A bundle-type auxiliary problem method for solving generalized variational-like inequalities

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Received 21 September 2006; received in revised form 16 November 2007; accepted 17 November 2007

Abstract

For generalized variational-like inequalities, by combining the auxiliary principle technique with the bundle idea for nonconvex nonsmooth minimization, we present an implementable iterative method. To make the subproblem easier to solve, even though the preinvex function may not be convex, we still consider using the model similar to the one in [R. Mifflin, A modification and extension of Lemarechal’s algorithm for nonsmooth minimization, Mathematical Programming 17 (1982) 77–90] (which may not be under the preinvex function) to approximate locally the involved preinvex function, and prove that this local approximation is well defined at each iteration of the algorithm, i.e., the construction of this local approximation can terminate in finite steps at each iteration of the proposed algorithm. We not only explain how to construct the approximation, but also prove the weak convergence of the sequence generated by the corresponding algorithm under some conditions. The proposed algorithm is a generalization of the existing algorithm for generalized variational inequalities to generalized variational-like inequalities in some sense, see [T.T. Hue, J.J. Strodiot, V.H. Nguyen, Convergence of the approximate auxiliary problem method for solving generalized variational inequalities, Journal of Optimization Theory and Applications 121 (2004) 119–145].

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Keywords: Nonconvex nonsmooth optimization; Generalized variational-like inequality; Preinvex function; Auxiliary problem principle; Bundle method for nonconvex function

1. Introduction

Variational-like inequalities are an useful and important generalization of the variational inequalities, which was considered and studied by Parida and Sen [3]. Yao [4] and Tian [5] used the Berge maximum Theorem and KKM maps to study the existence of solutions of variational-like inequalities in a convex setting. The variational-like inequalities are closely related to the concepts of the invex and preinvex functions, which generalize the notion of convexity of functions. The invex functions were introduced by Hanson [6] in 1981. Noor [7], Weir and Mond [8] proved that many results in mathematical programming involving convex functions and convex sets actually hold for invex
(preinvex) functions and their generalizations. The auxiliary principle technique was once used to deal with variational inequalities, and this technique is mainly due to Glowinski, Lions and Tremolieres [9]. In 2000, Noor [10] used the auxiliary principle technique to suggest an iterative method for variational-like inequalities, and the convergence analysis of the iterative method was also given.

In general, the generalized variational inequality problem (GVIP) is of the form: finding $x^* \in C$ such that

$$
(F(x^*), x - x^*) + \varphi(x) - \varphi(x^*) \geq 0, \quad \forall x \in C,
$$

where $H$ is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. $C$ is a nonempty closed subset of $H$, $F : H \to H$ is a single-valued operator, $\varphi : H \to R \cup \{+\infty\}$ is a nondifferentiable l.s.c. proper convex function. The auxiliary problem framework for solving (GVIP) can be outlined as follows (see [11, 12]): a sequence $\{\lambda_k\}_{k \in N}$ of positive numbers and a sequence $\{h_k\}_{k \in N}$ of strongly convex auxiliary functions are introduced in order to approximate $F$ by $\lambda_k^{-1}\nabla h_k$ at iteration $k$. Then the error is taken into account by adding the term $F(x^k) - \lambda_k^{-1}\nabla h_k(x^k)$. More precisely, given $x^k \in C$, the next iterate $x^{k+1}$ is the unique solution of

$$
\min_{x \in C} [\lambda_k^{-1}h_k(x) + \varphi(x) + (F(x^k) - \lambda_k^{-1}\nabla h_k(x^k), x - x^k)].
$$

