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The threshold effects for a family of Friedrichs models under rank one perturbations

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Abstract

A family of Friedrichs models under rank one perturbations $h_\mu(p)$, $p \in (-\pi, \pi]^3$, $\mu > 0$, associated to a system of two particles on the three-dimensional lattice $\mathbb{Z}^3$ is considered. We prove the existence of a unique eigenvalue below the bottom of the essential spectrum of $h_\mu(p)$ for all non-trivial values of p under the assumption that $h_\mu(0)$ has either a threshold energy resonance (virtual level) or a threshold eigenvalue. The threshold energy expansion for the Fredholm determinant associated to a family of Friedrichs models is also obtained.

Keywords: Family of Friedrichs models; Eigenvalues; Energy resonance; Pair non-local potentials; Conditionally negative definite functions

1. Introduction

In the present paper we consider a family of Friedrichs models under rank one perturbations associated to a system of two particles on the lattice $\mathbb{Z}^3$ interacting via pair non-local potentials.

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The main goal of the paper is to give a thorough mathematical treatment of the spectral properties of a family of Friedrichs models in dimension three with emphasis on threshold energy expansions for the Fredholm determinant associated to the family (see, e.g., [1–4,11,18,27,28,30] for relevant discussions and [15,23,32] for the general study of the low-lying excitation spectrum for quantum systems on lattices).

These kind of models have been discussed in quantum mechanics [9,12], solid physics [13,22,24,25] and in lattice field theory [19–21].

Threshold energy resonances (virtual levels) for the two-particle Schrödinger operators have been studied in [1,3,4,14,18,30]. Threshold energy expansions for the resolvent of two-particle Schrödinger operators have been studied in [4,8,14,17,18,27,28,30] and have been applied to the proof of the existence of Efimov's effect in [4,18,27–29].

Similarly to the lattice Schrödinger operators and in contrast to the continuous Schrödinger operators the family of Friedrichs models $h_\mu(p)$, $p \in (-\pi, \pi]^3$, $\mu > 0$, depends parametrically on the internal binding p, the quasi-momentum, which ranges over a cell of the dual lattice and hence it has spectral properties analogous to those of lattice Schrödinger operators.

Let us recall that the spectrum and resonances of the original Friedrichs model and its generalizations have been studied and the finiteness of the eigenvalues lying below the bottom of the essential spectrum has been proven in [9,12,16,31].

In [19,20] a peculiar family of Friedrichs models was considered and the appearance of eigenvalues for values of the total quasi-momentum $p \in (-\pi, \pi]^d$, $d = 1, 2$, of the system lying in a neighborhood of some particular values of the parameter p has been proven.

For a wide class of the two-particle Schrödinger operators only the existence of eigenvalues of $h_\mu(p)$, $p \in (-\pi, \pi]^3$, for all non-zero values of the quasi-momentum $0 \neq p \in \mathbb{T}^3$ (under the assumption that $h(0)$ has either a zero energy resonance or a zero eigenvalue) has been proven in [3].

In the present paper two main results.

First of them gives the existence of a unique eigenvalue $e_\mu(p)$ of $h_\mu(p)$, $p \in (-\pi, \pi]^3$, for all non-zero values of the quasi-momentum $0 \neq p \in \mathbb{T}^3$ (provided that $h_\mu(0)$ has either a threshold energy resonance or a threshold eigenvalue) and lower and upper bounds on it. The monotonous dependence of the eigenvalue $e_\mu(p)$ on μ (Theorem 2.15).

The second one presents an expansion for the Fredholm determinant respectively the Birman–Schwinger operator in powers of the quasi-momentum $p \in (-\pi, \pi]^3$ in a small δ-neighborhood of the origin and proving that the Fredholm determinant respectively the Birman–Schwinger operator has a differentiable continuation to the bottom of the essential spectrum of $h_\mu(p)$ as a function of $w = (m - z)^{1/2} \geqslant 0$ for $z \leqslant m$, where $z \in \mathbb{R}^1$ is the spectral parameter (Theorem 2.16).

We notice that if the functions u respectively φ are analytic on $(\mathbb{T}^3)^2$ respectively $\mathbb{T}^3$, then one can obtain a precise expansion for the Fredholm determinant and the Birman–Schwinger operator (see [14,17]).

The structure of the paper is as follows. In Section 2 we state the problem and present the main results. Proofs are presented in Section 4 and are based on a series of lemmas in Section 3. In Appendix A for an important subclass of the family of Friedrichs models we shall show that all assumptions in Section 2 are fulfilled.

Throughout the present paper we adopt the following conventions: $\mathbb{T}^3$ denotes the three-dimensional torus, the cube $(-\pi, \pi]^3$ with appropriately identified sides. For each $\delta > 0$ the notation $U_\delta(0) = \{p \in \mathbb{T}^3\colon |p| < \delta\}$ stands for a sufficiently small δ-neighborhood of the origin. Denote by $L_2(\mathbb{T}^3)$ the Hilbert space of square-integrable functions on $\mathbb{T}^3$.

Let $\mathcal{B}(\theta, U_\delta(0))$ with $1/2 < \theta \leqslant 1$, be the Banach spaces of Hölder continuous functions on $\overline{U_\delta(0)}$ with exponent θ obtained by the closure of the space of smooth functions f on $U_\delta(0)$ with respect to the norm

$$\|f\|_\theta = \sup_{\substack{t,\ell \in U_\delta(0)\\ t\neq \ell}} \big[\big|f(t)\big| + |t-\ell|^{-\theta}\big|f(t)-f(\ell)\big|\big].$$

The set of functions $f:\mathbb{T}^3 \to \mathbb{R}$ having continuous partial derivatives up to order $\leqslant n$ will be denoted by $C^{(n)}(\mathbb{T}^3)$. In particular $C^{(0)}(\mathbb{T}^3) = C(\mathbb{T}^3)$.

2. The model operator $h_\mu(p)$, main assumptions and statement of the results

Let u be a real-valued essentially bounded function on $(\mathbb{T}^3)^2$ and φ be a real-valued function in $L_2(\mathbb{T}^3)$. Let μ be a positive real number.

We introduce the following family of bounded self-adjoint operators (the Friedrichs model) $h_\mu(p)$, $p \in \mathbb{T}^3$, acting in $L_2(\mathbb{T}^3)$ by

$$h_\mu(p) = h_0(p) - \mu v,$$

where

$$\big(h_0(p)f\big)(q) = u(q,p)f(q), \quad f \in L_2\big(\mathbb{T}^3\big),$$

and v is non-local interaction operator

$$(vf)(q) = \varphi(q)\int_{\mathbb{T}^3} \varphi(t)f(t)\,dt, \quad f \in L_2\big(\mathbb{T}^3\big).$$

Remark 2.1. In the case where the function u is of the form

$$u(p,q) = \varepsilon(p) + \varepsilon(p-q) + \varepsilon(q), \tag{2.1}$$

the operator $h_0(p)$ is associated to a system of two particles (bosons) moving on the three-dimensional lattice $\mathbb{Z}^3$ and is called the *free Hamiltonian*, where $\varepsilon(\cdot)$ is the *dispersion relations of normal modes* associated with the free particle in question.

