = 0 \quad \forall \ i \in I^{re}, j \in I, \quad (1.4)
\]

\[
\sum_{n=0}^{1-a_{ij}} (-1)^n \left[ \frac{1 - a_{ij}}{n} \right] F_{ik}^{1-a_{ij}-n} F_{jl} F_{ik}^n = 0 \quad \forall \ i \in I^{re}, j \in I, \quad (1.5)
\]

where as usual we put

\[
[n] = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [n]! = [2] \cdots [n], \quad \left[ \frac{n}{k} \right] = \frac{[n]!}{[n-k]![k]!}.
\]

The algebra $U_v(\mathfrak{g})$ is equipped with a Hopf algebra structure as follows (see [2,9]):

\[
\Delta(K_i) = K_i \otimes K_i,
\]

\[
\Delta(E_{ik}) = E_{ik} \otimes K_i^{-1} + 1 \otimes E_{ik}, \quad \Delta(F_{ik}) = F_{ik} \otimes 1 + K_i \otimes F_{ik},
\]

\[
\varepsilon(K_i) = 1, \quad \varepsilon(E_{ik}) = \varepsilon(F_{ik}) = 0,
\]

\[
S(K_i) = K_i^{-1}, \quad S(E_{ik}) = -E_{ik} K_i, \quad S(F_{ik}) = -K_i^{-1} F_{ik}.
\]
Put $A = Z[v, v^{-1}]$. The integral form $U_A(g)$ is the $A$-subalgebra of $U_v(g)$ generated by $K_j^\pm, E_{ik}, F_{ik}$ for $j \in I, i \in I^\text{im}$ and $k = 1, \ldots, m_i$, and the divided powers

$$E_{ik}^{(n)} = \frac{E_{ik}^n}{[n]!}, \quad F_{ik}^{(n)} = \frac{F_{ik}^n}{[n]!} \quad \text{for } i \in I^\text{re} \text{ and } n > 0.$$ 

It follows from the definitions that the subalgebra $U_\geq 0_v(g)$ of $U_v(g)$ generated by $K_i, E_{ik}, F_{ik}$ for $i \in I$ and $k = 1, \ldots, m_i$ is a Hopf subalgebra. We also denote by $U_{\leq 0_v}(g)$ the subalgebra generated by $F_{ik}$ for $i \in I$ and $k = 1, \ldots, m_i$. We will use similar notations for $U_A(g)$.

Finally, let $u \mapsto \bar{u}$ be the semilinear involution of $U_v(g)$ defined by $v = v^{-1}, F_{ik} = F_{ik}, E_{ik} = E_{ik}$ and $K_i = K_i^{-1}$.

### 1.3. Nondegenerate bilinear form

It follows from [11] that there exists a unique symmetric bilinear form $(\cdot, \cdot) : U_{\leq 0_v}(g) \otimes U_{\leq 0_v}(g) \to C(v)$ satisfying

$$(K_i, K_j) = v^{-aij}, \quad (F_{ik}, F_{jl}) = \delta_{ij} \delta_{kl}, \quad (F_{ik}, K_j) = 0$$

and the invariance condition

$$(a, bc) = \sum_n (a_n^{(1)}, b)(a_n^{(2)}, c).$$

Here, we have used Sweedler’s notation $\Delta(a) = \sum_n a_n^{(1)} \otimes a_n^{(2)}$.

**Proposition 1.1** (Sevenhant and Van den Bergh [14]). The restriction of $(\cdot, \cdot)$ to $U_{\leq 0_v}(g)$ is nondegenerate.

### 2. Quivers and Hall algebras

#### 2.1. Quivers

Let $Q$ be an arbitrary locally finite quiver with vertex set $I$ and (oriented) edge set $\Omega$. For $\sigma \in \Omega$ we denote by $o(\sigma)$ and $i(\sigma)$ the outgoing and incoming edges, respectively, and sometimes use the notation $o(\sigma) \xrightarrow{\sigma} i(\sigma)$. We will denote by $c_i$ the number of loops at $i$ (i.e., the number of edges $\sigma$ with $i(\sigma) = o(\sigma) = i$).

Recall that a representation of $Q$ over a field $k$ is a collection $(V_i, x_\sigma)_{i \in I, \sigma \in \Omega}$, where $V_i$ is a finite-dimensional $k$-vector space, $V_i = 0$ for almost all $i$ and $x_\sigma \in \text{Hom}_k(V_{o(\sigma)}, V_{i(\sigma)})$. We denote by $\text{Rep}_k(Q)$ the abelian category of representations of $Q$ over $k$. For any $M, N \in \text{Rep}_k(Q)$ the spaces $\text{Hom}(M, N)$ and $\text{Ext}^1(M, N)$ are finite-dimensional and we have $\text{Ext}^n(M, N) = 0$ for $n > 1$. 

Let us set $\dim(V_i, x_\sigma) = \dim(V_i) = (\dim_k V_i) \in \mathbb{N}^{\oplus I}$. The Euler form given by

$$\langle M, N \rangle = \dim_k \text{Hom}(M, N) - \dim_k \text{Ext}^1(M, N)$$

only depends on $\dim M$ and $\dim N \mathbb{Z}^{\oplus I}$ and yields the following nonsymmetric bilinear pairing $\langle , \rangle: \mathbb{Z}^{\oplus I} \otimes \mathbb{Z}^{\oplus I} \to \mathbb{Z}$ defined by

$$\langle (d_i), (d'_j) \rangle \mapsto \sum_i (1 - c_i)d_i d'_j - \sum_{i \neq j} r_{ij}d_id'_j,$$

where $r_{ij}$ denotes the number of edges going from $i$ to $j$. We also introduce the associated symmetric form

$$(a, b) = \langle a, b \rangle + \langle b, a \rangle.$$

Observe that there is a correspondence between locally finite quivers and generalized root datum: to $Q$ we associate the array $A_Q = (a_{ij})_{i,j \in I}$ with $a_{ij} = (e_i, e_j)$, where $(e_i)$ stands for the standard basis of $\mathbb{Z}^{\oplus I}$. In particular, $a_{ii} = 2 - 2c_i$, and $i \in I$ is real if and only if $c_i = 0$. If $V$ is an $I$-graded vector space, we will write $V = V^{re} \oplus V^{im}$ for its decomposition according to $I = I^{re} \sqcup I^{im}$.

2.2. Hall algebras

Let us now assume that $k$ is a finite field with $q$ elements and fix $v \in \mathbb{C}$ such that $v^2 = q^{-1}$. Let us denote by $\text{Iso}_k(Q)$ the set of isomorphism classes of objects in $\text{Rep}_k(Q)$, and let us write $[M] \in \text{Iso}_k(Q)$ for the class of an object $M$. The Hall algebra $H_k(Q)$ is by definition the associative $\mathbb{C}$-algebra with basis indexed by $\text{Iso}_k(Q)$ and equipped with the product

$$[M] \cdot [N] = v^{-\langle \dim M, \dim N \rangle} \sum_{[P] \in \text{Iso}_k(Q)} \mathcal{P}_{M,N}^P [P],$$

where

$$\mathcal{P}_{M,N}^P = \# \{ X \subset P \mid X \in [N], P/X \in [M] \}.$$
the following relations:

\[ K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \quad (2.2) \]

\[ K_i [M] K_i^{-1} = v^{-(\varepsilon_i, \dim M)} [M]. \quad (2.3) \]

It is easy to see that the multiplication map \( \mathbb{C}[K_i^{\pm 1}] \otimes \mathcal{H}_k(Q) \to \mathcal{H}_k(Q) \) is an isomorphism of vector spaces. Finally, we equip the algebra \( \mathcal{H}_k(Q) \) with a coproduct by setting \( \Delta(K_i) = K_i \otimes K_i \) for all \( i \in I \) and

\[ \Delta([M]) = \sum_{N \subset M} v^{-(\dim M/\dim N)} \mathcal{P}_{M/N} \frac{a_{M/N} a_N}{a_M} [M/N] \mathcal{K}^{\dim N} \otimes [N], \]

where \( a_M = \#\text{Aut}(M) \) and \( \mathcal{K}^{\dim N} = \prod_i K_i^{\dim N_i} \).

The following is one of the main results in [6]:

**Theorem 2.1 (Green [6]).** There exists a Hopf pairing \( \langle \cdot, \cdot \rangle_G \) on \( \mathcal{H}_k(Q) \) such that

\[ \langle K_i, K_j \rangle_G = v^{-(\varepsilon_i, \varepsilon_j)}, \quad \langle [M], [N] \rangle_G = \delta_{[M],[N]} \frac{1}{a_M}. \]

### 2.3. Composition algebra

If \( i \in I^{re} \), then there exists a unique simple object \( S_i \in \text{Rep}_k(Q) \) such that \( \dim S_i = \varepsilon_i \). On the other hand, if \( i \in I^{im} \) then the set of simple representations of dimension \( \varepsilon_i \) is in bijection with \( k^{c_i} \): if \( \sigma_1, \ldots, \sigma_{c_i} \) denote the simple loops at \( i \) then to \( \hat{\sigma} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{c_i}) \) corresponds the simple module \( S_i(\hat{\sigma}) = (V_j, x_\sigma) \) with \( \dim_k V_j = \delta_{ij} \) and \( x_{\sigma_l} = \hat{\lambda}_l \text{Id} \) for \( l = 1, \ldots, c_i \).