The convergence theorem for this algorithm has been established under some conditions, see [2]. Motivated by this method, we try to solve the generalized variational-like inequality by utilizing the similar technique: finding $u^* \in H$ such that

$$
(GVLIP) \quad \langle \psi(u^*), \eta(u, u^*) \rangle + \varphi(u) - \varphi(u^*) \geq 0, \quad \forall u \in H,
$$

where $\psi : H \to H$ is a single-valued mapping, $\eta$ is a mapping from $H \times H$ to $H$, $\varphi : H \to R \cup \{+\infty\}$ is a l.s.c. locally Lipschitzian proper (not necessarily differentiable) preinvex function with respect to the function $\eta(\cdot, \cdot)$. The iterative method suggested by Noor [10] for solving (GVLIP) has to face the problem of solving the following minimization problem

$$
\min_{u \in H} \{g(u) - g(u^*) - \langle \nabla g(u^*), \eta(u, u^*) \rangle + \rho \varphi(u) - \rho \varphi(u^*)\},
$$

for a given $u^* \in H$. Since we require that $\varphi$ is a nondifferentiable proper l.s.c. preinvex function with respect to the function $\eta(\cdot, \cdot)$, then the problem (1.2) may be hard to solve. To overcome this difficulty, we propose a simpler function to approximate $\varphi$. Our aim in this paper is to consider the approximation of $\varphi$ as in nonconvex nonsmooth minimization and to study a general algorithm for solving (GVLIP).

This paper is organized as follows: In Section 2, by combining the auxiliary principle technique with the bundle idea for nonconvex nonsmooth minimization we present an algorithm for (GVLIP). In Section 3 the weak convergence of the proposed algorithm is given under some conditions.

2. An algorithm for (GVLIP)

Consider the problem of finding $u^* \in H$ such that

$$
(GVLIP) \quad \langle \psi(u^*), \eta(u, u^*) \rangle + \varphi(u) - \varphi(u^*) \geq 0, \quad \forall u \in H.
$$

For solving (GVLIP), Noor [10] considered the problem of finding a unique $u \in H$ satisfying the auxiliary generalized variational-like inequality associated with (GVLIP)

$$

\langle \nabla g(u), \eta(v, u) \rangle \geq \langle \nabla g(u^*), \eta(v, u^*) \rangle + \rho \varphi(u) - \rho \varphi(v), \quad \forall v \in H,
$$

for given $u^* \in H$, where $\rho$ is a positive number, $\nabla g$ is the differential of a strongly preinvex function $g$ with respect to $\eta(\cdot, \cdot)$. The problem (2.1) has a unique solution due to the strong preinvexity of $g$. At the same time, Noor presented the following lemma.

**Lemma 2.1.** [10] Let $g$ be a differentiable preinvex function and $\varphi$ be a nondifferentiable preinvex function with respect to $\eta$. If for all $u, v \in H$, the operator $\eta$ satisfies $\eta(u, v) = \eta(u, w) + \eta(w, v)$ and $\eta(\cdot, \cdot)$ is prelinear with
respect to the first argument, then the solution \( u \) of (2.1) can be characterized by the minimizer of the following minimization problem

\[
\min_{u \in H} E_1(u) = g(u) - g(u^*) - \langle \nabla g(u^*) - \rho \psi(u^*), \eta(u, u^*) \rangle + \rho \psi(u) - \rho \psi(u^*). \tag{2.2}
\]

It is easy to see that if the solution \( u \) of (2.1) happens to be \( u^* \), then \( u \) is a solution of (GVLIP). On the basis of this observation, Noor suggested a fixed-point algorithm for solving (GVLIP) and also proved its convergence, see [10].

**Algorithm 2.1** (see [10]).

**Step 1** At \( k = 0 \), start with an initial point \( u^0 \in H \).

**Step 2** At step \( k \), solve the auxiliary problem (2.2) with \( u^* = u^k \). Let \( u^{k+1} \) denote the solution of (2.2).

**Step 3** If \( \|u^{k+1} - u^k\| \leq \varepsilon \), for given \( \varepsilon \geq 0 \), stop. Otherwise set \( k = k + 1 \), goto Step 2.