Throughout this paper we assume the following additional hypotheses.

Assumption 2.2.

(i) The function u is even on $(\mathbb{T}^3)^2$ with respect to (p,q), and has a unique non-degenerate minimum at the point $(0,0) \in (\mathbb{T}^3)^2$ and all third order partial derivatives of u are continuous on $(\mathbb{T}^3)^2$ and their restrictions in $(U_\delta(0))^2$ belong to $\mathcal{B}(\theta, (U_\delta(0))^2)$.

(ii) For some positive definite matrix U and real numbers l, l_1, l_2 ($l_1, l_2 > 0, l \neq 0$) the following equalities hold

$$\left(\frac{\partial^2 u(0,0)}{\partial p^{(i)}\partial p^{(j)}}\right)_{i,j=1}^3 = l_1 U, \qquad \left(\frac{\partial^2 u(0,0)}{\partial p^{(i)}\partial q^{(j)}}\right)_{i,j=1}^3 = lU,$$

$$\left(\frac{\partial^2 u(0,0)}{\partial q^{(i)}\partial q^{(j)}}\right)_{i,j=1}^3 = l_2 U.$$

Remark 2.3. The function u is even and has a unique non-degenerate minimum on $\mathbb{T}^3$ and hence without loss of generality we assume that the function u has a unique minimum at the point $(0,0) \in (\mathbb{T}^3)^2$.

Remark 2.4. It is easy to see that Assumption 2.2 implies the inequality $l_1 l_2 > l^2$.

Assumption 2.5. The continuous function φ is either even or odd on $\mathbb{T}^3$ and all second order partial derivatives of φ are continuous on $\mathbb{T}^3$.

Let $\mathbb{C}$ be the field of complex numbers and set

$$u_p(q) = u(p,q), \qquad m = \min_{p,q\in\mathbb{T}^3} u(p,q),$$
$$u_{\min}(p) = \min_{q\in\mathbb{T}^3} u_p(q), \qquad u_{\min}(p) = \max_{q\in\mathbb{T}^3} u_p(q)$$

and

$$\Lambda(p,z) = \int\limits_{\mathbb{T}^3} \frac{\varphi^2(t)\,dt}{u_p(t) - z}, \quad p \in \mathbb{T}^3,\ z \in \mathbb{C} \setminus \big[u_{\min}(p), u_{\max}(p)\big]. \tag{2.2}$$

Remark 2.6. By part (i) of Assumption 2.2 all third order partial derivatives of the function $\Lambda(\cdot,z)$, $z < m$, belong to $C^{(2)}(\mathbb{T}^3)$.

The function $\Lambda(p,\cdot)$, $p \in \mathbb{T}^3$, is increasing in $(-\infty, u_{\min}(p))$ and hence the following finite or infinite positive limit exists

$$\lim_{z\to u_{\min}(p)-0} \Lambda(p,z) = \Lambda\big(p, u_{\min}(p)\big). \tag{2.3}$$

Remark 2.7. Since for any $p \in U_\delta(0)$, $\delta > 0$ sufficiently small, the function $u_p(\cdot)$ has a unique non-degenerate minimum in $\mathbb{T}^3$ (see part (i) of Lemma 3.4) Lebesgue's dominated convergence theorem yields the equality

$$\Lambda\big(p, u_{\min}(p)\big) = \int\limits_{\mathbb{T}^3} \frac{\varphi^2(t)\,dt}{u_p(t) - u_{\min}(p)}, \quad p \in U_\delta(0).$$

The perturbation v of the multiplication operator $h_0(p)$ is of rank one and hence, in accordance with Weyl's theorem the essential spectrum of the operator $h_\mu(p)$ fills the following interval on the real axis:

$$\sigma_{\mathrm{ess}}\big(h_\mu(p)\big) = \big[u_{\min}(p), u_{\max}(p)\big].$$

Remark 2.8. We remark that for some $p \in \mathbb{T}^3$ the essential spectrum of $h_\mu(p)$ may degenerate to the set consisting of the unique point $[u_{\min}(p), u_{\min}(p)]$. Because of this we cannot state that the essential spectrum of $h_\mu(p)$ is absolutely continuous for any $p \in \mathbb{T}^3$. This is the case, e.g., for a function u of the form (2.1), where $p = (\pi,\pi,\pi) \in \mathbb{T}^3$, and

$$\varepsilon(q) = 3 - \cos q_1 - \cos q_2 - \cos q_3, \quad q = (q_1, q_2, q_3) \in \mathbb{T}^3. \tag{2.4}$$

Definition 2.9. Assume part (i) of Assumption 2.2 and $\varphi \in C^{(0)}(\mathbb{T}^3)$. The operator $h_\mu(0)$ is said to have a threshold energy resonance if the number 1 is an eigenvalue of the operator

$$(G\psi)(q) = \mu\varphi(q) \int_{\mathbb{T}^3} \frac{\varphi(t)\psi(t)\,dt}{u_0(t)-m}, \quad \varphi \in C^{(0)}\big(\mathbb{T}^3\big),$$

and the associated eigenfunction ψ (up to constant factor) satisfies the condition $\psi(0) \neq 0$.

Remark 2.10. Assume part (i) of Assumption 2.2 and Assumption 2.5.

(i) If $\varphi(0) \neq 0$ and the operator $h_\mu(0)$ has a threshold energy resonance, then the function

$$f(q) = \frac{\varphi(q)}{u_0(q)-m} \tag{2.5}$$

obeys the equation $h_\mu(0)f = mf$ and $f \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$ (see Lemma 3.2).

(ii) If $\varphi(0) = 0$ and the threshold $z = m$ is an eigenvalue of the operator $h_\mu(0)$, then the function f, defined by (2.5), obeys the equation $h_\mu(0)f = mf$ and $f \in L_2(\mathbb{T}^3)$ (see Lemma 3.3).

Set

$$\mu_0 = \Lambda^{-1}(0, m).$$

Remark 2.11. Notice that the conditions $\mu = \mu_0$ and $\varphi(0) \neq 0$ (respectively $\mu = \mu_0$ and $\varphi(0) = 0$) mean that the operator $h_\mu(0)$ has threshold energy resonance (see Lemma 3.2) (respectively a threshold eigenvalue of $h_\mu(0)$ (see Lemma 3.3)).

Remark 2.12. We note that the bottom $z = m$ of the essential spectrum $\sigma_{\mathrm{ess}}(h_{\mu_0}(0))$ of $h_{\mu_0}(0)$ is either a threshold energy resonance or an eigenvalue for the operator $h_{\mu_0}(0)$.