For each \( i \in I^{im} \), let us now assume that we are given a positive integer \( m_i \). We choose \( \hat{\lambda}_j^{(l)} \in k^{c_i} \) for \( l = 1, \ldots, m_i \) in such a way that if \( |k^{c_i}| \geq m_i \) then \( \hat{\lambda}_j^{(l)} \neq \hat{\lambda}_{j'}^{(l')} \) for \( l \neq l' \). Consider the composition subalgebra \( \mathcal{C}_k(Q) \subset \mathcal{H}_k(Q) \) generated by the classes \( [S_i] \) for \( i \in I^{re} \) and \( [S_{i,l}] := [S_i(\hat{\lambda}_j^{(l)})] \) for \( i \in I^{im} \) and \( l = 1, \ldots, m_i \). We also define \( \mathcal{C}_k(Q) \) as the subalgebra of \( \mathcal{H}_k(Q) \) generated by \( \mathcal{C}_k(Q) \) and \( \mathbb{C}[K_i^{\pm 1}] \).

Following Ringel, we now define a generic composition algebra: let \( K \) be an infinite set of (nonisomorphic) finite fields, and let us choose for each \( k \in K \) an element \( v_k \in \mathbb{C} \) such that \( v_k^2 = |k|^{-1} \). Consider the direct product

\[ \mathcal{H}(Q) = \prod_{k \in K} \mathcal{H}_k(Q). \]

We view \( \mathcal{H}(Q) \) as a \( \mathbb{C}[v, v^{-1}] \)-module by mapping \( v, v^{-1} \) to \((v_k)_k, (v_k^{-1})_k\), respectively. We define \( \mathcal{C}(Q) \) to be the \( \mathbb{A} \)-subalgebra of \( \mathcal{H}(Q) \) generated by \((K_i)_k, (K_i^{-1})_k\).
for \( i \in I^\text{re} \) and \((S_{i,l})_k \) for \( i \in I^\text{im} \) and \( l = 1, \ldots, m_i \). The subalgebra \( C(Q) \) is defined in a similar fashion.

Our first result is an extension of a well-known theorem of Green (in the case of a quiver without loop):

**Theorem 2.2.** The assignment \( K_i^{\pm 1} \mapsto (K_i^{\pm 1})_k, F_i \mapsto ([S_i])_k \) for \( i \in I^\text{re} \) and \( F_{il} \mapsto ([S_{i,l}])_k \) for \( i \in I^\text{im} \) extends to an isomorphism of bialgebras \( U_0(\mathfrak{g}_A, m) \simeq \hat{C}(Q) \otimes \mathbb{C}(v) \) where \( m = (m_i)_i \).

The proof of the theorem parallels the proof in [6] and is given in the next section.

### 3. Hall algebra construction of \( U^{-v} \mathfrak{g} \)

We first show that for any fixed field \( k \), the assignment \( K_i^{\pm 1} \mapsto K_i^{\pm 1}, F_i \mapsto [S_i] \), \( F_{il} \mapsto [S_{i,l}] \) extends to a bialgebra homomorphism \( \varphi_k : U^{-v}(\mathfrak{g}_A, m) \rightarrow \hat{C}_k(Q) \).

From the relations (1.1) and (1.2), Eqs. (2.2), (2.3) and the definitions of the coproduct, it is easy to see that it is enough to check that we have a morphism \( \varphi_k : U^{-v}(\mathfrak{g}) \rightarrow C_k(Q) \). The corresponding computations are similar to the ones in [13], which we reproduce for the reader’s convenience.

We will first check that if \( i \in I^\text{re} \) and \( j \in I \), then \([S_i]\) and \([S_j]\) satisfy the \( q \)-Serre relation (1.4). To simplify the notations, let us set \( r_1 = r_{ij}, r_2 = r_{ji} \) and \( r = r_1 + r_2 \).

An easy induction shows that \([S_i]^{(l)} := [S_i]^{l}_{[l]} = v^{-l(l-1)}[S_i^{\otimes l}] \). We have

\[
[S_i]^{(l)}[S_{j,t}] = v^{-l(l-1)+lr_1} \sum_{[T] \in \mathcal{I}_1} [T],
\]

where \( \mathcal{I}_1 = \{ [T] | \exists X \subset T \text{ s.t. } X \simeq S_{j,t}, T/X \simeq S_i^{\otimes l} \} \). For a representation \( P \) of \( Q \) of dimension \((r+1)\varepsilon_i \oplus \varepsilon_j\), we define

\[
U_P = \bigcap_{i \rightarrow j} \ker x_\sigma, \quad V_P = \sum_{j \rightarrow i} \im x_\sigma,
\]

and set \( u_P = \dim U_P, v_P = \dim V_P \). For \( a, b \in \mathbb{Z} \), let us denote by \( \text{Gr}^b_a(k) \) the Grassmanian of \( a \)-dimensional subspaces in \( k^b \) (the empty set if \( a < 0 \) or \( a > b \)). A direct computation now shows that

\[
[S_i]^{(l)}[S_{j,t}][S_i]^{(n)} = v^{nr_2 + lr_1 - nl - (l-1)n(n-1)} \sum_{[P]} \sigma_{P,n}[P],
\]
where $\sigma_{P,n} = 0$ unless $V_P \subset U_P$, in which case we have
\[
\sigma_{P,n} = \# \text{Gr}_{n-v_P}^{u_P-v_P} = v^{-(u_P-n)(n-v_P)} \left[ \frac{u_P-v_P}{n-v_P} \right].
\]

Setting $n = r + 1 - l$ and summing up, we obtain
\[
\sum_{l=0}^{r+1} (-1)^l [S_I]^{(l)} [S_{i,l}]^{(n)} = \sum_{[P] \text{ s.t. } V_P \subset U_P} \gamma_P[P],
\]
where
\[
\gamma_P = \sum_{l=0}^{k+1} (-1)^l v^{nr_2+lr_1-nl-l(l-1)-n(n-1)-(u_P-n)(n-v_P)} \left[ \frac{u_P-v_P}{n-v_P} \right]
\]
\[
= v^{-(r+1)r_2+u_P v_P} \sum_{n=0}^{r+1} (-1)^{r+1-n} v^{(2r_2+1-u_P-v_P)n} \left[ \frac{u_P-v_P}{n-v_P} \right].
\]

Observe that $u_P \geq r_2 + 1 > v_P$ for any $P$. We deduce that $1 - u_P - v_P \leq 2r_2 + 1 - u_P - v_P \leq u_P + v_P - 1$. The $q$-Serre relation is now a consequence of the following well-known identity (see, for example, [10, (3.2.8)]).

**Lemma 3.1.** Let $m \geq 1$ and let $1 - m \leq d \leq m - 1$ with $d \equiv m - 1 \pmod{2}$. Then
\[
\sum_{n=0}^{m} (-1)^n v^{dn} \left[ \frac{m}{n} \right] = 0.
\]

Finally, let $i, j \in I$ such that $(\varepsilon_i, \varepsilon_j) = 0$. There are two possibilities: either $i \neq j$ and $r_{ij} = r_{ji} = 0$, or $i = j$ and $c_i = 1$. In the first case, the relation (1.5) is obviously satisfied by $[S_{i,s}]$ and $[S_{i,l}]$. In the second case, let us denote by $\sigma$ the edge loop at $i$. If $\lambda_1 \neq \lambda_2$, then the decomposition into $x_\sigma$-eigenspaces shows that any short exact sequence
\[
0 \to S_i(\lambda_1) \to M \to S_i(\lambda_2) \to 0
\]
canonically splits; i.e., $[S_i(\lambda_1)][S_i(\lambda_2)] = [S_i(\lambda_1) \oplus S_i(\lambda_2)]$, and the relation (1.5) follows. Hence there is a well-defined algebra homomorphism $\varphi : U^\leq_0(v) \to \hat{C}(Q) \otimes \mathbb{C}(v)$.

It remains to prove that the map $\varphi$ is injective. For this, recall the following lemma of Green ([6]). Set $\mathcal{T} = \{(i, l) \mid 1 \leq l \leq m_i\}$. For $v \in \mathbb{N}^\mathcal{T}$, let $\mathcal{I}_v$ be the set of sequences $\underline{a} = (a_1, \ldots, a_n)$ of elements of $\mathcal{T}$ such that $\# \{ h \mid a_h = (i, l) \} = v(i,l)$ for every $(i, l) \in \mathcal{T}$. 
Lemma 3.2 (Green [6]). Let $v \in \mathbb{N}^T$, and let $a, b \in \mathcal{I}_v$. There exists a polynomial $M_{a,b}(t) \in \mathbb{Z}[t, t^{-1}]$ such that

$$\langle F_{a_1} \cdots F_{a_n}, F_{b_1} \cdots F_{b_m} \rangle = M_{a,b}(v)$$

and such that for every finite field $k \in K$,

$$\langle [S_{a_1}] \cdots [S_{a_n}], [S_{b_1}] \cdots [S_{b_m}] \rangle_G = M_{a,b}(v_k) \prod_{(i,l) \in T} \langle [S_{i,l}], [S_{i,l}] \rangle^v_{G(i,l)}.$$ 

Now let $x = \sum_a c_a(v) F_{a_1} \cdots F_{a_n} \in \text{Ker} \varphi$, so that for any $k \in K$ we have

$$\varphi_k(x) = \sum_a c_a(v_k) [S_{a_1}] \cdots [S_{a_n}] = 0.$$ 

In particular, for any $b$, we have

$$\sum_a c_a(v_k) M_{a,b}(v_k) = 0.$$ 

Since $K$ is infinite, this implies that

$$\sum_a c_a(v) M_{a,b}(v) = 0 \in \mathbb{C}[v, v^{-1}].$$

Thus $x$ lies in the radical of $\langle , \rangle$. But by Proposition 1.1, the form $\langle , \rangle$ is nondegenerate. Thus $x = 0$ and Theorem 2.2 is proved. □

4. The algebra of semisimple perverse sheaves

4.1. Quiver representation varieties

We keep the notations of Section 2.1., but assume that $k = \mathbb{F}_q$. For simplicity, we will only consider the case of a generalized Kac–Moody algebra $\mathfrak{g}_{A,G,m}$ with trivial charge; i.e., $m_i = 1$ for all $i \in I$ (in fact, this is not restrictive; see [8, Remark 1.3]).

