Existence and multiplicity of solutions for asymptotically linear noncooperative elliptic systems

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1. Introduction and main results

Consider the noncooperative elliptic systems of the form

\[
\begin{aligned}
-\Delta u + au + bv &= \mu u + g_1(x, u) - h_1(x), \quad \text{in } \Omega, \\
\Delta v + bu + dv &= \mu v + g_2(x, v) - h_2(x), \quad \text{in } \Omega, \\
u &= v = 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1)

where \(\Delta\) denotes the Laplacian operator, \(a, b, d, \mu \in \mathbb{R}\), \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\), \(h_1, h_2 \in L^2(\Omega)\), \(g_1, g_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) are Carathéodory functions, and satisfy

\((g_i)\) For every \(\rho > 0\), there exists a function \(L_\rho \in L^2(\Omega)\) such that

\[
|g_i(x, t)| \leq L_\rho(x)
\]

for all \(|t| \leq \rho\) and a.e. \(x \in \Omega\), \(i = 1, 2\).

As is well known, linear self-adjoint operator \(-\Delta : H^2(\Omega) \cap H^1_0(\Omega) \rightarrow L^2(\Omega)\) possesses an unbounded eigenvalue sequence \(0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots\). Let \(E := H^1_0(\Omega) \times H^1_0(\Omega)\) with the inner product and norm given by

\[
\langle z, w \rangle_E = \langle u, \phi \rangle_{H^1_0(\Omega)} + \langle v, \psi \rangle_{H^1_0(\Omega)}\]

\[
\|z\|_E = \sqrt{\|u\|_{H^1_0(\Omega)}^2 + \|v\|_{H^1_0(\Omega)}^2}
\]

for \(z = (u, v), w = (\phi, \psi) \in E\). Set

\[\tag*{\text{ARTICLE INFO}}\]

\[\text{ABSTRACT}\]

Using the minimax methods in critical point theory and a generalized Landesman–Lazer type condition, we establish two existence results of solutions for asymptotically linear noncooperative elliptic systems at resonance. Besides this, we obtain two solutions in the case of near resonance.

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\(-\vec{\Delta}z := \begin{pmatrix} -\Delta u \\ -\Delta v \end{pmatrix}, \quad A := \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \quad R := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\)

then system (1) can be written as

\[-\vec{\Delta}z + RAz = \mu Rz + R \begin{pmatrix} g_1(x, u) \\ g_2(x, v) \end{pmatrix} - R \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix}.\]

Under the assumption

\[\lim_{|t| \to \infty} \frac{g_i(x, t)}{t} = 0 \quad \text{uniformly for } x \in \Omega, \ i = 1, 2,\]

system (1) is called as a resonant problem if linear problem

\[-\vec{\Delta}z + RAz = \mu Rz\]

has a nonzero solution in \(E\). And problem (2) has a nonzero solution in \(E\) if, and only if, matrix \(-RA + \mu R - \lambda_k I_2\) is singular, where \(I_2\) is the identity matrix of order 2. So for every \(k \in \mathbb{N}^+\), solving equation \(\det(-RA + \mu R - \lambda_k I_2) = 0\) gives two eigenvalues of problem (2) as follows

\[
\mu_k = \frac{a + d}{2} \pm \sqrt{\left(\frac{a - d}{2} + \lambda_k\right)^2 + b^2}.
\]

Moreover, the eigenfunctions space \(N_{\mu_k} \cap \mathbb{Z}\) associated with \(\mu_k\) are of the form

\[N_{\mu_k} = \{\lambda_k \phi_k, \lambda_k^2 \phi_k\} = \{\lambda_k \phi_k, \lambda_k^2 \phi_k\} \quad \text{if } l \neq k.
\]

Forming a new sequence with all elements of set \(\{\mu_k : k \in \mathbb{N}^+\}\) according to their size, then remarking them as

\[
\begin{cases}
\cdots < \mu_{-2} < \mu_{-1} < \mu_0 = \frac{a + d}{2} < \mu_1 < \mu_2 < \cdots, & \text{if } b = 0 \text{ and } \frac{d - a}{2} = \lambda_k \text{ for some } k, \\
\cdots < \mu_{-2} < \mu_{-1} < \mu_0 < \frac{a + d}{2} < \mu_1 < \mu_2 < \cdots, & \text{for others,}
\end{cases}
\]

where if two elements of set \(\{\mu_k : k \in \mathbb{N}^+\}\) are of the same size, we only mark once in the new sequence. In fact, if there exist \(k, l \in \mathbb{N}^+\) with \(k \neq l\), such that \(\lambda_k + \lambda_l = d - a\), which implies that \(\mu_k = \mu_l\), thus we mark \(\mu_j = \mu_k = \mu_l\) for some \(j \in \mathbb{Z}\) in the new sequence, and the corresponding eigenfunctions space \(N_{\mu_j}\) of \(\mu_j\) is of the form \(N_{\mu_j} = N_{\mu_k} \oplus N_{\mu_l}^\perp\).