The auxiliary problem (2.2) with \( u^* = u^k \) has the form, by ignoring the constant terms and dividing by \( \rho \),

\[
\min_{u \in H} E_2(u) = (1/\rho)g(u) + \phi(u) + \langle \psi(u^k) - (1/\rho)\nabla g(u^k), \eta(u, u^k) \rangle. \tag{2.3}
\]

We substitute \( \rho \) and \( g \) by a positive number \( \mu_k \) and a strongly preinvex function \( g^k \), respectively, at each iteration \( k \), then (2.3) has the form

\[
(P^k) \quad \min_{u \in H} E(u) = \mu_k^{-1}g^k(u) + \phi(u) + \langle \psi(u^k) - \mu_k^{-1}\nabla g^k(u^k), \eta(u, u^k) \rangle.
\]

Since \( \phi(\cdot) \) is a nondifferentiable preinvex function with respect to \( \eta(\cdot, \cdot) \) and \( \phi \) may not be a convex function, the problem \( P^k \) is hard to solve. Therefore, we propose a simpler function to approximate \( \phi \) by employing bundle ideas from nonconvex nondifferentiable minimization. Even though the approximation may not be under the involved preinvex function, we still can prove the finite termination of this kind of construction. Our aim in this paper is to consider the approximation to \( \phi \) as in nonconvex nonsmooth minimization and to study the following general algorithm for (GVLIP).

**General algorithm.** Given \( u^k \in H \), choose a l.s.c. proper function \( \phi^k \) and solve the subproblem

\[
(AOP^k_{d}) \quad \min_{u \in H} \{\mu_k^{-1}g^k(u) + \phi^k(u) + \langle \psi(u^k) - \mu_k^{-1}\nabla g^k(u^k), \eta(u, u^k) \rangle\}
\]

to obtain \( u^{k+1} \in H \). If we let \( d = u - u^k \), then \( AOP^k_{d} \) has the form

\[
(AOP^k_{d}) \quad \min_{u \in H} \{\mu_k^{-1}g^k(u^d) + \phi^k(u^d) + \langle \psi(u^k) - \mu_k^{-1}\nabla g^k(u^k), \eta(u^k + u^d, u^d) \rangle\}.
\]

Moreover, to prove the convergence of this algorithm, we will impose that the function \( \phi^k \) is chosen or built such that the following property holds:

\[
\phi(y^l) - \phi^k(y^l) \leq \Delta_k, \tag{2.4}
\]

where \( y^l = u^k + t_l d^l, d^l \) is the solution of \( (AOP^k_{d}) \), \( t_l \in (0, 1) \), \( \Delta_k > 0 \) is a tolerance parameter which will be defined a priori.

The quadratic approximation to \( \phi \) at \( u^k \) can be constructed in this way: let

\[
\tilde{\phi}^k(y) = \phi^k(u^k) + \max_{j \in \{1, 2, \ldots, i - 1\}} \{-\alpha(u^k, y^j) + \langle g^k(y^j), y - u^k \rangle\} + \frac{1}{2}\|y - u^k\|^2, \tag{2.5}
\]

where \( y^j \in H, g^k(y^j) \in \partial \phi(y^j), j = 1, 2, \ldots, i - 1 \), \( \partial \phi(y^j) = \text{conv}\{\lim_{t \to \infty} \nabla \phi(y^j) | y^l \to y^j \text{ and } \phi \text{ is differentiable at each } y^j \} \) is the subdifferential (generalized subdifferential) of \( \phi \) at \( y^j \); see [13]. \( \alpha(u^k, y^j) = |\phi(u^k) - \phi(y^j) - \langle g^k(y^j), u^k - y^j \rangle| \) are the absolute linearization errors at \( u^k \) for \( j = 1, 2, \ldots, i - 1 \). The reason for using such an approximation arises from the ideas of Kiwiel and Mifflin, in which they proposed one method for the nonconvex nonsmooth minimization problem, see [14,15]. In order to let \( \phi^k \) satisfy (2.4), the quadratic
convex approximation $\varphi^k$ can be built step by step in this way. To be more precise, a sequence of convex functions $\hat{\varphi}^1, \hat{\varphi}^2, \ldots, \hat{\varphi}^l, \ldots$ are generated until the solution $d^l$ of the following subproblem