For all $d \in \mathbb{N}^{\oplus I}$, we fix an $I$-graded $k$-vector space $V_d = \bigoplus_i V_i$ such that $\text{dim}(V_d) = d$. Let $d^{\text{im}}$ and $d^{\text{re}}$ be the imaginary and real components of $d$, respectively. Denote by

$$E_d = \{ (x_{\sigma}) | x_{\sigma_1} \cdots x_{\sigma_N} = 0 \text{ for any } \sigma_i \in \Omega \text{ and } N \gg 0 \} \subset \bigoplus_{\sigma \in \Omega} \text{Hom} (V_{\sigma_{\text{im}}}, V_{i_{\text{re}}})$$
the set of nilpotent representations of $Q$ in $V_d$. Now let $i = (i_1, \ldots, i_r)$ be a sequence of vertices $i_l \in I$ such that $\sum \varepsilon_{i_l} = d$. Consider the variety of $I$-graded flags

$$F_i = \{ D_\bullet \mid 0 \subseteq D_1 \subseteq \cdots \subseteq D_r = V_d; \ dim(D_l/D_{l-1}) = \varepsilon_{i_l} \}.$$ 

Finally, we define the incidence varieties

$$\widetilde{F}_i = \{(x, D_\bullet) \mid x(D_l) \subseteq D_{l-1} \} \subset E_d \times F_i,$$

$$\widetilde{F}_i^{\text{im}} = \left\{ (x, D_\bullet) \mid x(D_l) \subseteq D_{l-1} \oplus \bigoplus_{i \in I^{\text{re}}} V_i \right\} \subset E_d \times F_i^{\text{im}},$$ 

where $F_i^{\text{im}}$ is defined as $F_i$ by replacing $d$ by $d^{\text{im}}$.

Thus we have a commutative diagram, with obviously defined projection maps:

$$\xymatrix{ \widetilde{F}_i \ar[r]^{\pi'_1} \ar[d]_{\pi'_2} & \widetilde{F}_i^{\text{im}} \ar[r]^{\pi_1} \ar[d]_{\pi_2} & E_d \ar[d] \ar[r] & E_d \times E_d \ar[d] \ar[r] & E_d \times E_d \ar[d] \ar[r] \ar[r] \ar[r] & E_d \times E_d. }$$

Note that $\pi_1$ and $\pi'_1$ are smooth proper, while $\pi_2$ and $\pi'_2$ are vector bundles.

4.2. Notations

We use the notations in [11, Chapter 8] regarding perverse sheaves. In particular, for an algebraic variety $X$ defined over $k$ we denote by $D(X)$ (resp., $Q(X)$, resp., $M(X)$) the derived category of $\mathbb{Q}_l$-constructible sheaves (resp., the category of semisimple $\mathbb{Q}_l$-constructible complexes, resp., the category of perverse sheaves) on $X$. The Verdier dual of a complex $P$ is denoted by $D(P)$ and perverse cohomology by $H^\bullet(P)$. If $G$ is a connected algebraic group acting on $X$, then we denote by, $Q_G(X)$ and $M_G(X)$ the corresponding categories of $G$-equivariant complexes. Finally, for $P_1, P_2 \in Q_G(X)$ and $j \in \mathbb{Z}$ we set $D_j(P_1, P_2)$ to be the dimension of the space denoted $D_j(X, G, P_1, P_2)$ in [11].

4.3. Induction and restriction functors

The group $G_d = \prod_i GL(V_i)$ naturally acts on $E_d$. Following Lusztig (see [11]), let us fix an embedding of $I$-graded vector spaces $V_{d_1} \to V_{d_1+d_2}$ and an isomorphism $i : V_{d_1+d_2}/V_{d_1} \cong V_{d_2}$, and consider -

$$E_{d_1+d_2} \xleftarrow{i} F \xrightarrow{\kappa} E_{d_1} \times E_{d_2},$$
where $F = \{ x \in E_{d_1 + d_2} | x(V_{d_1}) \subset V_{d_1} \}$. \( i \) is the canonical embedding and \( \kappa(x) = (x|_{V_{d_1}}, i_*(x|_{V_{d_1} + d_2}/V_{d_1})) \). It is clear that \( \kappa \) is a vector bundle.

Similarly, consider the diagram

$$
E_{d_1} \times E_{d_2} \xleftarrow{\mathcal{I}_{d_1, d_2}} E' \xrightarrow{p_2} E'' \xrightarrow{p_3} E_{d_1 + d_2},
$$

where

- \( E' \) is the variety of all quadruples \( (x, V, z, \beta) \) satisfying \( x \in E_{d_1 + d_2}; \ V \subset V_{d_1 + d_2}; \ x(V) \subset V; \ \alpha : V \simeq V_{d_1}; \ \beta : V_{d_1 + d_2}/V \simeq V_{d_2} \),

- \( E'' \) is the variety of pairs \( (x, V) \) satisfying \( x \in E_{d_1 + d_2}; \ V \subset V_{d_1 + d_2}; \ x(V) \subset V; \ \dim(V) = d_1 \),

- \( p_1(x, V, z, \beta) = (\alpha_*(x|_V), \beta_*(x|_{V_{d_1 + d_2}/V})) \),

- \( p_2(x, V, z, \beta) = (x, V) \) and \( p_3(x, V) = x \).

Note that \( G_{d_1} \times G_{d_2} \) acts on \( E_{d_1} \times E_{d_2} \) and on \( E' \); that \( G_{d_1 + d_2} \) acts on \( E', E'', E_{d_1 + d_2} \) and trivially on \( E_{d_1} \times E_{d_2} \); that all maps are equivariant for these groups. Also observe that \( p_1 \) is smooth with connected fibers, \( p_2 \) is a principal \( G_{d_1} \times G_{d_2} \)-bundle, while \( p_3 \) is proper.

Define the functor

$$
\widetilde{\text{Res}}_{d_1, d_2} = \kappa_1! \mathcal{I} : \mathcal{Q}_{G_{d_1 + d_2}}(E_{d_1 + d_2}) \to \mathcal{D}(E_{d_1} \times E_{d_2})
$$

and put \( \text{Res}_{d_1, d_2} = \widetilde{\text{Res}}_{d_1, d_2} [l_1 - l_2 - 2 \sum (d_1_i (d_2_i))], \) where \( l_1 \) and \( l_2 \) are the dimensions of the fibres of \( p_1 \) and \( p_2 \), respectively. Also, we define the functor (see [11, Section 9.2]):

$$
\text{Ind}_{d_1, d_2} = p_3! p_2^* p_1^* : \mathcal{Q}_{G_{d_1} \times G_{d_2}}(E_{d_1} \times E_{d_2}) \to \mathcal{D}(E_{d_1 + d_2}),
$$

and set \( \text{Ind}_{d_1, d_2} = \text{Ind}_{d_1, d_2} [l_1 - l_2]. \) With this shift, the functor Ind commutes with Verdier duality ([11, §9.2.5]).

### 4.4. A class of semisimple perverse sheaves

Let \( i \) be a sequence of vertices as in Section 4.1. The variety \( \widetilde{\mathcal{L}}_i \) being smooth, the constant sheaf \( (\mathcal{O}_i) \widetilde{\mathcal{L}}_i[\dim \widetilde{\mathcal{L}}_i] \) is perverse. The map \( \pi_1^i : \widetilde{\mathcal{L}}_i \to \widetilde{\mathcal{L}}_i^{im} \) is proper and \( G_d \)-equivariant, so by Beilinson et al. [1] the complex \( \mathcal{L}_i = \pi_1^i((\mathcal{O}_i) \widetilde{\mathcal{L}}_i)[\dim \widetilde{\mathcal{L}}_i] \) is semisimple, \( G_d \)-equivariant and satisfies \( D(\mathcal{L}_i) = \mathcal{L}_i \). Let \( \mathcal{T}_i \) be the set of all simple perverse sheaves appearing (possibly with a shift) in \( \mathcal{L}_i \).

**Proposition 4.1.** If \( P \in \mathcal{T}_i \), then \( \pi_1^{im}(P) \) is a semisimple perverse sheaf. Moreover, if \( c_i > 1 \) for all \( i \in \mathbb{I}^{im} \), then \( \pi_1^{im}(P) \) is simple.