In order to study the spectral properties of $h_\mu(p)$ precisely we assume the following

Assumption 2.13. Assume that:

(i) The function $\Lambda(\cdot, u_{\min}(\cdot))$ has a unique minimum at the origin, i.e., for all $0 \neq p \in \mathbb{T}^3$ the following inequality holds:

$$\Lambda\big(p, u_{\min}(p)\big) - \Lambda\big(0, u_{\min}(0)\big) > 0.$$

(ii) The function $\Lambda(\cdot, m)$ has a unique maximum at the origin such that for some $c > 0$ the following inequality holds:

$$\Lambda(0, m) - \Lambda(p, m) > c|p|^2, \quad 0 \neq p \in U_\delta(0).$$

Remark 2.14. If for all $0 \neq p \in \mathbb{T}^3$ and a.e. $q \in \mathbb{T}^3$ the inequality

$$u_p(q) - u_{\min}(p) < u_0(q) - u_{\min}(0)$$

holds, then part (i) of Assumption 2.13 is obviously fulfilled. In Appendix A we shall show that for the functions of the form (2.1) Assumption 2.13 is fulfilled.

The following theorem presents a result characteristic for the two-particle Hamiltonians on lattices (see [3]).

Theorem 2.15. *Assume Assumptions* 2.2, 2.5 *and* 2.13. *Then*

(i) *For all* $p \in \mathbb{T}^3 \setminus \{0\}$ *the operator* $h_{\mu_0}(p)$ *has a unique eigenvalue* $e_{\mu_0}(p)$. *One has*

$$m < e_{\mu_0}(p) < u_{\min}(p), \quad 0 \neq p \in \mathbb{T}^3.$$

(ii) *For any* $\mu > \mu_0$ *the operator* $h_\mu(p)$, $p \in \mathbb{T}^3$, *has a unique eigenvalue* $e_\mu(p)$. *One has*

$$e_\mu(p) < e_{\mu_0}(p) < u_{\min}(p), \quad 0 \neq p \in \mathbb{T}^3,$$

and

$$e_\mu(0) < m.$$

For any $p \in \mathbb{T}^3$ we define an analytic function $\Delta_\mu(p, \cdot)$ (the Fredholm determinant associated to the operator $h_\mu(p)$) in $\mathbb{C} \setminus [u_{\min}(p), u_{\max}(p)]$ by

$$\Delta_\mu(p, z) = 1 - \mu \int_{\mathbb{T}^3} \big(u(p, t) - z\big)^{-1} \varphi^2(t)\, dt.$$

Now we formulate a result (threshold energy expansion for the Fredholm determinant) of the paper, which is important in the spectral analysis for a model operator associated to a system of three particles on the lattice $\mathbb{Z}^3$ [5].

Theorem 2.16. *Assume Assumptions* 2.2 *and* 2.5.

For any $z < u_{\min}(p)$ *the function* $\Delta_\mu(\cdot, z)$ *is of class* $C^{(2)}(\mathbb{T}^3)$ *and the following decomposition*

$$\Delta_\mu(p, z) = \Delta_\mu(0, m) + \frac{2\sqrt{2}\pi^2 \mu \varphi^2(0)}{l_1^{3/2} \det(U)^{1/2}} \sqrt{u_{\min}(p) - z} + \Delta_\mu^{\mathrm{res}}\big(u_{\min}(p) - z\big) + \Delta_\mu^{\mathrm{res}}(p, z), \quad z \leqslant u_{\min}(p),\ p \in U_\delta(0),$$

holds, where $\Delta_\mu^{\mathrm{res}}(u_{\min}(p) - z) = O((u_{\min}(p) - z)^{(1+\theta)/2})$ *as* $z \to u_{\min}(p)$ $(z < u_{\min}(p))$ *and* $\Delta_\mu^{\mathrm{res}}(p, z) = O(p^2)$ *as* $p \to 0$ *uniformly in* $z \leqslant u_{\min}(p)$.

Remark 2.17. An analogue result has been proven in [4] in the case where $\varphi(\cdot) \equiv \mathrm{const}$ and the function $u(\cdot, \cdot)$ is of the form (2.1), (2.4).

Corollary 2.18. *Assume Assumptions* 2.2 *and* 2.5.

(i) *Let the operator* $h_{\mu_0}(0)$ *have a threshold energy resonance. Then for all* $p \in U_\delta(0)$ *and* $z \leqslant m$ *the following decomposition holds*

$$\Delta_{\mu_0}(p, z) = \frac{4\sqrt{2}\pi^2 \mu_0 \varphi^2(0)}{l_1^{3/2} \det(U)^{1/2}} \sqrt{u_{\min}(p) - z} + \Delta_{\mu_0}^{\mathrm{res}}\big(u_{\min}(p) - z\big) + \Delta_{\mu_0}^{\mathrm{res}}(p, z).$$

(ii) *Let the threshold $z = m$ be an eigenvalue of $h_{\mu_0}(0)$. Then for any $p \in U_\delta(0)$ and $z \leqslant m$ the following decomposition holds*

$$\Delta_{\mu_0}(p, z) = \Delta^{\mathrm{res}}_{\mu_0}\big(u_{\min}(p) - z\big) + \Delta^{\mathrm{res}}_{\mu_0}(p, z).$$

Remark 2.19. We see that Corollary 2.18 gives a threshold energy expansions for the Fredholm determinant, leading to different behaviors for a threshold energy resonance respectively a threshold eigenvalue.

The following Corollary 2.20 (respectively Corollary 2.21) plays a crucial role in the proof of the infiniteness (respectively finiteness) of the number of eigenvalues lying below the bottom of the essential spectrum for a model operator associated to a system of three particles on three-dimensional lattice $\mathbb{Z}^3$ [5].

Corollary 2.20. *Assume Assumptions* 2.2 *and* 2.5. *Let the operator $h_{\mu_0}(0)$ have a threshold energy resonance. Then for some $c_1, c_2 > 0$ the following inequalities hold*

$$c_1|p| \leqslant \Delta_{\mu_0}(p, m) \leqslant c_2|p|, \quad p \in U_\delta(0), \tag{2.6}$$

$$\Delta_{\mu_0}(p, m) \geqslant c_1, \quad p \in \mathbb{T}^3 \setminus U_\delta(0). \tag{2.7}$$

Corollary 2.21. *Assume Assumptions* 2.2, 2.5 *and* 2.13. *Let $z = m$ be an eigenvalue of $h_{\mu_0}(0)$. Then for some $c > 0$ the following inequality holds*

$$\Delta_{\mu_0}(p, m) \geqslant cp^2, \quad p \in U_\delta(0).$$

3. Spectral properties of the operator $h_\mu(p)$

In this section we study some spectral properties of the family $h_\mu(p)$, $p \in \mathbb{T}^3$, with emphasis on the threshold energy resonance and a threshold eigenvalue.

