JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 62, 81–89 (1978)

On Solving Differential and Difference Equations with Variable Coefficients*

MOURAD EL-HOUSSIENY ISMAIL

Department of Applied Mathematics, McMaster University, Hamilton, Ontario, Canada

Submitted by K. L. Cooke

We introduce a method of solving the functional equation $\sum_{j=0}^{n} a_j L^j f(x) = 0$ where the a's are constants and L is a differential or finite difference operator that reduces the degree of a polynomial by 1.

INTRODUCTION

Linear differential equations with constant coefficients are the simplest differential equations to solve. The next equations to consider are linear differential equations with polynomial, or in general analytic, coefficients. The situation here is quite different. As a matter of fact, there are only very few special equations that we know how to solve. In most cases, the existence of the solution(s) follows from general criteria and the solution, if unique, is computed numerically. One way of solving linear differential equations with constant coefficients is by Mikusinski's operational calculus, see Erdelyi [13] for details. Recently Ditkin and Prudnikov [12] constructed an operational calculus to solve linear differential equations with constant coefficients in DxD, $D \equiv d/dx$, that is, of the type

$$\sum_{j=0}^{n} a_j L^j f(x) = 0, \tag{1.1}$$

with $L = DxD$, and their work was generalized by Meller [16] to solve (1.1) when $L = x^{-\alpha} Dx^{\alpha+1} D$. Later in a series of papers by Dimovski [8–11], this was further generalized to solve (1.1) with

$$L = x^{\alpha_0} \frac{d}{dx} x^{\alpha_1} \frac{d}{dx} \cdots x^{\alpha_{m-1}} \frac{d}{dx} x^{\alpha_m},$$

* This research was sponsored by NSF Grant GP-33897X2 and in part by the United States Army under Contract No. DA-31-124-ARO-D-462 and Grant A4522 of the Canadian NRC.

0022-247X/78/0621–0081$02.00/0

where

$$\alpha_0 + \alpha_1 + \cdots + \alpha_m - m < 0. \tag{1.3}$$

We felt that these extensions are getting complicated; for example, the convolution products in Dimovski's works [9–12] are quite cumbersome to work with. Others used the approach of changing the independent variable to reduce the differential equation to a linear equation with constant coefficients. Euler's equation is, of course, the model for this approach. Breuer and Gottlieb [6] characterized the nth order differential equations of the form

$$\sum_{k=0}^{n} P_k(x)\, D^k y(x) = 0, \tag{1.4}$$

that can be reduced to linear differential equations with constant coefficients in a new independent variable. Later Breuer [5] did the same for the finite difference analog of (1.4).

In the present paper we attempt to develop a theory for solving equations of type (1.1) analogous to the theory of linear differential equations with constant coefficients in the most direct, simple, and obvious way. Our theory, however, does not determine all the linear independent solutions of a given equation. Our methods determine only the totality of admissible solutions; see below. This situation is also encountered in solving differential equations using the Laplace transform. In Section 2 we develop our new method, while Section 3 is devoted to treating some special differential and difference equations in details.

2. Main Results

Let L be an operator defined on all polynomials so that Lx^n is a polynomial of degree $n-1$ if $n > 0$ and is identically zero if $n = 0$. Sheffer [20, Sect. 1] proved that such operator L has the representation

$$L = \sum_{k=0}^{\infty} b_k(x)\, D^{k+1}, \qquad b_0 \neq 0, \tag{2.1}$$

where $b_k(x)$ is a polynomial of degree at most k. Sheffer also proved the existence of a unique polynomial set $\{P_n(x)\}_0^\infty$, $P_n(x)$ is of precise degree n, satisfying

$$LP_n(x) = nP_{n-1}(x), \qquad n = 0, 1, \ldots, P_{-1}(x) = 0, \tag{2.2}$$

and

$$P_0(0) = 1, \qquad \text{and} \qquad P_n(0) = 0 \quad \text{for } n > 0. \tag{2.3}$$

Such a set is called the basic set of L. We define R_L by

$$R_L = \inf_x\{\overline{\lim_{n\to\infty}} \mid n!/P_n(x)|^{1/n}\}. \tag{2.4}$$