**Proof.** The semisimplicity of \( \pi_1^{im}(P) \) follows from the Decomposition Theorem in [1] and the fact that \( \pi_1 \circ \pi_1^i \) is proper. Let us prove that \( \pi_1^{im}(P) \) is in addition perverse.
Let $E_d' \subset \bigoplus_{i \neq j} \text{Hom}(V_i, V_j)^{r_{ij}}$ and $E_d'' \subset \bigoplus_i \text{Hom}(V_i, V_i)^{c_i}$ be the set of nilpotent representations and let $u : E_d \to E_d'$, $t : E_d \to E_d''$ be the projections. Finally, we set $G_i = (u \times \text{Id})(\tilde{F}_i) \subset E_d' \times \mathcal{F}_i$ and $G_i^{\text{im}} = (u \times \text{Id})(\tilde{F}_i^{\text{im}}) \subset E_d' \times \mathcal{F}_i^{\text{im}}$, so that we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{F}_i & \xrightarrow{\pi'_i} & \tilde{F}_i^{\text{im}} \\
\downarrow u \times \text{Id} & & \downarrow u \times \text{Id} \\
G_i & \xrightarrow{s} & G_i^{\text{im}}
\end{array}
\]

Observe that the vertical maps are vector bundles, and that $u \times \text{Id} : \tilde{F}_i \to G_i$ is the pullback by $s$ of the bundle $u \times \text{Id} : \tilde{F}_i^{\text{im}} \to G_i^{\text{im}}$. Hence $\pi'_i((\tilde{F}_i)) = \pi'_i((u \times \text{Id})^* s_i((\tilde{F}_i))$. In particular, any of the simple perverse sheaves in $\mathcal{T}_i$ is of the form $\text{IC}(X, \mathcal{L})$ with $X = (u \times \text{Id})^{-1}(Y)$ for a smooth irreducible subvariety $Y \subset G_i^{\text{im}}$ and $\mathcal{L} = (u \times \text{Id})^* \mathcal{R}$ for an irreducible local system $\mathcal{R}$ on $Y$. Let us show that the restriction of $\pi_1$ to $\overline{X}$ is semismall; i.e., $\dim \overline{X} \times \overline{X} = \dim \overline{X}$. Then standard arguments would show that $\pi_{11}(\text{IC}(X, \mathcal{L}))$ is perverse (see [5]).

Let us denote by $\mathcal{O}_\lambda \subset \mathfrak{sl}_n$ the nilpotent orbit associated to a partition $\lambda$ of an integer $n$, and set $s_\lambda = \dim \{D_\bullet \in \tilde{F}_n \mid x(D_i) \subset D_{i-1}\}$ to be the dimension of the Springer fiber over any point $x \in \mathcal{O}_\lambda$. We say that a nilpotent element $x_i = (x^j_i)_{j=1}^{c_i} \in \text{Hom}(V_i, V_i)^{c_i}$ is of type $\tilde{\lambda}_i = (\lambda_i^1, \ldots, \lambda_i^{c_i})$ if $x^j_i \in \mathcal{O}_{\lambda_i^j}$. Finally, we say that $x = (x_i)_{i} \in \bigoplus_i \text{Hom}(V_i, V_i)^{c_i}$ is of type $\tilde{\lambda} = (\tilde{\lambda}_i)_{i}$ if $x_i$ is of type $\tilde{\lambda}_i$ for all $i$.

The space $\overline{X}$ admits a finite partition $\overline{X} = \bigsqcup \overline{X}_{\lambda}$, where

\[
\overline{X}_{\lambda} = \{(x, D_\bullet) \in \overline{X} \mid t(x) \text{ is of type } \lambda\}.
\]

Thus $\overline{X} = \bigsqcup_{\lambda} \overline{X}_{\lambda}$. For each $\lambda = (\lambda_i)_{i}$ let us fix partitions $\lambda^{(i)}$ such that

(i) $\dim \mathcal{O}_{\lambda^{(j)}_i} \leq \dim \mathcal{O}_{\lambda^{(j)}_i}$ for all $j = 1, \ldots, c_i$,

(ii) $\lambda^{(i)}_i = \lambda^{j}_i$ for at least one $j \in \{1, \ldots, c_i\}$.

It is clear that

\[
\dim \overline{X}_{\lambda} \leq \sum_{i} s_{\lambda^{(i)}}.
\] (4.1)

On the other hand, since $\overline{X} = (u \times \text{Id})^{-1}(Y)$, we have

\[
\text{codim} \overline{X}_{\lambda} \geq \sum_{i,j} \text{codim}_{\mathcal{O}_{\lambda^{j}_i} \cap \mathcal{O}_{\lambda^{j}_i}}.
\]
where \( n_i \subset \mathfrak{sl}(V_i) \) is the nilpotent radical of any Borel subalgebra. Moreover, by a well-known result of Spaltenstein [15],

\[
\text{codim}_{n_i}(O_{\lambda_i} \cap n_i) = \dim n_i - \frac{1}{2} \dim O_{\lambda_i} = s_{\lambda_i},
\]

which yields

\[
\text{codim}_X \frac{X}{E} \geq \sum_{i,j} s_{\lambda_i} \geq \sum_{i} c_i s_{\lambda(i)}.
\]

(4.2)

Thus, combining (4.1) with (4.2) we obtain

\[
\dim \frac{X \times X}{E} = \sup \dim \frac{X}{E} \times \frac{X}{E} \leq \sup \dim \frac{X}{E} + \sum_i s_{\lambda(i)} \leq \sup \dim \frac{X}{E} + \sum_i (1 - c_i)s_{\lambda(i)} \leq \dim \frac{X}{E}
\]

(4.3)

as desired. Now assume that \( c_i > 1 \) for all \( i \in I^{im} \). Then \( \dim \frac{X}{E} \times \frac{X}{E} < \dim X \) as soon as \( O_{\lambda(i)} \) is not regular for some \( i \). On the other hand, if \( O_{\lambda(i)} \) is regular for all \( i \), then we have \( \frac{X}{E} \times \frac{X}{E} \subset \Delta_{\overline{X}} \), where \( \Delta_{\overline{X}} \) is the diagonal of \( \overline{X} \). Hence \( \pi_1 \) is small (see [5]) and thus \( \pi_1!^!(P) \) is simple. □

**Definition.** Set \( P_1 = \{ \pi_1!(P) \mid P \in \mathcal{P}_1 \} \). By the previous Lemma, \( P_1 \) consists of semisimple \( G_d \)-equivariant perverse sheaves. Let \( \mathcal{P}_d = \bigcup_1 \mathcal{P}_1 \) where the sum ranges over all sequences \( \mathbf{i} \) such that \( \sum_1 \mathbf{e}_i = d \). Further, denote by \( \mathcal{Q}_d \) the category of complexes which are sums of shifts of elements in \( \mathcal{P}_d \). Thus \( \mathcal{Q}_d \) is a full subcategory of \( \mathcal{Q}_{G_d}(E_d) \). Finally, for \( \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{N}^{\otimes I} \) we let \( \mathcal{Q}_{\mathbf{d}_1} \boxtimes \mathcal{Q}_{\mathbf{d}_2} \) be the full subcategory of \( \mathcal{Q}_{G_{\mathbf{d}_1}} \times G_{\mathbf{d}_2}(E_{\mathbf{d}_1} \times E_{\mathbf{d}_2}) \) consisting of sums of complexes \( P_1 \boxtimes P_2 \) with \( P_1 \in \mathcal{Q}_{\mathbf{d}_1}, P_2 \in \mathcal{Q}_{\mathbf{d}_2} \).

**Lemma 4.1.** The functors \( \text{Ind} \) and \( \text{Res} \) restrict to functors

\[
\text{Ind}^{d_1+d_2}_{d_1,d_2} : \mathcal{Q}_{\mathbf{d}_1} \boxtimes \mathcal{Q}_{\mathbf{d}_2} \rightarrow \mathcal{Q}_{\mathbf{d}_1+d_2},
\]

\[
\text{Res}^{d_1+d_2}_{d_1,d_2} : \mathcal{Q}_{\mathbf{d}_1+d_2} \rightarrow \mathcal{Q}_{\mathbf{d}_1} \boxtimes \mathcal{Q}_{\mathbf{d}_2}.
\]
Proof. To prove the first statement, it is enough to show that for any \( P_1 \in T_1, P_2 \in T_2 \) we have \( \text{Ind}_{d_1+d_2}^{d_1+d_2}(\pi_1 \!\otimes\! \pi_1; P_1, P_2) \in \mathcal{Q}_{d_1+d_2} \). For this, consider the following diagram:

\[
\begin{array}{c}
\tilde{\mathcal{F}}_{\mathbf{i}_1}^{\text{im}} \times \tilde{\mathcal{F}}_{\mathbf{i}_2}^{\text{im}} \xrightarrow{\rho_1} E'_{\text{im}} \xrightarrow{\rho_2} E''_{\text{im}} \xrightarrow{\rho_3} \tilde{\mathcal{F}}_{\mathbf{i}_1 \mathbf{i}_2}^{\text{im}},
\end{array}
\]

where \( \mathbf{i}_1 \mathbf{i}_2 \) is the concatenation of the sequences \( \mathbf{i}_1 \) and \( \mathbf{i}_2 \), and

- \( E' \) is the variety of quintuples \((x, V, D_\bullet, x, \beta)\) such that \( x \in E_{d_1+d_2}; V \subset V_{d_1+d_2}; x(V) \subset V; (x, D_\bullet) \in \tilde{\mathcal{F}}_{\mathbf{i}_1 \mathbf{i}_2}^{\text{im}}; \dim x(V) \leq d_1; \beta : V_{d_1+d_2}/V \cong V_{d_2}, \)
- \( E'' \) is the variety of triples \((x, V, D_\bullet)\) satisfying \( x \in E_{d_1+d_2}; V \subset V_{d_1+d_2}; x(V) \subset V; (x, D_\bullet) \in \tilde{\mathcal{F}}_{\mathbf{i}_1 \mathbf{i}_2}^{\text{im}}; \dim V = d_1, \)
- \( \rho_1(x, V, D_\bullet) = (x, V_{d_1+d_2}/V, D_\bullet/V) \).
- \( \rho_2(x, V, D_\bullet, x, \beta) = (x, V, D_\bullet) \) and \( \rho_3(x, V, D_\bullet) = (x, D_\bullet) \).