The associated functional of system (1) is

\[
J(u, v) = \frac{1}{2} \int_\Omega \left( R \left( -\Delta u \right), \left( u \right) \right) dx + \frac{1}{2} \int_\Omega \left( A \left( u \right), \left( u \right) \right) dx - \frac{1}{2} \int_\Omega \left( \left( u \right), \left( u \right) \right) dx
\]

for \((u, v) \in E\), where \(G_1(x, u) = \int_0^u g_1(x, s) ds, G_2(x, v) = \int_0^v g_2(x, s) ds\). Under conditions \((g_\infty)\) and \((g_\infty)\), it is not difficult to check that \(J \in C^1(E, \mathbb{R})\), maps bounded sets into bounded sets in \(E\), and

\[
\langle J'(u, v), (\phi, \psi) \rangle = \int_\Omega \left( R \left( -\Delta u \right), \left( \phi \right) \right) dx + \int_\Omega \left( A \left( u \right), \left( \phi \right) \right) dx - \int_\Omega \left( \left( u \right), \left( \phi \right) \right) dx
\]

for \((u, v), (\phi, \psi) \in E\). Furthermore, the weak solutions of system (1) are exactly the critical points of \(J\) in \(E\).

Noncooperative elliptic systems, arising naturally a steady states in reaction-diffusion process that appear in chemical and biological phenomena, have been extensively investigated in last two decades. For instance, the readers are referred to [8,11] for superlinear cases, and asymptotically linear cases were considered in [5–7,9,10,24,27] and references therein.
To be specific, in the case of \( h_1 = h_2 = 0 \), \([5–7]\) established the existence of solutions for system \((1)\) via the so-called nonquadraticity conditions, \([9,10,24]\) proved system \((1)\) admits a nontrivial solution via Morse theory or index theory, \([27]\) obtained multiple solutions for system \((1)\) with even potential.

Multiplicity of solutions for single equation approaching the first eigenvalue of corresponding linear problem have been studied by many authors since the works of Mawhin and Schmitt \([16,17]\), \([17]\), as well as \([11,12]\), considered the one-dimensional case, \([3,4]\) discussed the higher dimension case. All papers mentioned above are based on bifurcation theory. Using variational methods, \([23,14]\) has proved that there exist at least three solutions for semilinear elliptic equation near resonant at the first eigenvalue, subsequently, these results were extended to \( p \)-Laplacian equation in \([18,13]\), and to cooperative systems in \([20]\). Results for higher eigenvalues were obtained in \([12,16,21]\), where \([12]\) used bifurcation from infinity and degree theory, but only for the one-dimensional case and making use of the fact that in this case all eigenvalues are simple. In \([16]\) they also used bifurcation theory to deal with the eigenvalues of odd multiplicity. The authors of \([21]\) used variational techniques to study semilinear elliptic equation in any spatial dimension for all eigenvalues above the first one.

In \([25]\), Tang first introduced a generalized Landesman–Lazer type condition to study existence of solutions for two-point boundary value problem at resonance, since then, this existence result has been extended to semilinear elliptic equation in \([26]\), \( p \)-Laplacian equation in \([2]\), cooperative elliptic systems in \([19]\). In the present paper, first of all, we will extend these results to system \((1)\). In addition, we will use this kind of technique to study the multiplicity of solutions for system \((1)\) near resonance. Motivated by \([25]\), we define

\[
F_i(x, t) = \begin{cases} 
2G_i(x, t) - \frac{g_i(x)}{4}t^4, & t \neq 0, \\
g_i(x, 0), & t = 0,
\end{cases}
\]

where \( G_i(x, t) = \int_0^t g_i(x, s) ds \), \( i = 1, 2 \). By assuming

\[
(F_-) \quad \liminf_{t \to -\infty} F_i(x, t) = F_i^1(\infty), \quad \limsup_{t \to +\infty} F_i(x, t) = F_i(\infty, +\infty)
\]

uniformly for \( x \in \Omega, \ i = 1, 2 \),

\[
(F_+) \quad \liminf_{t \to +\infty} F_i(x, t) = F_i(\infty, +\infty), \quad \limsup_{t \to -\infty} F_i(x, t) = F_i(\infty, -\infty)
\]

uniformly for \( x \in \Omega, \ i = 1, 2 \), we will prove the following results.