We shall restrict ourselves to operators L of type (2.1) whose R_L is positive and that makes $\sum_{n=0}^{\infty} (P_n(x)/n!)\, t^n$ an entire function of x for every t with $|t| < R_L$. It is clear that $\sum_{n=0}^{\infty} P_n(x)\,(t^n/n!)$ is an analytic function of t in $|t| < R_L$ and for all x. A solution of (x) to

$$\sum_{j=0}^{n} a_j L^j f(x) = g(x) \tag{2.5}$$

is called admissible if $f(x) = \sum_{k=0}^{\infty} \lambda_k (P_k(x)/k!)$ with $L^j f(x) = \sum_{k=0}^{\infty} \lambda_{k+j}(P_k(x)/k!)$, $j = 1, 2,..., n$. Clearly the notion of admissibility depends on the equation under consideration. The reader must always keep in mind the analogy between the present theory and the theory of linear differential equations, with constant coefficients, which is available in any elementary text on differential equations. We now present our results.

THEOREM 1. *Assume* $|t| < R_L$. *The general admissible solution of*

$$(L - t) f(x) = b \sum_{k=0}^{\infty} P_{j+k}(x)\, t^k/k! \tag{2.6}$$

is

$$f(x) = a \sum_{k=0}^{\infty} \frac{P_k(x)}{k!} t^k + \frac{b}{j+1} \sum_{k=0}^{\infty} \frac{P_{k+j+1}(x)}{k!} t^k. \tag{2.7}$$

Proof. Let $\varphi(x) = \sum_{k=0}^{\infty} \mu_k(t)\,(P_k(x)/k!)$ satisfy (2.6) with $b = 0$. A direct substitution shows that $\mu_{k+1}(t) = t\mu_k(t)$, $k = 0, 1,...,$ hence $\mu_k(t) = at^k$, $k = 0, 1,....$ We now claim that $[b/(j+1)] \sum_{k=0}^{\infty} (P_{k+j+1}(x)/k!)\, t^k$ satisfies (2.6). This is so, since

$$\begin{aligned}(L - t) \sum_{k=0}^{\infty} \frac{P_{j+k+1}(x)}{k!} t^k &= \sum_{k=0}^{\infty} \frac{(j+k+1)}{k!} P_{j+k}(x)\, t^k - \sum_{0}^{\infty} \frac{P_{j+k+1}(x)}{k!} t^{k+1} \\ &= (j+1) P_j(x) + \sum_{k=0}^{\infty} \frac{P_{k+j+1}}{(k+1)!} t^{k+1}(j+k+2-k-1) \\ &= (j+1) \sum_{k=0}^{\infty} \frac{P_{j+k}(x)}{k!} t^k.\end{aligned}$$

Therefore (2.7) is the general admissible solution of (2.6) because it is the sum of the general solution of the homogeneous equation and a particular solution of (2.6).

Before proceeding to Theorem 2 we introduce the finite difference operators E and Δ defined as

$$Ef(x) = f(x+1), \qquad \Delta f(x) = f(x+1) - f(x).$$

Clearly,

$$\Delta^n f(x) = (E-1)^n f(x) = \sum_{s=0}^{n} \binom{n}{s} (-1)^{n-s} f(x+s). \tag{2.8}$$

THEOREM 2. *The general admissible solution to*

$$(L-t)^m f(x) = 0 \tag{2.9}$$

when $|t| < R_L$ *is*

$$f(x) = \sum_{j=0}^{m-1} \lambda_j \sum_{k=0}^{\infty} P_{k+j}(x) \frac{t^k}{k!}, \tag{2.10}$$

where the λ*'s are independent of* x.

Proof. Let $\phi_j(x) = \sum_{k=0}^{\infty} P_{j+k}(x)\,(t^k/k!)$. To show that $\phi_j(x)$ satisfies (2.9) for $j = 0, 1,..., m-1$ it suffices to show that

$$(L-t)^{j+1} \phi_j(x) = 0, \tag{2.11}$$

since $m > j$. Therefore we have, since $P_{-1}(x)/(-1)! = 0$,

$$(L-t)^{j+1}\phi_j(x) = \sum_{l=0}^{j+1} \binom{j+1}{l} (-t)^l \sum_{k=0}^{\infty} \frac{t^k(k+j)!}{k!(k+l-1)!} P_{k+l-1}(x)$$

$$= \sum_{r=0}^{\infty} P_r(x) \frac{t^{r+1}}{r!} \sum_{l=0}^{j+1} \binom{j+1}{l} (-1)^l (r+2-l)_j,$$