We set

\[
\tilde{\text{Ind}}_{\mathbf{i}_1 \mathbf{i}_2}^{d_1+d_2} = \rho_3 \circ \rho_2 \circ \rho_1^{\ast} : \mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2}(\tilde{\mathcal{F}}_{\mathbf{i}_1}^{\text{im}} \times \tilde{\mathcal{F}}_{\mathbf{i}_2}^{\text{im}}) \to \mathcal{Q}_{d_1+d_2}(\tilde{\mathcal{F}}_{\mathbf{i}_1 \mathbf{i}_2}^{\text{im}}).
\]

Claim. We have \( \pi_1! \circ (\tilde{\text{Ind}}_{\mathbf{i}_1 \mathbf{i}_2}^{d_1+d_2}) = \tilde{\text{Ind}}_{d_1+d_2}^{d_1+d_2} \circ (\pi_1! \otimes \pi_1!). \)

Proof of claim. We have a commutative diagram

\[
\begin{array}{c}
\tilde{\mathcal{F}}_{\mathbf{i}_1}^{\text{im}} \times \tilde{\mathcal{F}}_{\mathbf{i}_2}^{\text{im}} \xrightarrow{\rho_1} E'_{\text{im}} \xrightarrow{\rho_2} E''_{\text{im}} \xrightarrow{\rho_3} \tilde{\mathcal{F}}_{\mathbf{i}_1 \mathbf{i}_2}^{\text{im}}
\end{array}
\]

\[
\begin{array}{c}
\downarrow \pi_1 \times \pi_1
\end{array}
\]

\[
\begin{array}{c}
E_{d_1} \times E_{d_2} \xleftarrow{p_1} E' \xrightarrow{p_2} E'' \xrightarrow{p_3} E_{d_1+d_2}
\end{array}
\]

where \( p_4 \) and \( p_5 \) are the obvious projections. The leftmost commutative square is a cartesian (pull-back) diagram, and therefore \( p_4! p_1^{\ast} = \pi_1! (\pi_1! \otimes \pi_1!) \). Similarly, we have \( p_2! p_5^{\ast} = \pi_2! \pi_2^{\ast} \), and hence \( p_2! p_4! = p_5! p_2! \). Thus we get

\[
\tilde{\text{Ind}}_{d_1+d_2}^{d_1+d_2}(\pi_1!(P_1) \otimes \pi_1!(P_2)) = p_3! p_2! p_1^{\ast} (\pi_1!(P_1) \otimes \pi_1!(P_2))
\]

\[
= p_3! p_2! p_4! (\pi_1^{\ast} (P_1 \otimes P_2))
\]

\[
= p_3! p_5! p_2! (\rho_1^{\ast} (P_1 \otimes P_2))
\]

\[
= \pi_1! p_3! p_5! (\rho_1^{\ast} (P_1 \otimes P_2))
\]

\[
= \pi_1! (\tilde{\text{Ind}}_{\mathbf{i}_1 \mathbf{i}_2}^{d_1+d_2}(P_1 \otimes P_2)).
\]

This proves the claim. □
Now, as in [11, Lemma 9.2.3], one can prove that \( \widetilde{\text{Ind}}_{i_1,i_2}^{j_1,j_2}(L_{i_1}' \boxtimes L_{i_2}') = L_{i_1}^{i_2}. \) Thus \( \widetilde{\text{Ind}}_{i_1,i_2}^{j_1,j_2}(P_1 \boxtimes P_2) \) is a sum of shifts of simple perverse sheaves appearing in \( L_{i_1}^{i_2}. \) It follows that \( \pi_1! \widetilde{\text{Ind}}_{i_1,i_2}^{j_1,j_2}(P_1 \boxtimes P_2) \in \mathcal{Q}_{d_1+d_2}. \) This proves the first part of Lemma 4.2.

The second statement is proved in a similar way: we consider the diagram

\[
\begin{array}{ccc}
\tilde{F}_i & \xleftarrow{i''} & F'' \\
\downarrow \pi'_i & & \downarrow r \\
\tilde{F}''_i & \xleftarrow{i'} & F' \\
\downarrow \pi_1 & & \downarrow q \\
E_{d_1+d_2} & \xleftarrow{i} & F & \xrightarrow{\kappa} & E_{d_1} \times E_{d_2}
\end{array}
\]

where \( F' \) (resp., \( F'' \)) is the variety of pairs \((x, D_\bullet) \in \tilde{F}''_i\) (resp., \((x, D_\bullet) \in \tilde{F}_i\)) such that \( x(V_{d_1}) \subset V_{d_1}. \) Let \( P \in \mathcal{T}_i. \) We have \( \text{Res}_{d_1+d_2}(\pi_1|P) = \kappa_1 q_1 (i')^* P. \)

Note that by Lusztig [11, Section 9.2], \( \text{Res}_{d_1+d_2}(\pi_1|L'_i) \) is semisimple, hence so is \( \kappa_1 q_1 (i')^* P. \)

For a pair of complementary subsequences \( j_1 \) and \( j_2 \) in \( j = i^{\text{im}}, \) set

\[ F'(j_1,j_2) = \{(x, D_\bullet) \in F' | D_\bullet \cap V_{d_1} \in \tilde{F}_{j_1}, D_\bullet/D_\bullet \cap V_{d_1} \in \tilde{F}_{j_2}\}, \]

and

\[ \kappa_{j_1,j_2} : F'(j_1,j_2) \to \tilde{F}''_{j_1} \times \tilde{F}''_{j_2} \]

\[ (x, D_\bullet) \mapsto ((x|_{V_{d_1}}, D_\bullet \cap V_{d_1}), (x|_{V_{d_2}/V_{d_1}}, D_\bullet/D_\bullet \cap V_{d_1})). \]

Note that \( (F'(j_1,j_2))_{j_1,j_2} \) is a smooth stratification of \( F', \) and that \( \kappa_{j_1,j_2} \) is a vector bundle. The map \( \kappa q \) decomposes as follows: on \( F'(j_1,j_2), \) it is equal to the composition

\[ F'(j_1,j_2) \xrightarrow{\kappa_{j_1,j_2}} \tilde{F}''_{j_1} \times \tilde{F}''_{j_2} \xrightarrow{\pi_1 \times \pi_1} E_{d_1} \times E_{d_2}. \]

Similarly, we define a smooth stratification \( (F''(i_1,i_2))_{i_1,i_2} \) of \( F'' \) for \( i_1, i_2 \) running in the set of complementary subsequences in \( i, \) together with the vector bundles \( \kappa_{i_1,i_2} : F''(i_1,i_2) \to \tilde{F}_{i_1} \times \tilde{F}_{i_2}. \) Moreover, one has

\[ r^{-1}(F'(j_1,j_2)) = \bigsqcup_{(i_1,i_2)^{\text{im}}=(j_1,j_2)} F''(i_1,i_2). \]
Considering the diagram

\[ \begin{array}{cccc}
\mathcal{F}_{i_1} & \xrightarrow{i''} & \bigcup_{(i_1, i_2)^{im} = (j_1, j_2)} F''(i_1, i_2) \\
\downarrow \pi_1 & & \downarrow r \\
\mathcal{F}_{i_1}^{im} & \xleftarrow{i'} & F'(j_1, j_2) & \xrightarrow{k_{j_1, j_2}} \mathcal{F}_{j_1}^{im} \times \mathcal{F}_{j_2}^{im} \xrightarrow{\pi_1 \times \pi_1} E_{d_1} \times E_{d_2}
\end{array} \]

and reasoning as in [11, Lemma 9.2.4], we see that \( k_{j_1, j_2}(i') P_{F'(j_1, j_2)} \) belongs to \( Q_{j_1} \boxtimes Q_{j_2} \) (in particular, it is semisimple). Finally, applying [11, §8.1.6] (see also [12, §4.6, Lemma 4.7]), we obtain for all \( n \)

\[ H^n((kq)_!(i')^* P) \simeq \bigoplus_{j_1, j_2} H^n(\pi_1! \times \pi_1! k_{j_1, j_2}(i')^* P_{F'(j_1, j_2)}). \]

Now, \( \pi_1!(\bigoplus_{j_1, j_2} k_{j_1, j_2}(i')^* P_{F'(j_1, j_2)}) \) and \( (kq)_! i'^* P \) are two semisimple complexes with isomorphic perverse cohomology, so they are isomorphic and the claim follows. \( \square \)