**Theorem 1.** Suppose that \((g_+), (g_\infty)\) and \((F_-)\) hold, and assume that \( F_i(x, -\infty), F_i(x, +\infty) \in L^2(\Omega), \ i = 1, 2 \), and satisfy

\[
\int_\Omega h_1 u \ dx + \int_\Omega h_2 v \ dx < \int_\Omega F_1(x, -\infty) u^+ - \int_\Omega F_1(x, +\infty) u^- + \int_\Omega F_2(x, -\infty) v^+ - \int_\Omega F_2(x, +\infty) v^-
\]

(3)

for \((u, v) \in N_{\mu} \setminus \{(0, 0)\}\), where \( u^+ := \max(0, u) \) and \( u^- := \max(0, -u) \), \( u \in H_0^1(\Omega) \). Then system \((1)\) has at least a weak solution.

**Remark 1.** There exist functions \( h_1, h_2, g_1, g_2 \) satisfying our assumptions. For example (cf. \([25]\)), let \( h_1 = h_2 = 0 \) and

\[
g_1(x, t) = g_2(x, t) = \begin{cases} 
e^{-t^4 \sin t} \ln(1 + t^2) - 1, & t \geq 0 \text{ and } x \in \Omega, \\
1 - 2e^t, & t \leq 0 \text{ and } x \in \Omega.
\end{cases}
\]

It is not difficult to check that \( F_1(x, -\infty) = 1, F_1(x, +\infty) = -1, \ i = 1, 2 \), which implies that \((3)\) holds.

**Theorem 2.** Suppose that \((g_+), (g_\infty)\) and \((F_+)\) hold, and assume that \( F_i(x, +\infty), F_i(x, -\infty) \in L^2(\Omega), \ i = 1, 2 \), and satisfy

\[
\int_\Omega h_1 u \ dx + \int_\Omega h_2 v \ dx < \int_\Omega F_1(x, +\infty) u^+ - \int_\Omega F_1(x, -\infty) u^- + \int_\Omega F_2(x, +\infty) v^+ - \int_\Omega F_2(x, -\infty) v^-
\]

for \((u, v) \in N_{\mu} \setminus \{(0, 0)\}\). Then system \((1)\) has at least a weak solution.

**Remark 2.** There exist functions \( h_1, h_2, g_1, g_2 \) satisfying our assumptions. For example, let \( h_1 = h_2 = 0 \) and

\[
g_1(x, t) = g_2(x, t) = \begin{cases} 1 - e^{-t^4 \sin t} \ln(1 + t^2), & t \geq 0 \text{ and } x \in \Omega, \\
2e^t - 1, & t \leq 0 \text{ and } x \in \Omega.
\end{cases}
\]

The reason is similar to Remark 1.
Theorem 3. Suppose that \( (g_+, \infty) \) and \((F_+) \) hold, and assume that \( F_i(x, -\infty), F_i(x, +\infty) \in L^2(\Omega), i = 1, 2, \) and satisfy
\[
\int_{\Omega} h_1 u \, dx + \int_{\Omega} h_2 v \, dx < \int_{\Omega} F_1(x, -\infty) u^+ - \int_{\Omega} F_1(x, +\infty) u^- + \int_{\Omega} F_2(x, -\infty) v^+ - \int_{\Omega} F_2(x, +\infty) v^- \tag{4}
\]
for \((u, v) \in \mathcal{N}_{\mu_j} \setminus \{(0, 0)\} \). Then there exists \( \delta_1 > 0 \) such that for \( \mu \in (\mu_j, \mu_j + \delta_1) \), system (1) has at least two weak solutions.

Theorem 4. Suppose that \( (g_+, \infty) \) and \((F_+) \) hold, and assume that \( F_i(x, +\infty), F_i(x, -\infty) \in L^2(\Omega), i = 1, 2, \) and satisfy
\[
\int_{\Omega} h_1 u \, dx + \int_{\Omega} h_2 v \, dx < \int_{\Omega} F_1(x, +\infty) u^+ - \int_{\Omega} F_1(x, -\infty) u^- + \int_{\Omega} F_2(x, +\infty) v^+ - \int_{\Omega} F_2(x, -\infty) v^- \tag{5}
\]
for \((u, v) \in \mathcal{N}_{\mu_j} \setminus \{(0, 0)\} \). Then there exists \( \delta_2 > 0 \) such that for \( \mu \in (\mu_j - \delta_2, \mu_j) \), system (1) has at least two weak solutions.