where $(x)_j$ is 1 if $j = 0$ and $x(x+1)\cdots(x+j-1)$ if $j > 0$. Note that $(x)_j$ is a polynomial of degree j and since the operator Δ decreases the degree of a polynomial by one, we get, by (2.8),

$$\sum_{l=1}^{j+1} \binom{j+1}{l} (-1)^l (r+2-l)_j = \Delta^{j+1}(x)_j \qquad \text{when} \qquad x = r-j+1,$$

hence is zero for all r and j and (2.11) follows. Therefore, $f(x)$ as given by (2.10) satisfies (2.9). To show that (2.10) represents the general admissible solution to (2.9) we use induction. When $m = 1$, this holds by Theorem 1.

Now assume that (2.6) is such a solution for some m and let $\varphi(x)$ be a solution of (2.9) with m replaced by $m + 1$. Hence

$$(L - t)\,\varphi(x) = \sum_{j=0}^{m-1} \lambda_{j+1} \sum_{k=0}^{\infty} P_{k+j}(x)\,\frac{t^k}{k!}\,,$$

and, by Theorem 1, we obtain

$$\varphi(x) = \lambda_0 \sum_{0}^{\infty} P_k(x)\,\frac{t^k}{k!} + \sum_{j=1}^{m} \frac{\lambda_j}{j} \sum_{k=0}^{\infty} P_{j+k}(x)\,\frac{t^k}{k!}$$

and the induction is complete.

THEOREM 3. *The functions* $\phi_j(x) = \sum_{k=0}^{\infty} P_{k+j}(x)\,(t^k/k!)$, $j = 0, 1,\dots, m$, $|\,t\,| < R_L$ *are linearly independent.*

Proof. Let

$$\sum_{j=0}^{m} a_j \phi_j(x) = 0. \tag{2.12}$$

Therefore, the coefficient of $P_k(x)$ in the left-hand side of (2.12) must vanish. However, this coefficient is the same as the coefficient of x^k in $\sum_{j=0}^{m} a_j x^j e^{tx}$. This shows that $\phi_0(x), \phi_1(x),\dots, \phi_m(x)$ are linearly independent if and only if e^{tx}, $xe^{tx},\dots, x^m e^{tx}$ are so. But e^{tx}, $xe^{tx},\dots, x^m e^{tx}$ are clearly linearly independent. This completes the proof.

Using the same argument of Theorem 3 one can easily prove the following therem.

THEOREM 4. *The functions* $\sum_{k=0}^{\infty} P_{k+j}(x)\,((t_l)^k/k!)$, $j, l = 0, 1, 2,\dots$ *are linearly independent if and only if the t's are distinct.*

The following theorem can be easily verified.

THEOREM 5 (The method of undetermined coefficients). *A particular solution of*

$$(L - t)^m f(x) = \sum_{k=0}^{\infty} P_{k+j}(x)\,\frac{s^k}{k!} \tag{2.13}$$

is either

$$\sum_{l=0}^{j} \lambda_l \sum_{k=0}^{\infty} P_{k+l}(x)\,\frac{s^k}{k!}\,, \qquad \text{when} \qquad s \neq t, \tag{2.14}$$

or

$$\sum_{l=m}^{m+j} \lambda_l \sum_{k=0}^{\infty} P_{k+l}(x) \frac{s^k}{k!}, \qquad \text{when} \qquad s = t, \tag{2.15}$$

where the coefficients λ_l can be determined by substituting for (2.14) *or* (2.15) *in* (2.13).

Following the analogy with differential equations, we call the equation

$$\sum_{k=0}^{n} a_k t^k = 0 \tag{2.12}$$

the characteristic equation of (2.5) *and call its roots characteristic values.*

THEOREM 6. *Let $t_1, \ldots, t_r$ be the characteristic values of* (2.5). *Assume $|t_j| < R_L$ for $j = 1, \ldots, r$. If t_j has multiplicity α_j, then the general admissible solution of* (2.5) *when $g(x) \equiv 0$ is*

$$f(x) = \sum_{j=1}^{r} \sum_{l=0}^{\alpha_j - 1} \lambda_{j,l} \left\{ \sum_{k=0}^{\infty} P_{l+k}(x) \frac{(t_j)^k}{k!} \right\}.$$

Theorem 6 follows from Theorems 2 and 5.