$$(P^{k}_{l}) \min_{d \in H} \{\mu_k^{-1} g^k(u^k + d) + \hat{\varphi}^l(u^k + d) + \langle \psi(u^k) - \mu_k^{-1} \nabla g^k(u^k), \eta(u^k + d, u^k) \rangle\}$$

is such that $\varphi(\bar{y}^i) - \hat{\varphi}^j(\bar{y}^i) \leq \Delta_k$, where $\bar{y}^i = u^k + t_i d^i$, $t_i \in (0, 1)$. In this case, we set $\varphi^k = \hat{\varphi}^j, u^{k+1} = y^i = u^k + d^i$.

**Algorithm 2.2** (A bundle-type auxiliary problem method). Let $u^0 \in H$ be an initial point, two positive number sequences $\{\mu_k\}_{k \in N}$, $\{\Delta_k\}_{k \in N}$ and one sequence $\{g^k\}_{k \in N}$ of strongly preconvex functions are given. Set $y^0 = u^0, t_1 \in (0, 1), k = 0, i = 1, \kappa \in (0, 1)$.

**Step 1** Choose a convex quadratic function $\hat{\varphi}^i$ and solve problem $(P^{k}_{l})$ to obtain $y^i = u^k + d^i$. Let $\bar{y}^i = u^k + t_i d^i$.

**Step 2** If

$$\varphi(\bar{y}^i) - \hat{\varphi}^j(\bar{y}^i) \leq \Delta_k,$$

then set $\varphi^k = \hat{\varphi}^j, u^{k+1} = y^i$ and let $k = k + 1$.

**Step 3** Set $t_{i+1} = \kappa t_i$ and let $i = i + 1$, goto Step 1.

**Proposition 2.1.** If the stopping test is suppressed in the bundle algorithm (Algorithm 2.2) after some outer iterate $u^k$ has been reached, the optimal solution $d^i$ of $(P^k_{l})$ is bounded for each $i$, then $\varphi(\bar{y}^i) \to \hat{\varphi}^j(\bar{y}^i)$ as $i \to +\infty$.

**Proof.** Since $\bar{y}^i = u^k + t_i d^i, d^i$ is bounded for each $i$ and $t_i \to 0$ as $i \to +\infty$ because $t_{i+1} = \kappa t_i, \kappa \in (0, 1)$, so one has $\bar{y}^i \to u^k$ as $i \to +\infty$. In view of the definition of $\hat{\varphi}^j(\cdot)$,

$$\hat{\varphi}^j(\bar{y}^i) = \varphi(u^k) + \max_{j \in \{1, 2, \ldots, i-1\}} \{- \alpha(u^k, y^j) + \langle g^i(y^j), \bar{y}^i - u^k \rangle \} + \frac{1}{2} \|\bar{y}^i - u^k\|^2,$$

due to the local boundedness of $\partial \varphi(\cdot)$ and $\alpha(u^k, y^j) \to 0, j \in \{1, 2, \ldots, i-1\}$ as $i \to +\infty$, see [1], we have $\hat{\varphi}^j(\bar{y}^i) \to \varphi(u^k)$ as $i \to +\infty$. The proof is completed. \(\boxdot\)

**Proposition 2.1** indicates that (2.4) can be satisfied after finitely many inner iterations since $\varphi(\bar{y}^i) \to \hat{\varphi}^j(\bar{y}^i)$ and $\Delta^k > 0$. Algorithm 2.2 is well defined.

3. Convergence analysis

We need the following definitions and assumption about the function $\eta : H \times H \to H$, which plays an important role in obtaining our results.