4.5. The algebra \( K_v \)

We define an \( \mathbb{A} \)-module \( K(Q_d) \) as follows: \( K(Q_d) \) is generated by symbols \( [P] \) for \( P \in Q_d \) subject to the relations \([P_1 \oplus P_2] = [P_1] + [P_2] \) and \([P[n]] = v^n[P]\). We also set \( K(Q) = \bigoplus_d K(Q_d) \), where by convention \( K(Q_0) = \mathbb{A} \). The Verdier duality induces a semilinear involution \([P] \mapsto [\overline{P}] = [D(P)] \). The \( \mathbb{A} \)-module \( K(Q_d_1 \boxtimes Q_d_2) \) is defined in a similar fashion, and there is a canonical isomorphism \( K(Q_{d_1} \boxtimes Q_{d_2}) \simeq K(Q_{d_1}) \otimes_{\mathbb{A}} K(Q_{d_2}) \). Note that the functors Ind and Res commute with direct sums and shifts, and hence induce \( \mathbb{A} \)-linear maps

\[ \text{ind}_{d_1 + d_2}^{d_1, d_2} : K(Q_{d_1}) \otimes_{\mathbb{A}} K(Q_{d_2}) \to K(Q_{d_1 + d_2}), \]

\[ \text{res}_{d_1 + d_2}^{d_1, d_2} : K(Q_{d_1 + d_2}) \to K(Q_{d_1}) \otimes_{\mathbb{A}} K(Q_{d_2}). \]

Summing up over \( d_1 \) and \( d_2 \) yields the maps \( m = \bigoplus_{d_1, d_2} \text{ind}_{d_1 + d_2}^{d_1, d_2} : K(Q) \otimes_{\mathbb{A}} K(Q) \to K(Q) \) and \( \Delta' = \bigoplus_{d_1, d_2} \text{res}_{d_1 + d_2}^{d_1, d_2} : K(Q) \to K(Q) \otimes_{\mathbb{A}} K(Q) \). Finally, we set \( \Delta = - \circ \Delta' \circ - \).

**Proposition 4.2.** The space \( (K(Q), m) \) is an associative algebra. Equip the tensor product \( K(Q) \otimes K(Q) \) with a twisted algebra structure by setting \((x \otimes y)(x' \otimes y') = v^{(\dim x', \dim y)}(xx' \otimes yy')\). Then \( \Delta \) is a morphism of algebras.
Proof. This can be proved exactly in the same way as in [11, Chapter 13]. □

Proposition 4.3. The set \{[P] \mid P \in \mathcal{P}_d\} is an $A$-basis of $\mathcal{K}(Q_d)$.

Proof. By definition, \{[P] \mid P \in \mathcal{P}_d\} is a generating set of $\mathcal{K}(Q_d)$ over $A$. We will show that these elements are linearly independent. We use the notations and results in the proof of Proposition 4.1. Let us call $x \in E_d$ regular if for each $i \in \mathcal{I}$, at least one of the nilpotent elements in $t(x)_i \in \text{Hom}(V_i, V_i)$ is regular. We denote by $E_d^{\text{reg}}$ the dense open subset of regular elements in $E_d$, and extend this notation to $\tilde{\mathcal{F}}_i$ and $\tilde{\mathcal{F}}_i^{\text{reg}}$. Recall that any of the simple perverse sheaves in $\mathcal{T}_i$ is of the form $IC(X, \mathcal{L})$ with $X = (u \times \text{Id})^{-1}(Y)$ for a smooth irreducible subvariety $Y \subset G^{\text{im}}_i$ and $\mathcal{L} = (u \times \text{Id})^* \mathfrak{R}$ for an irreducible local system $\mathfrak{R}$ on $Y$. Put $X^{\text{reg}} = X \cap \tilde{\mathcal{F}}_i^{\text{reg}}$.

There is a unique complete flag in a vector space compatible with a given regular nilpotent element. Hence $\pi_1 : \tilde{\mathcal{F}}_i^{\text{reg}} \rightarrow \pi_1(\tilde{\mathcal{F}}_i^{\text{reg}})$ is an isomorphism. Consequently, $\pi_1!(IC(X, \mathcal{L})) = IC(\pi_1(X^{\text{reg}}, \mathcal{L})) = IC(\pi_1(X^{\text{reg}})) + P'$, where $P'$ is a direct sum of simple perverse sheaves supported on $E_d^{\text{reg}}$.

Finally, assume that $\sum_i z_i [P_i] = 0$ is an $\mathbb{A}$-linear relation between elements $P_i = \pi_1!(Q_i) \in \mathcal{P}_d$. Restricting to the open set $E_d^{\text{reg}}$ we deduce a similar linear relation between the perverse sheaves $Q_i$. But these are simple perverse sheaves, hence $\mathbb{A}$-linearly independent. Therefore $z_i = 0$ for all $i$ as desired. □

Define a bilinear form on $\mathcal{K}(Q)$ by the formula

$$((P), (Q)) = \sum_{j \in \mathbb{Z}} D_j(D(P), D(Q))v^j$$

for all $P, Q \in \bigsqcup_d \mathcal{P}_d$. Then $(\ ,\ )$ is a Hopf pairing, i.e we have

$$(xy, z) = (x \otimes y, \Delta(z)) \quad \forall x, y, z \in \mathcal{K}(Q). \quad (4.4)$$

Assume $i \in \mathcal{I}$ and let $n \geq 1$ with $n = 1$ if $i$ is imaginary. It is easy to see that the space $E_{n_{\text{re}}}$ is a point, and that the constant sheaf $(\mathcal{O}_i)_{E_{n_{\text{re}}}}$ belongs to $\mathcal{P}_{n_{\text{re}}}$. The following theorem is our main result and will be proved in Section 5.

Theorem 4.1. The assignment

$$F_i^{(n)} \mapsto [(\mathcal{O}_i)_{E_{n_{\text{re}}}}] \quad \text{for} \quad i \in \mathcal{I}_{\text{re}}, n \geq 1,$$

$$F_i \mapsto [(\mathcal{O}_i)_{E_{\text{im}}}] \quad \text{for} \quad i \in \mathcal{I}_{\text{im}}$$
extends to an isomorphism of $\mathcal{A}$-algebras $\Theta : U^−_{\mathcal{A}}(g_{AQ, m}) \sim \mathcal{K}(Q)$ where $A_Q$ is the Cartan matrix associated to $Q$ (see Section 2.1) with $m_i = 1$ for all $i \in I$.

**Definition.** The set $B = \{\Theta^{-1}([P]) | P \in \bigsqcup_d \mathcal{P}_d\}$ is the canonical basis of $U^−_{\mathcal{A}}(g)$.

**Corollary 4.1.** If $b, b' \in B$, then $b \cdot b' = \sum_{c \in B} a_{cc} c$ with $a_{cc} \in \mathbb{N}[v, v^{-1}]$.

**5. Geometric realization of $U^−_v(g)$**

Note that each imaginary generator $F_j$ appears in each of the defining relations of $U^−_{\mathcal{A}}(g)$ with multiplicity at most one. In particular, all the relations that we need to check in order to show that the map $\Theta : U^−_{\mathcal{A}}(g_{AQ, m}) \to \mathcal{K}(Q)$ is well-defined occur in spaces $E_d$ with $d_j \in \{0, 1\}$ whenever $j \in I^m$. But in these cases, all maps associated to edge loops at an imaginary vertex are necessarily zero (by the nilpotency condition) and our construction coincides with Lusztig’s original construction for the quiver obtained from $Q$ by removing all edge loops. Hence, the relations (1.4) follow from [11, Theorem 13.2.11].

It now remains to check that the map $\Theta$ is an isomorphism. On generators one checks that $\Theta^∗(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ coincide (up to renormalization). Using Proposition 4.2 and the fact that both forms are Hopf pairings (see (4.4)), we deduce that $\Theta^∗(\cdot, \cdot)$ coincides (up to renormalization) with $\langle \cdot, \cdot \rangle$ on $U^−_{\mathcal{A}}(g)$. In particular, $\text{Ker} \Theta$ belongs to the radical of $\langle \cdot, \cdot \rangle$ which is trivial by Proposition 1.1. Thus $\Theta$ is injective.

In the rest of this section, we prove that $\Theta$ is surjective. We argue by induction on the dimension vector $d$. Assume that $[Q] \in \text{Im} \Theta$ for all $Q \in \mathcal{P}_d$ with $d'_j \leq d_j$ for all $j$ and $d'_k < d_k$ for at least one $k \in I$.