Denote by $r_0(p, z) = (h_0(p) - zI)^{-1}$, where I is identity operator on $L_2(\mathbb{T}^3)$, the resolvent of the operator $h_0(p)$, that is, the multiplication operator by the function

$$\big(u_p(\cdot) - z\big)^{-1}, \quad z \in \mathbb{C} \setminus \big[u_{\min}(p), u_{\max}(p)\big].$$

From the equality (2.3) we have

$$\Delta_\mu(p, m) = \lim_{z \to m-0} \Delta_\mu(p, z), \quad p \in U_\delta(0).$$

Lemma 3.1. *Assume part* (i) *of Assumption* 2.2 *and $\varphi \in C^{(0)}(\mathbb{T}^3)$. For any $\mu > 0$ and $p \in \mathbb{T}^3$ the following statements are equivalent*:

(i) *The operator $h_\mu(p)$ has an eigenvalue $z \in \mathbb{C} \setminus [u_{\min}(p), u_{\max}(p)]$ below the bottom of the essential spectrum.*
(ii) $\Delta_\mu(p, z) = 0$, $z \in \mathbb{C} \setminus [u_{\min}(p), u_{\max}(p)]$.
(iii) $\Delta_\mu(p, z') < 0$ *for some* $z' \leqslant u_{\min}(p)$.

Proof. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (ii). From the positivity of v it follows that the positive square root of v exists, we shall denote it by $v^{1/2}$. For any $\mu > 0$ and $p \in \mathbb{T}^3$ the number

$z \in \mathbb{C} \setminus [u_{\min}(p), u_{\max}(p)]$ is an eigenvalue of $h_\mu(p)$ if and only if $\lambda = 1$ is an eigenvalue of the operator

$$G_\mu(p,z) = \mu v^{\frac{1}{2}} r_0(p,z) v^{\frac{1}{2}}$$

(this follows from the Birman–Schwinger principle).

Since the operator $v^{1/2}$ is of the form

$$\big(v^{\frac{1}{2}} f\big)(q) = \|\varphi\|^{-1}\varphi(q) \int\limits_{\mathbb{T}^3} \varphi(t) f(t)\,dt, \qquad f \in L_2\big(\mathbb{T}^3\big),$$

the operator $G_\mu(p,z)$ has the form

$$\big(G_\mu(p,z) f\big)(q) = \frac{\mu \Lambda(p,z)}{\|\varphi\|^2}\varphi(q) \int\limits_{\mathbb{T}^3} \varphi(t) f(t)\,dt, \qquad f \in L_2\big(\mathbb{T}^3\big),$$

where $\Lambda(p,z)$ is defined by (2.2).

According to the Fredholm theorem the number $\lambda = 1$ is an eigenvalue of the operator $G_\mu(p,z)$ if and only if $\Delta_\mu(p,z) = 0$. (i) $\Leftrightarrow$ (ii) is proven.

(ii) $\Leftrightarrow$ (iii). Let $\Delta_\mu(p,z_0) = 0$ for some $z_0 \in \mathbb{C} \setminus [u_{\min}(p), u_{\max}(p)]$. The operator $h_\mu(p)$ is self-adjoint and hence by (i) $\Leftrightarrow$ (ii) we conclude that z_0 is real. For all $z > u_{\max}(p)$ we have $\Delta_\mu(p,z) > 1$ and hence $z_0 \in (-\infty, u_{\min}(p))$. Since for any $p \in \mathbb{T}^3$ the function $\Delta_\mu(p,\cdot)$ is decreasing in $z \in (-\infty, u_{\min}(p))$ we have $\Delta_\mu(p,z') < \Delta_\mu(p,z_0) = 0$ for some $z_0 < z' \leqslant u_{\min}(p)$.

(iii) $\Leftrightarrow$ (ii). Suppose that $\Delta_\mu(p,z') < 0$ for some $z' \leqslant u_{\min}(p)$. For any $p \in \mathbb{T}^3$ we have $\lim_{z\to-\infty} \Delta_\mu(p,z) = 1$, $\Delta_\mu(p,\cdot)$ is continuous in $z \in (-\infty, u_{\min}(p))$ and hence there exists $z_0 \in (-\infty, z')$ such that $\Delta_\mu(p,z_0) = 0$. This completes the proof. $\square$

The following lemmas describe whether the bottom of the essential spectrum of $h_{\mu_0}(0)$ is threshold energy resonance or a threshold eigenvalue.

Lemma 3.2. *Assume part* (i) *of Assumption* 2.2 *and* $\varphi \in \mathcal{B}(\theta, \mathbb{T}^3)$, $\frac{1}{2} < \theta \leqslant 1$. *The following statements are equivalent*:

(i) *The operator* $h_\mu(0)$ *has a threshold energy resonance and*

$$\varphi(q)\big(u_0(q) - m\big)^{-1} \in L_1\big(\mathbb{T}^3\big) \setminus L_2\big(\mathbb{T}^3\big). \tag{3.1}$$

(ii) $\varphi(0) \neq 0$ *and* $\Delta_\mu(0,m) = 0$.

(iii) $\varphi(0) \neq 0$ *and* $\mu = \mu_0$.

Proof. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Let the operator $h_\mu(0)$ have threshold energy resonance for some $\mu > 0$. Then by Definition 2.9 the equation

$$\psi(q) = (G\psi)(q), \qquad \psi \in C\big(\mathbb{T}^3\big), \tag{3.2}$$

has a simple solution $\psi(\cdot)$ in $C(\mathbb{T}^3)$, such that $\psi(0) \neq 0$.

This solution is equal to the function φ (up to a constant factor) and hence $\Delta_\mu(0,m) = 0$ and so $\mu = \mu_0$.

Let $\varphi(0) \neq 0$ and $\mu = \mu_0$, hence the equality $\Delta_\mu(0,m) = 0$ holds. Then the function φ is a solution of Eq. (3.2), that is, the operator $h_\mu(0)$ has a threshold energy resonance. Since $u_0(\cdot)$ has a unique non-degenerate minimum at $p = 0 \in \mathbb{T}^3$ and $\varphi(0) \neq 0$ the inclusion (3.1) holds. $\square$

Lemma 3.3. *Assume part* (i) *of Assumption* 2.2 *and* $\varphi \in \mathcal{B}(\theta, \mathbb{T}^3)$, $\frac{1}{2} < \theta \leqslant 1$. *The following statements are equivalent*:

(i) *The operator* $h_\mu(0)$ *has a threshold eigenvalue.*
(ii) $\varphi(0) = 0$ *and* $\Delta_\mu(0, m) = 0$.
(iii) $\varphi(0) = 0$ *and* $\mu = \mu_0$.