**Assumption 3.1.** For all $u, v, w \in H$, the operator $\eta : H \times H \to H$ satisfies the condition $\eta(u, v) = \eta(u, w) + \eta(w, v)$ and is weakly continuous with respect to the first argument.

From Assumption 3.1 we have

$$\begin{align*}
(1) & \quad \eta(u, u) = 0, \quad \forall u \in H; \\
(2) & \quad \eta(u, v) = -\eta(v, u), \quad \forall u, v \in H.
\end{align*}$$

**Definition 3.1.** For all $u, v \in H$ and a given operator $\eta : H \times H \to H$, the operator $\psi$ is said to be $\eta$-monotone if

$$\langle \psi(u) - \psi(v), \eta(u, v) \rangle \geq 0.$$

**Definition 3.2.** For all $u, v \in H$, the operator $\eta : H \times H \to H$ is said to be Lipschitzian continuous if there exists a constant $\beta_1 > 0$ such that

$$\|\eta(u, v)\| \leq \beta_1 \|u - v\|.$$

Before presenting our convergence result, we have the following lemma.
Lemma 3.1. If $\psi$ is $\eta$-monotone and weakly continuous on $H$, then
\[ z^k \rightharpoonup z \implies \lim_{k \to \infty} \langle \psi(z^k), \eta(z^k, 0) \rangle \geq \langle \psi(z), \eta(z, 0) \rangle. \]

Proof. By Assumption 3.1 and that $\psi$ is $\eta$-monotone, we have
\[ \langle \psi(z^k) - \psi(z), \eta(z^k, z) \rangle = \langle \psi(z^k) - \psi(z), \eta(z^k, 0) - \eta(z, 0) \rangle \geq 0, \]
that is,
\[ \langle \psi(z^k), \eta(z^k, 0) \rangle - \langle \psi(z), \eta(0, 0) \rangle - \langle \psi(z), \eta(z^k, 0) \rangle + \langle \psi(z), \eta(z, 0) \rangle \geq 0. \]

By the Frechet–Riesz Representation theorem, for $\eta(z, 0) \in H$, there exists a unique $f_{\eta(z, 0)} \in H$ such that
\[ f_{\eta(z, 0)}(x) = \langle x, \eta(z, 0) \rangle, \text{ for any } x \in H. \]
Particularly, we let $x = \psi(z^k)$ and $x = \psi(z)$, then
\[ f_{\eta(z, 0)}(\psi(z^k)) = \langle \psi(z^k), \eta(z, 0) \rangle, \]
\[ f_{\eta(z, 0)}(\psi(z)) = \langle \psi(z), \eta(z, 0) \rangle. \]
Since $\psi$ is weakly continuous and $z^k \rightharpoonup z$, we have
\[ f_{\eta(z, 0)}(\psi(z^k)) \to f_{\eta(z, 0)}(\psi(z)) \text{ as } k \to \infty, \]
i.e., $\langle \psi(z^k), \eta(z, 0) \rangle \to \langle \psi(z), \eta(z, 0) \rangle$ (as $k \to \infty$). Similarly, because $\eta$ is weakly continuous with respect to the first argument, it can be proved that $\langle \psi(z), \eta(z^k, 0) \rangle \to \langle \psi(z), \eta(z, 0) \rangle$ (as $k \to \infty$). The conclusion follows from the inequality (**) by letting $k \to \infty$. □