Let $k \in I$ be a sink (i.e., the only arrows leaving the vertex $k$ are edge loops). If $x \in E_d$, we put

$$n_k(x) = \text{codim} V_k \left( \frac{\mathbb{F}_q[x_{\sigma_1}, \ldots, x_{\sigma_{c_k}}]}{\sum_{j \neq k; j \rightarrow k} x_{\sigma_j}(V_j)} \right),$$

where $\sigma_1, \ldots, \sigma_{c_k}$ are the edge loops at $k$, and if $P$ is any complex, we set

$$n_k(P) = \inf_{x \in \text{supp}(P)} n_k(x).$$

**Proposition 5.1.** If $P \in \mathcal{P}_d$ is such that $n_k(P) > 0$, then $[P] \in \text{Im} \Theta$.

**Proof.** We will prove this by descending induction on $n_k(P)$. The statement is empty for $n_k(P) > 0$. Let us assume that it holds for all $Q$ with $n_k(Q) > n$ and let us fix some $P$ with $n_k(P) = n$. Choose $i$ and $R \in \mathcal{T}_i$ such that $P = \pi_{1!} R$. We
also set
\[ O = \{ x \in E_d | n_k(x) = n \}, \]
\[ O_i = \{ (x, D_o) \in \tilde{F}_i^{\text{im}} | x \in O \}. \]

We will first describe \( \text{Res}_{d-n_{\theta_k},n_{\theta_k}}^d (P) \). For this, fix an embedding \( V_{d-n_{\theta_k}} \subset V_d \) and consider the diagram

\[
\begin{array}{ccc}
\mathcal{O}_i & \xleftarrow{i_i} & \mathcal{O}_i' \\
\pi_i & \downarrow & \pi_1 \\
\mathcal{O} & \xleftarrow{j_i} & \mathcal{O}' \\
\downarrow & & \downarrow \\
E_d & \xleftarrow{i} & F \\
\kappa & \downarrow & \kappa \\
\mathcal{Y}_{d-n_{\theta_k}} \times E_{n_{\theta_k}} & \longrightarrow & \mathcal{Y}_{d-n_{\theta_k}} \times E_{n_{\theta_k}} \\
\end{array}
\]

where the notations are as follows: the bottom row is as in Section 4.3.; \( \mathcal{O}' = \mathcal{O} \cap F \) and \( \mathcal{O}'_i = \{ (x, D_o) \in \mathcal{O}_i | x \in F \} \);

\[
\mathcal{Y}_{d-n_{\theta_k}} = \left\{ x \in E_{d-n_{\theta_k}} \mid \bar{F}[x_{\sigma_1}, \ldots x_{\sigma_{\theta_k}}] \cdot \sum_{j \neq k, j \rightarrow k} x_{\sigma}(V_j) = (V_{d-n_{\theta_k}})_k \right\}
\]

and all maps are obvious ones (with \( p = \kappa \pi_1 \)). Note that all squares are cartesian. Let \( I : \mathcal{O}_i \rightarrow \tilde{F}_i^{\text{im}} \) be the inclusion. We have \( j^* \pi_1! (R) = \pi_1! l^* (R) \). Moreover, \( l^* (R) \) is perverse since \( \text{supp}(R) \subset \overline{O_i} \) and \( \text{supp}(R) \cap O_i \neq \emptyset \). On the other hand,

\[
\kappa_1 ! l^* \pi_1! (R) = \kappa_1 (j')^* i^* \pi_1! (R)
\]

\[
= \kappa_1 (i')^* j^* \pi_1! (R)
\]

\[
= \kappa_1 (i')^* \pi_1! l^* (R)
\]

\[
= \kappa_1 \pi_1! i_1^* l^* (R)
\]

\[
= p_1 i_1^* l^* (R).
\]

Let us denote for simplicity \( i_1^* l^* (R) \) by \( R' \). Let \( H \) be the parabolic subgroup of \( G_d \) stabilizing \( V_{d-n_{\theta_k}} \). We have \( O_i = G_d \times_H O_i' \) so that \( (i)^* : \mathcal{M}_{G_d}(O_i) \rightarrow \mathcal{M}_H(O_i') \) is an equivalence of categories. In particular, \( R'[\cdot \dim G/H] \) is perverse.

Set \( O_{\text{reg}} = \mathcal{O} \cap E_{\text{reg}}^d \) and \( O_{\text{reg}}' = (\pi_1)^{-1} (\mathcal{O}_{\text{reg}}) \). Note that from the proof of Proposition 4.3 it follows that \( \text{supp}(R) = X = (u \times \text{Id})^{-1} (Y) \) for some irreducible \( Y \subset O_{\text{reg}} \). Then
\(X^{\text{reg}}\) is open in \(X\) and thus \(X^{\text{reg}} \cap \mathcal{O}_1\) is open in \(X\). Thus \(R = \text{IC}(X^{\text{reg}} \cap \mathcal{O}_1)\) hence \(\text{supp}(l^* R) \subset \text{supp}(R) \subset \mathcal{O}_1\). Now, if \((x, D_{\bullet}) \in \mathcal{O}_1\) then \(D_{\bullet} \cap V_k\) is completely determined by \(x\). On the other hand, the subspace \(F_q[x_{\sigma_1}, \ldots, x_{\sigma_{c_k}}] \cdot \sum_{j \neq k, j \neq k} x_{\sigma}(V_j)\) is stable under \(x_{\sigma_i}\) for \(i = 1, \ldots, c_k\), so it has to belong to \(D_{\bullet} \cap V_k\). Define a closed subset of \(\mathcal{O}_1(i) \subset \mathcal{O}_1\) by the condition

\[
\overline{F}_q[x_{\sigma_1}, \ldots, x_{\sigma_{c_k}}] \cdot \sum_{j \neq k, j \neq k} x_{\sigma}(V_j) \in D_{\bullet} \cap V_k. \quad (*)
\]

Then we have \(\text{supp}(l^* R) \subset \mathcal{O}_1(i)\). We define \(\mathcal{O}_1'(i)\) similarly and we have \(\text{supp}(R') \subset \mathcal{O}_1'(i)\). Construct a sequence \(j\) by deleting the last \(d_k - n\) entries equal to \(k\) in \(i\) (so that if \((x, D_{\bullet}) \in \mathcal{O}_1'(i)\), then \(D_{\bullet} \cap V_{d-nk}\) is of type \(j\)). Now consider the diagram

\[
\begin{array}{cccccc}
\mathcal{O}_1(i) & \xrightarrow{\pi'_1} & Y_j \times \tilde{F}_{nk} & \xrightarrow{h'} & \tilde{F}_j \times \tilde{F}_{nk} \\
\mathcal{O}_1'(i) & \xrightarrow{\pi_1} & Y^\text{im}_j \times \tilde{F}^\text{im}_{nk} & \xrightarrow{h} & \tilde{F}^\text{im}_j \times \tilde{F}^\text{im}_{nk} \\
\mathcal{O}' & \xrightarrow{\kappa} & Y_{d-nk} \times E_{nk} & \xrightarrow{\tilde{f}} & E_{d-nk} \times E_{nk} \\
\end{array}
\]

with \(\mathcal{O}_1(i) = (\pi'_1)^{-1}(\mathcal{O}_1'(i))\), \(Y^\text{im}_j = \pi_1^{-1}(Y_{d-nk}) \subset \tilde{F}^\text{im}_j\), \(Y_j = (\pi'_1)^{-1}(Y^\text{im}_j)\) and with obviously defined maps (\(h\) and \(h'\) are open embeddings). Note that \(\kappa, \kappa', \kappa''\) are all vector bundles of rank \(c_k(d_k - n)n\) by the condition \((*)\) together with the fact that \(k\) is a sink. From this one deduces that the two squares in the above diagram are cartesian. By construction, \(R'\) appears in \(\pi'_{1!}((\mathbb{C}^k_1) \mathcal{O}_1(i)) = \pi'_{1!}(\kappa'^*)((\mathbb{C}^k_1) Y_j \times \tilde{F}_{nk}^k) = (\kappa'^*)^*\pi'_{1!}((\mathbb{C}^k_1) Y_j \times \tilde{F}_{nk}^k) = (\kappa'^*)^* \pi'_{1!}((\mathbb{C}^k_1) Y_j \times \tilde{F}_{nk}^k)\). In particular, there exists \(R'' \in \mathcal{M}(\tilde{F}^\text{im}_j \times \tilde{F}^\text{im}_{nk})[\dim G_{d}/H + c_k(d_k - n)n]\) such that \(R' = (\kappa'^*)^*h^*R''\). Hence \(\pi_{1!}R'' = \kappa'^*\pi_{1!}h^*R''\) and altogether we obtain

\[
\tilde{j}^* \text{Res}_{d-nk, nk}^d (P) = p_{1!}(R') = \kappa'^* \pi_{1!}h^*R'' = \pi_{1!}h^*(R'')[-2c_k(d_k - n)n],
\]

which is an object in \(\mathcal{M}(Y_{d-nk} \times E_{nk})[\dim G_{d}/H - c_k(d_k - n)n]\) by Proposition 4.1. Further, note that \(\pi_{1!}h^*(R'') = \tilde{j}^* \pi_{1!}(R'')\) and \(\pi_{1!}(R'') \in \mathcal{Q}_{d-nk} \boxtimes Q_{nk}\) by construction. Hence \(\pi_{1!}(R'') \in \Im \Theta \otimes \Im \Theta\) by our first induction hypothesis. On the other hand,
all squares in the following diagram are cartesian:

\[
\begin{array}{ccc}
  E_d & \xleftarrow{p_3} & E'' & \xrightarrow{p_2} & E' & \xrightarrow{p_1} & E_{d-n\ell_k} \times E_{n\ell_k} \\
  \downarrow j' & & \downarrow & & \downarrow j & & \\
  \mathcal{O}' & \xleftarrow{\text{Id}} & \mathcal{O}' & \xleftarrow{\mathcal{O}' \times G_{d-n\ell_k} \times GL(n)} & \Upsilon_{d-n\ell_k} \times E_{n\ell_k}
\end{array}
\]

It then follows from the definitions that

\[
(j')^* j^* \widetilde{\text{Ind}}_{d-n\ell_k,n\ell_k}^d (\pi_{1!} R'') = \kappa^* j^* \pi_{1!} (R'') = \kappa^* \pi_{1!} h^* (R'') = \pi_{1!} (R') = (i')^* j^* (P).
\]