Proof. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Suppose $f \in L_2(\mathbb{T}^3)$ is an eigenfunction of the operator $h_\mu(0)$ associated with the eigenvalue m. Then

$$\big(u_0(q) - m\big) f(q) - \mu\varphi(q) \int_{\mathbb{T}^3} \varphi(t) f(t)\, dt = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \varphi(t) f(t)\, dt \neq 0. \tag{3.3}$$

Hence (3.3) yields $\varphi(0) = 0$. We find that f, except for an arbitrary factor, is given by

$$f(q) = \big(u_0(q) - m\big)^{-1} \varphi(q). \tag{3.4}$$

Thus (3.3) implies $\Delta_\mu(0, m) = 0$ and $\mu = \mu_0$.

Let $\varphi(0) = 0$ and $\mu = \mu_0$, then $\Delta_\mu(0, m) = 0$ and the function f, defined by (3.4), obeys the equation $h_\mu(0) f = mf$. Since $u_0(\cdot)$ has a unique non-degenerate minimum at $p = 0 \in \mathbb{T}^3$ and $\varphi(0) = 0$ we have $f \in L_2(\mathbb{T}^3)$. $\square$

Lemma 3.4. *Assume Assumption* 2.2 *be fulfilled. Then there exist a* δ*-neighborhood* $U_\delta(0) \subset \mathbb{T}^3$ *of the point* $p = 0$ *and a function* $q_0(\cdot) \in C^{(2)}(U_\delta(0))$ *such that*:

(i) *for any* $p \in U_\delta(0)$ *the point* $q_0(p)$ *is a unique non-degenerate minimum of* $u_p(\cdot)$ *and*

$$q_0(p) = -\frac{l_2}{l_1} p + O\big(|p|^{2+\theta}\big) \quad \text{as } p \to 0; \tag{3.5}$$

(ii) *the function* $u_{\min}(\cdot) = U(\cdot, q_0(\cdot))$ *is even, of class* $C^{(3)}(U_\delta(0))$ *and has the asymptotics*

$$u_{\min}(p) = m + \frac{l_1^2 - l_2^2}{2l_1} (Up, p) + O\big(|p|^{3+\theta}\big) \quad \text{as } p \to 0; \tag{3.6}$$

(iii) *let for some* $p \in \mathbb{T}^3$ *the point* $q_0(p)$ *be a minimum of* $u_p(\cdot)$ (*if the minimum value of* $\hat{u}_p(\cdot) = \hat{u}(p, \cdot)$ *is attained in several points* $q_0(p)$ *as nearest to* $0 \in \mathbb{T}^3$), *that is,* $u_p(q_0(p)) = \min_{q \in \mathbb{T}^3} u_p(q)$. *Then* $q_0(-p) = -q_0(p)$.

Proof. (i) By the implicit function theorem there exist $\delta > 0$ and a function $q_0(\cdot) \in C^{(2)}(U_\delta(0))$ such that for any $p \in U_\delta(0)$ the point $q_0(p)$ is the unique non-degenerate minimum point of $u_p(\cdot)$ (see [17, Lemma 3]).

Since $u(\cdot,\cdot)$ is even with respect to $(p, q) \in (\mathbb{T}^3)^2$ we have

$$\begin{aligned} u_{\min}(-p) &= \min_{q \in \mathbb{T}^3} u_{-p}(q) = \min_{q \in \mathbb{T}^3} u_p(-q) = \min_{-q \in \mathbb{T}^3} u_p(q) \\ &= \min_{q \in \mathbb{T}^3} u_p(q) = u_{\min}(p), \quad p \in \mathbb{T}^3, \end{aligned} \tag{3.7}$$

and hence for all $p \in U_\delta(0)$ the equality

$$u_p\big(q_0(p)\big) = \min_{q \in \mathbb{T}^3} u_p(q) = u_{\min}(p) = u_{\min}(-p) = u_{-p}\big(q_0(-p)\big) = u_p\big(-q_0(-p)\big) \quad (3.8)$$

holds. For each $p \in U_\delta(0)$ the point $q_0(p)$ is the unique non-degenerate minimum of the function $u_p(\cdot)$ and hence the equality (3.8) yields $q_0(-p) = -q_0(p)$, $p \in U_\delta(0)$.

The asymptotics (3.5) follows from the fact that $q_0(\cdot)$ is an odd function and its coefficient $-\frac{l_2}{l_1}$ is calculated using the identity $\nabla u(p, q_0(p)) \equiv 0$, $p \in U_\delta(0)$.

(ii) The equality $u_{\min}(p) = u_p(q_0(p))$ and asymptotics (3.5) yield asymptotics (3.6).

(iii) Since the function $u_{\min}(\cdot)$ is even (see (3.7)) we conclude that if $q_0(p) \in \mathbb{T}^3$ is the minimum point of $u_p(\cdot)$ then $-q_0(p) \in \mathbb{T}^3$ is the minimum point of $u_{-p}(\cdot)$. Hence $q_0(-p) = -q_0(p)$. $\square$

Set

$$\mathbb{C}_+ = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}, \qquad \mathbb{R}_+ = \{x \in \mathbb{R}\colon x > 0\}, \qquad \mathbb{R}_+^0 = \mathbb{R}_+ \cup \{0\}.$$

Let $u(\cdot,\cdot)$ be the function defined on $U_\delta(0) \times \mathbb{T}^3$ as

$$\tilde{u}(p,q) = u_p\big(q + q_0(p)\big) - u_{\min}(p).$$

For any $p \in \mathbb{T}^3$ we define an analytic function $D(p,\cdot)$ in $\mathbb{C}_+$ by

$$D(p,w) = \int_{\mathbb{T}^3} \frac{\varphi^2(q + q_0(p))\,dq}{\tilde{u}(p,q) + w^2}.$$

Lemma 3.5. *Assume Assumptions* 2.2 *and* 2.5. *Then for any $w \in \mathbb{R}_+^0$ the function $D(\cdot, w)$ is of class $C^{(2)}(U_\delta(0))$, the right-hand derivative of $D(0,\cdot)$ at $w = 0$ exists and the following decomposition*

$$D(p,w) = D(0,0) - \frac{4\sqrt{2}\pi^2\varphi^2(0)}{l_1^{3/2}(\det U)^{1/2}} w + D^{\mathrm{res}}(w) + D^{\mathrm{res}}(p,w)$$

holds, where $D^{\mathrm{res}}(w) = O(w^{1+\theta})$ as $w \to +0$ and $D^{\mathrm{res}}(p,w) = O(p^2)$ as $p \to 0$ uniformly in $w \in \mathbb{R}_+^0$.

Proof. (i) For any $p \in U_\delta(0)$ the point $q_0(p)$ is the non-degenerate minimum of the function $u_p(\cdot)$ (see Lemma 3.4) and $q_0(\cdot) \in C^{(2)}(U_\delta(0))$. Since $u_{\min}(\cdot) \in C^{(2)}(U_\delta(0))$ by definition of $D(\cdot,\cdot)$ and Assumptions 2.2 and 2.5 we obtain that the function $D(\cdot, w)$ is of class $C^{(2)}(U_\delta(0))$ for any $w \in \mathbb{R}_+^0$, where $C^{(n)}(U_\delta(0))$ can be defined in the same way as $C^{(n)}(\mathbb{T}^3)$.