Remark. The above theory could be easily extended to operators J such that Jx^n is a polynomial of degree $n - r$ if $n \geqslant r$, and is zero if $n = 0, 1, \ldots, r - 1$ by showing that such operator has the representation

$$J = \sum_{l=0}^{\infty} b_l(x) D^{l+r},$$

with $b_l(x)$ a polynomial of degree at most l and with b_0 is a nonzero constant. This shows that $J = L^r$, where L reduces the degree of a polynomial by 1 and then we can apply our theory to L.

3. EXAMPLES AND REMARKS

Let us take the operator L to satisfy

$$Lx^n = c_n x^{n-1}, \tag{3.1}$$

where $c_0 = 0$, $c_n \neq 0$ for $n > 0$, and $c_n \to \infty$ as $n \to \infty$. We follow Ward's notation [22] where $[0]! = 1$ and $[n]! = c_1 c_2 \cdots c_n$ for $n > 0$. In this case $P_k(x) = (k!/[k]!)\, x^k$, $R_L = \infty$, and the theory developed in Section 2 is applicable. Typical examples of these operators are the generalized Bessel operator

$x^{-\alpha} D x^{\alpha+1} D$, see [16] when $c_n = n(n+\alpha)$ or Dimovski's (1.2) after a change of both the dependent and independent variable to replace (1.3) by $\alpha_m = 0$ and $\alpha_0 + \alpha_1 + \cdots + \alpha_{m-1} - m = -1$, which corresponds to $c_n = n(n+\alpha_1) \cdots (n + \alpha_{m-1})$. See also the remark at the end of Section 2. The q-difference operator D_q defined by $D_q f(x) = (f(qx) - f(x))/(q-1)x$ also belongs to the above class with $c_n = (q^n - 1)/(q-1)$ when $q > 1$. The case $q < 1$ could also be hendled if in (2.5) x is replaced by xq^{-n}. This reduces (2.5) to a q-difference equation with $q > 1$. Abdi [1] showed how q-Laplace transforms can be used to solve equations of the type (2.5) with $L = D_q$ and $g(x) \equiv 0$ while McLeod [15] and Ismail [14, Chap. 5] developed operational calculi to do the same. However, there is no nkown transform technique or operational calculus that handles equations of type (2.5) with L defined by (3.1) and $g(x) \equiv 0$.

Let us now study the finite difference operator Δ. The polynomial $P_k(x)$ are $P_0(x) = 1$ and $P_k(x) = x(x-1) \cdots (x-k+1)$ for $k > 0$. In this example x takes only the values 0, 1, 2,... and so we have only to assume that $\sum_{k=0}^{\infty} P_{k+j}(x)\,(t^k/k!)$ is an entire function in t. This is true because $\sum_{k=1}^{\infty} P_{k+j}(x)\,(t^k/k!)$ is in fact a polynomial in t. The theory of linear difference equations with constant coefficients is well known. Berg [4] developed an operational calculus to solve these equations, while Al-Salam and Ismail [3] introduced operational calculi and transforms that can solve (1.1) when

$$L \binom{x}{k} = c_k \binom{x}{k-1}, \qquad k = 0, 1, \ldots, \tag{3.2}$$

with $\binom{x}{-1} = 0$. Again here $x = 0, 1, 2, \ldots$, and there is no restriction on $\sum_{k=0}^{\infty} P_{k+j}(x)\,(t^k/k!)$ because these will be polynomials. These are finite difference analogs of (3.1). Al-Salam and Ismail's transforms have an advantage over our present approach because their use of formal series avoids solving a polynomial fo degree n, the characteristic polynomial, which may be difficult for large n.

Note that while our approach of polynomial expansions may seem to be easy, a very important question arises. How can one recognize an equation of type (2.5) because differential and difference equations are usually written in terms of derivatives or difference operators. By expanding powers of $x^{-\alpha} D x^{\alpha+1} D$ in terms of powers of D. Osipov [17] characterized the general form of (1.1) when $L = x^{-\alpha} D x^{\alpha+1} D$; see also Osipov [18]. Expansions of powers or operators of type (3.1) and (3.2) when $c_n = n(n+\alpha_1) \cdots (n+\alpha_k)$ in terms of dericatives and finite difference operators, respectively, follow from general operational formulas developed by Al-Salam and Ismail [2]. However, this general problem is not settled yet.