Since \((i')^* : \mathcal{M}_{G_d}(\mathcal{O}_i) \to \mathcal{M}_H(\mathcal{O}_j) [\dim G_d/H]\) is fully faithful, we deduce

\[
j^* (P) = j^* \widetilde{\text{Ind}}_{d-n\ell_k,n\ell_k}^d (\pi_{1!} R'').
\] (5.1)

But it is obvious that \(\text{supp}(\widetilde{\text{Ind}}_{d-n\ell_k,n\ell_k}^d (\pi_{1!} R'')) \in \overline{\mathcal{O}}\). Thus from (5.1) we deduce that, in \(\mathcal{Q}_d\), \([P] = [\widetilde{\text{Ind}}_{d-n\ell_k,n\ell_k}^d (\pi_{1!} R'')] + [P']\), where \(P'\) has support in \(\overline{\mathcal{O}} \setminus \mathcal{O}\). Such complex necessarily satisfies \(n_k (P') > n\). Therefore, by our second induction hypothesis \([P'] \in \text{Im } \Theta\) and thus \([P] \in \text{Im } \Theta\). This completes the proof of Proposition 5.1. \(\square\)

The rest of the proof of Theorem 4.1 goes as in [11]. Namely, let \(P \in \mathcal{P}_1\) with \(i = (i_1, \ldots, i_l, k)\). Using the Fourier–Deligne transform, we may assume that \(k\) is a sink in our quiver. But then it is easy to see that \(n_k (P) > 0\) and from Proposition 5.1 it follows that \([P] \in \text{Im } \Theta\) as desired. \(\square\)

**Corollary 5.1.** (i) Fix \(k \in I\) and \(n \geq 1\). Set \(F^{(n)}_k = F^n_k\) if \(k \in I^\text{im}\). Then there exist subsets \(B_{\geq n,k}, B'_{\geq n,k} \subset B\) such that

\[
F^{(n)}_k U_{\mathcal{A}} = \bigoplus_{c \in B_{\geq n,k}} \mathcal{A} c, \quad U_{\mathcal{A}} F^{(n)}_k = \bigoplus_{c \in B'_{n,i}} \mathcal{A} c.
\]

(ii) For every \(b \in B\), there exists \(k\) such that \(b \in B_{\geq 1,k}\).
(iii) If \( b \in B \geq n,k \setminus B \geq n+1,k \), then there exists \( b' \in B \) such that

\[
F_k^{(n)} b' \in b \oplus \bigoplus_{c \in B \geq n+1,k} \mathbb{A}c.
\]

Proof. We use the same notations as in the proof of Theorem 4.1. In (i), we only prove the statement concerning the space \( U^+_{F_k^{(n)}} \). The other part is proved in an analogous way. Note that \( P_{n_k} = \{ P_{n,k} \} \) consists of a single element. Let \( x \in U^+_{F_k^{(n)}} \). Write \( \Theta(x) = \sum x_P [P] \), so that \( \Theta(x F_k^{(n)}) = \sum x_P [\text{Ind}_{d,n_k}^d (P \boxtimes P_{n,k})] \). By construction, it is clear that \( n_k (\text{Ind}_{d,n_k}^d (P \boxtimes P_{n,k})) \geq n \) for any \( P \). Thus

\[
\Theta(U^+_{F_k^{(n)}}) \subset \bigoplus_{n_k(Q) \geq n} \mathbb{A}[Q].
\]

We now prove the opposite inclusion. Fix \( d \in \mathbb{N}^I \). We argue by descending induction on \( n_k(Q) \). The statement is empty for \( n_k(Q) > d_k \). Fix \( R \in P_d \) with \( n_k(P) > 0 \) and assume that \( [Q] \in \Theta(U^+_{F_k^{(n_k(Q))}}) \) for all \( Q \in P_d \) such that \( n_k(Q) > n_k(R) \). The proof of Theorem 4.1 shows the existence of \( T \in P \) such that

\[
[R] = [T] \cdot [P_{n_k(R),k}] + [P'] \quad \text{with} \quad n_k(P') > n_k(R).
\]

By induction \( [P'] \in \Theta(U^+_{F_k^{(n_k(P'))}}) \subset \Theta(U^+_{F_k^{(n_k(R))}}) \) and we are done.

The statement (ii) follows from the Fourier–Deligne transform (see the end of the proof of Theorem 4.1). The statement (iii) follows from (5.2) and the dual version of (5.2). □

Corollary 5.2. For any \( b \in B \) we have \( \overline{b} = b \). Assume that \( c_i > 1 \) for all \( i \in I^{im} \). Then the following holds:

\[
\forall b \in B, \quad \langle b, b \rangle \in 1 + vZ[v], \quad \text{(5.3)}
\]

\[
\forall b \neq b' \in B, \quad \langle b, b' \rangle \in vZ[v]. \quad \text{(5.4)}
\]

Proof. The first statement is proved in the same way as in [7, §13]. When \( c_i > 1 \) for all \( i \in I^{im} \) the elements of \( \sqcup_d P_d \) are all simple perverse sheaves by Proposition 4.1. The relations (5.3) and (5.4) then follow from [11, §8.1.10] together with the fact that the forms \( \Theta^* (, , ) \) and \( \langle , , \rangle \) differ (on each weight space) by a factor in \( 1 + vZ[v] \). □

Proposition 5.2. Let \( B \) be the global basis of \( U^-_{F_k^{(g)}} \) defined in [8]. Following [8] we denote the elements of \( B \) by \( G(\beta) \). Then the following statements hold:
For every $G(\beta) \in B$ we have $G(\overline{\beta}) = G(\beta)$.

Assume that $c_i > 1$ for all $i \in I^m$. Then we have

$$
\forall G(\beta) \in B \quad \langle G(\beta), G(\beta) \rangle \in 1 + v\mathbb{Z}[v],
$$

$$
\forall G(\beta) \neq G(\beta') \in B \quad \langle G(\beta), G(\beta') \rangle \in v\mathbb{Z}[v].
$$

For every $i \in I$ and $n \geq 1$, there exists a subset $B_{\geq n,i} \subset B$ such that

$$
F_{i}^{(n)} U_{\Lambda_i}^{-} \bigoplus_{G(\gamma) \in B_{\geq n,i}} \mathbb{A} G(\gamma).
$$

For any $G(\beta) \in B_{\geq n,i} \setminus B_{\geq n+1,i}$ there exists $G(\beta') \in B$ such that

$$
F_{i}^{(n)} G(\beta') \in G(\beta) \bigoplus_{G(\gamma) \in B_{\geq n+1,i}} \mathbb{A} G(\gamma).
$$

**Proof.** The first three statements can be found in [8], in Sections 9.3, 7.38 and 10.2, respectively. We prove the last statement. Let $(L_\infty, B_\infty)$ denote the crystal basis of $U_{\Lambda_i}(q)$ defined in [8] and let $\tilde{e}_i, \tilde{f}_i, e'_i$ stand for the Kashiwara operators. Let $G(\beta) \in B_{\geq n,i} \setminus B_{\geq n+1,i}$. Denote by $\beta \in B_\infty$ the corresponding element in the crystal graph. Then there exists $\beta_0$ in $B_\infty$ such that $\tilde{e}_i \beta_0 = 0$ and $\tilde{f}_i \beta_0 = \beta$. Let $G(\beta_0) \in B \setminus B_{\geq 1,i}$ be the global basis element associated to $\beta_0$. By (iii), there exists $G(\beta_i) \in B$ and $a_i \in \mathbb{A}$ such that

$$
F_{i}^{(n)} G(\beta_0) \equiv \sum_i a_i G(\beta_i) \pmod{F_{i}^{(n+1)} U_{\Lambda_i}^{-}(g)}.
$$

Moreover, we have $\overline{a_i} = a_i$ by invariance of $G(\beta_0), G(\beta_i)$ under the bar involution. Let $P_i$ denote the projection of $U_{\Lambda_i}^{-}(g) = \ker e'_i \oplus F_i U_{\Lambda_i}^{-}(g)$ onto $\ker e'_i$. We have

$$
F_{i}^{(n)} G(\beta_0) \equiv F_{i}^{(n)} P_i G(\beta_0) \pmod{F_{i}^{(n+1)} U_{\Lambda_i}^{-}(g)}
$$

$$
\equiv \tilde{f}_i^n P_i G(\beta_0) \pmod{F_{i}^{(n+1)} U_{\Lambda_i}^{-}(g)}
$$

$$
\equiv \tilde{f}_i^n G(\beta_0) \pmod{F_{i}^{(n+1)} U_{\Lambda_i}^{-}(g)}
$$

$$
\equiv G(\beta) \pmod{F_{i}^{(n+1)} U_{\Lambda_i}^{-}(g) \oplus vL_\infty}.
$$

Thus we deduce that $a_i \equiv 1 \pmod{v\mathbb{Z}[v]}$ if $\beta_i = \beta$ and $a_i \in v\mathbb{Z}[v]$ otherwise. But from $a_i = \overline{a_i}$ it now follows that $a_i = 1$ for $\beta_i = \beta$ and $a_i = 0$ otherwise. \[\square\]

From Corollaries 5.1, 5.2 and Proposition 5.2 we deduce, using the method in [7], the following result.
Theorem 5.1. If \( c_i > 1 \) for all \( i \in I^{im} \) then \( B \) coincides with the global basis \( B \).

We conjecture that Theorem 5.1 holds unconditionally.

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