One can easily see that

$$u_p\big(q + q_0(p)\big) - u_{\min}(p) = \frac{l_1}{2}(Uq,q) + o\big(|p||q|^2\big) + o\big(|q|^2\big) \quad \text{as } |p|, |q| \to 0.$$

Therefore for some $C > 0$ and all $w \in \mathbb{R}_+^0$ and $i, j = 1,2,3$ the inequalities hold

$$\left| \frac{\partial^2}{\partial p_i \partial p_j} \frac{\varphi^2(q + q_0(p))}{\tilde{u}(p,q) - w^2} \right| \leqslant C|q|^{-2}, \quad p, q \in U_\delta(0), \quad (3.9)$$

and

$$\left|\frac{\partial^2}{\partial p_i \partial p_j} \frac{\varphi^2(q+q_0(p))}{\tilde{u}(p,q)-w^2}\right| \leqslant C, \quad p \in U_\delta(0),\ q \in \mathbb{T}^3 \setminus U_\delta(0). \tag{3.10}$$

Lebesgue's dominated convergence theorem implies that

$$\frac{\partial^2}{\partial p_i \partial p_j} D(p,0) = \lim_{w\to 0+} \frac{\partial^2}{\partial p_i \partial p_j} D(p,w), \quad p \in U_\delta(0).$$

Repeatedly applying Hadamard's lemma (see [33, V.1, p. 512]) we obtain

$$D(p,w) = D(0,w) + \sum_{i=1}^{3} \frac{\partial}{\partial p_i} D(0,w) p_i + \sum_{i,j=1}^{3} H_{ij}(p,w) p_i p_j,$$

where for any $w \in \mathbb{R}^0_+$ the functions $H_{ij}(\cdot,w)$, $i,j = 1,2,3$, are continuous in $U_\delta(0)$ and

$$H_{ij}(p,w) = \frac{1}{2} \int_0^1 \int_0^1 \frac{\partial^2}{\partial p_i \partial p_j} D(x_1 x_2 p, w)\, dx_1\, dx_2.$$

Estimates (3.9) and (3.10) give

$$\begin{aligned} \left|H_{i,j}(p,w)\right| &\leqslant \frac{1}{2} \int_0^1 \int_0^1 \left|\frac{\partial^2}{\partial p_i \partial p_j} D(x_1 x_2 p, w)\right| dx_1\, dx_2 \\ &\leqslant C\left(1 + \int_{U_\delta(0)} q^{-2} \varphi^2\big(q+q(p)\big)\, dq\right) \end{aligned}$$

for any $p \in U_\delta(0)$ uniformly in $w \in \mathbb{R}^0_+$.

Since for any $w \in \mathbb{R}^0_+$ the function $D(\cdot,w)$ is even in $U_\delta(0)$ we have

$$\left.\frac{\partial}{\partial p_i} D(p,w)\right|_{p=0} = 0, \quad i = 1,2,3.$$

(ii) Now we show that there exists a right-hand derivative of $D(0,\cdot)$ at $w=0$ and for some $C>0$ the following relations hold

$$\lim_{w\to 0+} w^{-1}\big(D(0,w) - D(0,0)\big) = \frac{2\sqrt{2}\pi^2\, \mu\varphi^2(0)}{l_1^{3/2}(\det U)^{1/2}}, \tag{3.11}$$

$$\left|\frac{\partial}{\partial w} D(0,w) - \frac{\partial}{\partial w} D(0,0)\right| < C w^\theta, \quad w \in \mathbb{R}^0_+. \tag{3.12}$$

Indeed, the function $D(0,\cdot)$ can be represented as

$$D(0,w) = D_1(w) + D_2(w)$$

with

$$D_1(w) = \int_{U_\delta(0)} \frac{\varphi^2(q)}{\tilde{u}(0,q)+w^2}\, dq, \quad w \in \mathbb{C}_+,$$

and

$$D_2(w) = \int\limits_{\mathbb{T}^3 \setminus U_\delta(0)} \frac{\varphi^2(q)}{\tilde{u}(0,q) + w^2}\, dq, \quad w \in \mathbb{C}_+.$$

Since the function $\tilde{u}(0,\cdot)$ is continuous on the compact set $\mathbb{T}^3 \setminus U_\delta(0)$ and has a unique minimum at $q = 0$ there exists $M_1 > 0$ such that $|\tilde{u}(0,q)| > M_1$ for all $q \in \mathbb{T}^3 \setminus U_\delta(0)$.

Then by $\varphi(\cdot) \in C^{(2)}(\mathbb{T}^3)$ we have

$$\bigl|D_2(w) - D_2(0)\bigr| \leqslant C w^2, \quad w \in \mathbb{R}^0_+,$$

for some $C = C(\delta) > 0$.

Now, let us consider

$$D_1(w) - D_1(0) = \int\limits_{U_\delta(0)} w^2 \bigl[\bigl(\tilde{u}(0,q) + w^2\bigr)\tilde{u}(0,q)\bigr]^{-1} \varphi^2(q)\, dq. \tag{3.13}$$

The function $\tilde{u}(0,\cdot)$ has a unique non-degenerate minimum at $q = 0$. Therefore, by virtue of the Morse lemma (see [10]) there exists a one-to-one mapping $q = \psi(t)$ of a certain ball $u_\gamma(0)$ of radius $\gamma > 0$ with the center at $t = 0$ to a neighborhood $\tilde{W}(0)$ of the point $q = 0$ such that

$$\tilde{u}\bigl(0, \psi(t)\bigr) = t^2 \tag{3.14}$$

with $\psi(0) = 0$ and for the Jacobian $J_\psi(t) \in \mathcal{B}(\theta, U_\delta(0))$ of the mapping $q = \psi(t)$ the equality holds

$$J_\psi(0) = \sqrt{2} l_1^{-\frac{3}{2}} (\det U)^{-\frac{1}{2}}.$$

In the integral in (3.13) making a change of variable $q = \psi(t)$ and using the equality (3.14) we obtain

$$D_1(w) - D_1(0) = -\frac{w^2}{2} \int\limits_{u_\gamma(0)} \frac{\varphi^2(\psi(t)) J_\psi(t)}{t^2(t^2 + w^2)}\, dt. \tag{3.15}$$

Going over in the integral in (3.15) to spherical coordinates $t = r\omega$, we reduce it to the form

$$D_1(w) - D_1(0) = -\frac{w^2}{2} \int\limits_0^\gamma \frac{F(r)}{r^2 + w^2}\, dr$$

with

$$F(r) = \int\limits_{\mathbb{S}^2} \varphi^2\bigl(\psi(r\omega)\bigr) J_\psi(r\omega)\, d\omega,$$

where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ and $d\omega$ is the element of the unit sphere in this space.