We note that while Equation (2.5) is of degree n in L and has n linear-independent admissible solutions it might be of a higher degree as a differential or difference equation and hence will have nonadmissible solutions. This situation is not new, and as a matter of fact, it also arises in solving differential equations using the Laplace transform, see Doetsch [7, Sect. 38].

In conclusion, we would like to mention that using polynomial expansions to solve functional equations is ceratinly an old idea. For example, Pincherle [19] used it to solve

$$\sum_{j=1}^{k} a_j y(x + w_j) = F(x).$$

For details and related results, see Sheffer [21]. Our approach is quite different even though we also use polynomial expansions.

REFERENCES

1. W. H. ABDI, On certain q-difference equations and q-Laplace transform, *Proc. Nat. Inst. Sci. India Ser. A* **28** (1962), 1–15.
2. W. AL-SALAM AND M. E. H. ISMAIL, Some operational formulas, *J. Math. Anal. Appl.* **51** (1975), 208–218.
3. W. AL-SALAM AND M. E. H. ISMAIL, A family of operational calculi, to appear.
4. L. BERG, Zur operatorenrechnung fur funktionen einer discreten veranderlichen, *Studia Math.* **20** (1961), 227–243.
5. S. BREUER, Effective reduction of linear ordinary difference equations to equations with constant coefficients, *Ann. Polon. Math.* **28** (1973), 239–284.
6. S. BREUER AND D. GOTTLIEB, The reduction of linear ordinary differential equations to equations with constant coefficients, *J. Math. Anal. Appl.* **33** (1970), 62–76.
7. G. DOETSCH, "Introduction to the Theory and Applications of the Laplace Transformation," (English translation), Springer–Verlag, New York, 1974.
8. I. DIMOVSKI, Operational calculus for a class of differential operators, *C. R. Acad. Bulg. Sci.* **19** (1966), 1111–1114.
9. I. DIMOVSKI, On an operational calculus for a differential operator, *C. R. Acad. Bulg. Sci.* **21** (1968), 513–516.
10. I. DIMOVSKI, On the uniqueness of the operational field of operators $t^{a_0} \prod_k \frac{dt}{d} t^{a_k}$, *C. R. Acad. Bulg. Sci.* **21** (1968), 1255–1258.
11. I. DIMOVSKI, A transform approach to operational calculus for the general Bessel type differential operator, *C. R. Acad. Bulg. Sci.* **27** (1974), 155–158.
12. V. DITKIN AND A. PRUDNIKOV, On the theory of operational calculus for the Bessel equation, *U.S.S.R. Comput. Math. and Math. Phys.* **3** (1963), 296–315.
13. A. ERDELYI, "Operational Calculus and Generalized Functions," Holt, Reinhart, and Winston, New York, 1962.
14. M. E. H. ISMAIL, "A Family of Operational Calculi," Doctoral dissertation, University of Alberta, Edmonton, Alberta, Canada, 1974.
15. B. MACLEOD, "An Outline of a Basic Operational Calculus," unpublished manuscript, University of Adelaide, Australia.
16. N. MELLER, On an operational calculus for the operator B_α (in Russian), *Vuch. Math.* No. 6, *Izv. Akad. Nauk SSSR* (1960), 161–168.
17. S. I. OSIPOV, On the expansion of a polynomial of the operator B_α, *U.S.S.R. Comput. Math. and Math. Phys.* **3** (1963), 250–256.
18. S. I. OSIPOV, On the expansion of a polynomial from Bessel operators, *U.S.S.R. Comput. Math. and Math. Phys.* **4** (1964), 204–213.

19. S. PINCHERLE, Sur la résolution de l'équation fonctionnelle $\sum h_\nu \varphi(x + a_\nu) = f(x)$ a' coefficients constants, *Acta Math.* **48** (1926), 279–304.
20. I. M. SHEFFER, Some properties of polynomial sets of type zero, *Duke Math. J.* **5** (1939), 590–622.
21. I. M. SHEFFER, Some applications of certain polynomial sets, *Bull. Amer. Math. Soc.* **47** (1941), 885–898.
22. M. WARD, Calculus of sequences, *Amer. J. Math.* **48** (1930), 255–266.