Theorem 3.1. Assume that the sequence $\{u^k\}_{k \in N}$ is generated by Algorithm 2.2 and the following conditions hold true:
(i) $\psi : H \to H$ is $\eta$-monotone and weakly continuous on $H$;
(ii) $\{\nabla g^k\}_{k \in N}$ is a sequence of Lipschitz continuous mappings with Lipschitz constants $\gamma_k \leq \Lambda$ for all $k \in N$;
(iii) $\mu_k \geq \mu > 0$, $\forall k \in N$;
(iv) the operator $\eta : H \times H \to H$ is Lipschitz continuous with $\beta_1 > 0$ (Definition 3.2);
(v) for each weak limit point $u^*$ of the sequence $\{u^k\}_{k \in N}$, there exists a subset $K \subset N$ such that
\[ \limsup_{k \in K} [\varphi^k(x^k) - \varphi^k(u^{k+1})] \leq \varphi(x) - \varphi(u^*), \]
where $\varphi^k$ is constructed as described in Section 2, $x \in H$, $\{x^k\}_{k \in K} \subset H$ is a sequence constructed such that
\[ x^k \to x; \]
(vi) the sequence $\{u^k\}_{k \in N}$ is bounded and such that the sequence $\{\|u^{k+1} - u^k\|\}_{k \in N}$ converges to zero.

Then every weak limit point of the sequence $\{u^k\}_{k \in N}$ is a solution of problem (GVLP).

Proof. Let $u^*$ be a weak limit point of $\{u^k\}_{k \in N}$, and without loss of generality we suppose that $\{u^k\}_{k \in K \subseteq N}$ is a subsequence weakly converging to $u^*$, i.e., $\{u^k\}_{k \in K} \rightharpoonup u^*$. Since $\|u^{k+1} - u^k\| \to 0$, we have that $\{u^{k+1}\}_{k \in K} \rightharpoonup u^*$. Moreover, for each $x \in H$, we can construct a sequence $\{x^k\}_{k \in K}$ such that
\[ x^k \to x. \]

According to Assumption 3.1, we have
\[ \langle \psi(u^k), \eta(x^k, u^{k+1}) \rangle = \langle \psi(u^k), \eta(x^k, x) \rangle + \langle \psi(u^k), \eta(x, 0) \rangle \]
\[ + \langle \psi(u^k), \eta(u^k, u^{k+1}) \rangle - \langle \psi(u^k), \eta(u^k, 0) \rangle. \]
By Lemma 3.1, we obtain
\[ \lim_{k \in K} \langle \psi(u^k), \eta(x^k, u^{k+1}) \rangle \leq \langle \psi(u^*), \eta(x, u^*) \rangle. \]
Now by the definition of \( \{u^k\}_{k \in \mathbb{N}} \), we have that, for all \( k \in \mathbb{N} \),
\[
\mu_k^{-1} g^k(u^{k+1}) + \phi^k(u^{k+1}) + \langle \psi(u^k) - \mu_k^{-1} \nabla g^k(u^k), \eta(u^{k+1}, u^k) \rangle
\leq \mu_k^{-1} g^k(x^k) + \phi^k(x^k) + \langle \psi(u^k) - \mu_k^{-1} \nabla g^k(u^k), \eta(x^k, u^k) \rangle,
\]  
(3.5)
i.e.,
\[
0 \leq \langle \psi(u^k) + \mu_k^{-1} (\nabla g^k(u^{k+1}) - \nabla g^k(u^k)), \eta(x^k, u^{k+1}) \rangle + \phi^k(x^k) - \phi^k(u^{k+1}).
\]  
(3.6)
Taking the superior limit on \( k \in K \) in the above inequality, using (3.4) and the assumption (v), and observing that
\[
\limsup_{k \to \infty} \mu_k^{-1} \langle \nabla g^k(u^{k+1}) - \nabla g^k(u^k), \eta(x^k, u^{k+1}) \rangle \leq \mu^{-1} \Lambda \limsup_{k \to \infty} \| u^{k+1} - u^k \| \cdot \| \eta(x^k, u^{k+1}) \| = 0,
\]  
(3.7)
then we obtain
\[
0 \leq \langle \psi(u^*), \eta(x, u^*) \rangle + \phi(x) - \phi(u^*), \quad \forall x \in H,
\]  
which means that \( u^* \) is a solution of (GVLIP). \( \square \)

References