Using $\varphi \in C^{(2)}(\mathbb{T}^3)$ and $J_\psi \in \mathcal{B}(\theta, U_\delta(0))$ we see that

$$\bigl|F(r) - F(0)\bigr| \leqslant C r^\theta, \quad r \in [0, \delta]. \tag{3.16}$$

Applying the inequality (3.16) it is easy to see that

$$\lim_{w \to 0+} \frac{D_1(w) - D_1(0)}{w} = 2\sqrt{2}\pi l_1^{-\frac{3}{2}} \varphi^2(0) (\det U)^{-\frac{1}{2}}.$$

Hence we have that there exists a right-hand derivative of $D_1(\cdot)$ at $w = 0$ and the equality (3.11) and the inequality (3.12) hold. □

4. Proof of the main results

Proof of Theorem 2.15. (i) Part (i) respectively part (ii) of Assumption 2.13 yields

$$\Delta_{\mu_0}\big(p, u_{\min}(p)\big) < \Delta_{\mu_0}(0, m) = 0, \quad 0 \neq p \in \mathbb{T}^3,$$

respectively

$$\Delta_{\mu_0}(p, m) > \Delta_{\mu_0}(0, m) = 0, \quad 0 \neq p \in \mathbb{T}^3.$$

Since $\lim_{z\to-\infty} \Delta_{\mu_0}(p, z) = 1$ and $\Delta_{\mu_0}(p, \cdot)$ is monotonously decreasing on $(-\infty, u_{\min}(p))$ we conclude that the function $\Delta_{\mu_0}(p, z)$ has a unique solution $e_{\mu_0}(p)$ in $(m, u_{\min}(p))$. Lemma 3.1 completes the proof of part (i) of Theorem 2.15.

(ii) Let $\mu > \mu_0$. We have

$$\Delta_{\mu}(p, z) < \Delta_{\mu_0}(p, z) \tag{4.1}$$

for all $p \in \mathbb{T}^3$, $z \leqslant u_{\min}(p)$.

Set $e_{\mu_0}(0) = m$. Assertion (i) of Theorem 2.15 and Lemma 3.1 yield that the value of the function $e_{\mu_0}(\cdot)$ at the point $p \in \mathbb{T}^3$ satisfies $m < e_{\mu_0}(p) < u_{\min}(p)$, $p \neq 0$, and $m = e_{\mu_0}(0)$, $p = 0$, and $\Delta_{\mu_0}(p, e_{\mu_0}(p)) = 0$.

By (4.1) we have $\Delta_{\mu}(p, e_{\mu_0}(p)) < \Delta_{\mu_0}(p, e_{\mu_0}(p)) = 0$, $p \in \mathbb{T}^3$, and hence by Lemma 3.1 for any $p \in \mathbb{T}^3$ there exists the number $e_{\mu}(p)$ such that

$$e_{\mu}(p) \in \big(-\infty, e_{\mu_0}(p)\big) \quad \text{and} \quad \Delta_{\mu}\big(0, e_{\mu}(p)\big) = 0.$$

Hence Lemma 3.1 completes the proof of Theorem 2.15. □

Proof of Theorem 2.16. Proof follows from Lemma 3.5 if we take into account the equality $\Delta_{\mu}(p, z) = 1 - \mu \Lambda(p, z) = 1 - \mu D(p, \sqrt{u_{\min}(p) - z})$ and that $w = (u_{\min}(p) - z)^{1/2} \geqslant 0$ for $z \leqslant u_{\min}(p)$. □

Proof of Corollary 2.18. Proof follows from Theorem 2.16 and Lemmas 3.2, 3.3. □

Proof of Corollary 2.20. Let the operator $h_{\mu_0}(0)$ have a threshold energy resonance then $\varphi(0) \neq 0$ (see Lemma 3.2). One has the asymptotics (see part (ii) of Lemma 3.4)

$$u_{\min}(p) = m + \big(l_1 l_2 - l^2\big)(2l)^{-1}(Up, p) + o\big(|p|^2\big) \quad \text{as } p \to 0,$$

and Corollary 2.18 yields (2.6) for some positive numbers c_1, c_2.

The positivity and continuity of the function $\Delta_{\mu_0}(\cdot, m)$ on the compact set $\mathbb{T}^3 \setminus U_{\delta}(0)$ imply (2.7). □

Proof of Corollary 2.21. By Lemma 3.3 we have $\varphi(0) = 0$ and $\Delta_{\mu_0}(0, m) = 0$. Taking into account that $\mu_0 = \Lambda^{-1}(0, m)$, where the function $\Lambda(\cdot,\cdot)$ is defined by (2.2), we get the equality

$$\Delta_{\mu_0}(p, m) = \mu_0\big(\Lambda(0, m) - \Lambda(p, m)\big).$$

Then Assumption 2.13 completes the proof. □

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Appendix A

Here we show that there are some important subclasses of the family of Friedrichs models (see, e.g., [3,6]) and for these subclasses the assumptions in Section 2 are fulfilled.

Let

$$\hat{u}(p,q) = \varepsilon(p) + \varepsilon(p-q) + \varepsilon(q), \tag{A.1}$$

where $\varepsilon(p)$ is a real-valued conditionally negative definite function on $\mathbb{T}^3$ and hence

(i) ε is an even function,
(ii) $\varepsilon(p)$ has a minimum at $p = 0$.

Recall that a complex-valued bounded function $\varepsilon : \mathbb{T}^3 \to \mathbb{C}$ is called conditionally negative definite if $\varepsilon(p) = \overline{\varepsilon(-p)}$ and

$$\sum_{i,j=1}^{n} \varepsilon(p_i - p_j) z_i \bar{z}_j \leqslant 0 \tag{A.2}$$

for any $n \in \mathbb{N}$, for all $p_1, p_2, \dots, p_n \in \mathbb{T}^3$ and all $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ satisfying $\sum_{i=1}^{n} z_i = 0$ (see, e.g., [3,26]).

Assumption A.1. Assume that $\varepsilon(\cdot)$ is a real-valued conditionally negative definite function on $\mathbb{T}^3$ having a unique non-degenerate minimum at the origin and all third order partial derivatives of $\varepsilon(\cdot)$ are continuous and belong to $\mathcal{B}(\theta, U_\delta(0))$.

Let $q_0(p) \in \mathbb{T}^3$ be a minimum point of $u_p(\cdot)$ defined in Lemma 3.4.

Lemma A.2. *Assume Assumption* A.1 *and* $u(p,q) = \hat{u}(p, q + q_0(p))$. *Then part* (i) *of Assumption* 2.13 *is fulfilled.*

Proof. (i) By the definition of $\Lambda(p, u_{\min}(p))$ we have

$$\Lambda\big(p, u_{\min}(p)\big) - \Lambda\big(0, u_{\min}(0)\big) = \int_{\mathbb{T}^3} \frac{2(u_0(t) - u_{\min}(0)) - [u_p(t) + u_{-p}(t) - 2u_{\min}(p)]}{(u_p(t) - u_{\min}(p))(u_{-p}(t) - u_{\min}(p))(u_0(t) - u_{\min}(0))} \varphi^2(t)\, dt.$$

According to $u_p(0) = u_{\min}(p)$ for any $0 \neq p \in \mathbb{T}^3$ we arrive to the inequality

$$u_0(q) - u_{\min}(0) > \frac{u_p(q) + u_p(-q)}{2} - u_{\min}(p), \quad \text{a.e. } q \in \mathbb{T}^3,$$

which is equivalent to the inequality in [3, Lemma 5] and proves part (i) of Assumption 2.13. □

Lemma A.3. *Let* $u(p,q)$ *be of the form* (A.1). *Then part* (ii) *of Assumption* 2.13 *is fulfilled.*

Proof. The real-valued (even) conditionally negative definite function ε admits the (Lévy–Khinchin) representation (see, e.g., [3,7])

$$\varepsilon(p) = \varepsilon(0) + \sum_{s \in \mathbb{Z}^3 \setminus \{0\}} \big(\cos(p,s) - 1\big)\hat{\varepsilon}(s), \quad p \in \mathbb{T}^3, \tag{A.3}$$

which is equivalent to the requirement that the Fourier coefficients $\hat{\varepsilon}(s)$ with $s \neq 0$ are non-positive, that is,

$$\hat{\varepsilon}(s) \leqslant 0, \quad s \neq 0, \tag{A.4}$$

and the series $\sum_{s\in\mathbb{Z}^3\setminus\{0\}} \hat{\varepsilon}(s)$ converges absolutely.

Since u and $|\varphi|$ are even the function $\Lambda(\cdot)$ is also even. Hence the equality

$$u_0(t) - \frac{u_p(t) + u_p(-t)}{2} = \sum_{s\in\mathbb{Z}^3\setminus\{0\}} \hat{\varepsilon}(s)\bigl(1 + \cos(t,s)\bigr)\bigl(1 - \cos(p,s)\bigr)$$

yields the representation

$$\Lambda(0,m) - \Lambda(p,m) = \frac{1}{2} \sum_{s\in\mathbb{Z}^3\setminus\{0\}} \bigl(-\hat{\varepsilon}(s)\bigr)\bigl(1 - \cos(p,s)\bigr) \int_{\mathbb{T}^3} \bigl(1 + \cos(t,s)\bigr) F(p,t)\,dt + \tilde{B}(p), \tag{A.5}$$

where

$$F(p,\cdot) = \frac{[u_p(\cdot) + u_{-p}(\cdot) - 2m]}{(u_p(\cdot) - m)(u_{-p}(\cdot) - m)(u_0(\cdot) - m)} \varphi^2(\cdot)$$

and

$$\tilde{B}(p) = \frac{1}{4} \int_{\mathbb{T}^3} \frac{[u_p(t) - u_{-p}(t)]^2}{(u_p(t) - m)(u_{-p}(t) - m)(u_0(t) - m)} \varphi^2(t)\,dt.$$

Set

$$B(p,s) = \int_{\mathbb{T}^3} \bigl(1 + \cos(t,s)\bigr) F(p,t)\,dt.$$

We rewrite the function $B(p,s)$ as a sum of two functions

$$B_\delta^{(1)}(p,s) = \int_{\mathbb{T}^3\setminus U_\delta(0)} \bigl(1 + \cos(t,s)\bigr) F(p,t)\,dt$$

and

$$B_\delta^{(2)}(p,s) = \int_{U_\delta(0)} \bigl(1 + \cos(t,s)\bigr) F(p,t)\,dt.$$

Let $\chi_\delta(\cdot)$ be the characteristic function of $U_\delta(0)$. Choose $\delta > 0$ such that

$$\operatorname{mes}\bigl\{\bigl(\mathbb{T}^3 \setminus U_\delta(0)\bigr) \cap \operatorname{supp}\varphi\bigr\} > 0.$$

Set $F_\delta(p,\cdot) = (1 - \chi_\delta(\cdot))F(p,\cdot)$. Then for all $p \in \mathbb{T}^3$ and a.e.

$$t \in \bigl(\mathbb{T}^3 \setminus U_\delta(0)\bigr) \cap \operatorname{supp}\varphi(\cdot)$$

the function $F_\delta(\cdot,\cdot)$ is strictly positive. Since the function u has a unique minimum at $(0,0)$ and φ is continuous on $\mathbb{T}^3$ we have that $F_\delta(p,\cdot)$, $p \in \mathbb{T}^3$, belongs to the Banach space $L^1(\mathbb{T}^3)$. Then for some (sufficiently large) $R > 0$ and (sufficiently small) $c_1(\delta) > 0$ and for all $|s| \leqslant R$, $p \in \mathbb{T}^3$ we have the inequality

$$B_\delta^{(1)}(p,s) = \int\limits_{\mathbb{T}^3} \bigl(1 + \cos(t,s)\bigr) F_\delta(p,t)\,dt > c_1(\delta) > 0.$$

The Riemann–Lebesgue lemma yields

$$B_\delta^{(1)}(p,s) \to \int\limits_{\mathbb{T}^3} F_\delta(p,t)\,dt > 0, \quad p \in \mathbb{T}^3 \text{ as } s \to \infty.$$

The continuity of the function

$$\tilde{F}(p) = \int\limits_{\mathbb{T}^3} F_\delta(p,t)\,dt$$

on the compact set $\mathbb{T}^3$ yields that for all $p \in \mathbb{T}^3$ and $|s| > R$ the inequality $B_\delta^{(1)}(p,s) \geqslant c_2(\delta)$ holds.

Thus for $c(\delta) = \min\{c_1(\delta), c_2(\delta)\}$ the inequality $B_\delta^{(1)}(p,s) \geqslant c$ holds for all $s \in \mathbb{Z}^3$, $p \in \mathbb{T}^3$. So $B_\delta^{(2)}(p,s) \geqslant 0$, $s \in \mathbb{Z}^3$, $p \in \mathbb{T}^3$, yields $B(p,s) > c$, $s \in \mathbb{Z}^3$, $p \in \mathbb{T}^3$. Taking into account the inequalities $\tilde{B}(p) \geqslant 0$, $p \in \mathbb{T}^3$, and $\hat{\varepsilon}(s) \leqslant 0$, $s \in \mathbb{Z}^3 \setminus \{0\}$ (see (A.4)), from the representations (A.3) and (A.5) we have

$$\Lambda(0,m) - \Lambda(p,m) \geqslant c\bigl(\varepsilon(p) - \varepsilon(0)\bigr).$$

This together with the assumptions on $\varepsilon(\cdot)$ completes the proof of Lemma A.3. □

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