2. Preliminaries

For \( z = (u, v) \), let
\[
I(z) = \frac{1}{2} \int_{\Omega} \left( \mathcal{R} \left( -\Delta u \right), \left( u \atop v \right) \right) \, dx + \frac{1}{2} \int_{\Omega} \left( A \left( u \atop v \right), \left( u \atop v \right) \right) \, dx,
\]
thus there exists a positive constant \( \nu \) dependent of \( \mu \) such that
\[
I(z) = \frac{\mu}{2} \int_{\Omega} |z|^2 \, dx \begin{cases} \geq \frac{\nu}{2} \|z\|^2_E, & \text{for } z \in \bigoplus_{\mu_j > \mu} \mathcal{N}_{\mu}, \\ = 0, & \text{for } z \in \mathcal{N}_{\mu}, \\ \leq -\frac{\nu}{2} \|z\|^2_E, & \text{for } z \in \bigoplus_{\mu_j < \mu} \mathcal{N}_{\mu}, \end{cases}
\]
where \( |z|^2 = u^2 + v^2 \). (See [6, Proposition 2.2].)

In addition, we introduce an abstract notion and result in the following, see [22].

Definition 1. (See [22].) Let \( X \) be a real Banach space with \( X = X_1 \oplus X_2 \), where both \( X_1 \) and \( X_2 \) may be infinite-dimensional. Let \( P_1, P_2 \) be the projectors of \( X \) onto \( X_1, X_2 \), associated with the given splitting of \( X \). Let \( S, Q \subset X \) with \( Q \subset \tilde{X} \), a given subspace of \( X \), and \( \partial Q \) will refer to the boundary of \( Q \) in \( \tilde{X} \). We say \( S \) and \( \partial Q \) link if whenever \( \sigma \in S \) and \( \sigma (t, \partial Q) \cap S = \emptyset \) for all \( t \in [0, 1] \), then \( \sigma (t, Q) \cap S \neq \emptyset \) for all \( t \in [0, 1] \), where
\[
S := \{ \sigma \in C([0, 1] \times X, X) \mid \sigma (0, u) = u \text{ and } P_2 \sigma (t, u) = P_2 u - K(t, u), \text{ where } K : [0, 1] \times X \to X_2 \text{ is compact} \}.
\]

Remark 3. (See [22].) Let \( X, X_1, X_2 \) as in Definition 1, \( Q := \mathcal{O} \cap X_2 \), where \( \mathcal{O} \) is a neighborhood of \( 0 \) in \( X \), \( \tilde{X} := X_2 \) and \( S := X_1 \). Then \( S \) and \( \partial Q \) link.

Theorem 5. (See [22].) Let \( X \) be a Hilbert space with \( X = X_1 \oplus X_2 \) and \( X_2 = X_1^\perp \). Suppose that \( J \in C^1(X, \mathbb{R}) \), satisfies the \((PS)\) condition, and
\[
\begin{align*}
(J_1) & \quad J(u) = \frac{1}{2} (Lu, u) + \Psi(u), \text{ where } L u = L_1 P_1 u + L_2 P_2 u, L_i : X_i \to X_i \text{ is bounded and selfadjoint, } i = 1, 2, \\
(J_2) & \quad \Psi' \text{ compact.}
\end{align*}
\]

There exist a subspace \( \tilde{X} \subset E \) and sets \( S \subset X \), \( Q \subset \tilde{X} \) and constants \( \nu > 0 \) such that
\[
\begin{align*}
(i) & \quad S \subset X_1 \text{ and } J_S \geq \nu, \\
(ii) & \quad Q \text{ is bounded and } J_{|Q} \leq \alpha, \\
(iii) & \quad S \text{ and } \partial Q \text{ link.}
\end{align*}

Then \( J \) possesses a critical value \( c \geq \nu \).

3. Proofs of theorems

Lemma 1. Suppose that \( (g_+) \) and \( (g_\infty) \) hold, further assume that \((F_-) \) and \((3) \) hold if there is resonance. Then \( J \) satisfies the \((PS)\) condition.

Proof. For any sequence \( \{z_n = (u_n, v_n) \} \subset E \) such that
\[
\begin{align*}
|J(z_n)| < \infty & \quad \text{for all } n, \\
& \quad \text{and } J'(z_n) \to 0 \quad \text{as } n \to \infty,
\end{align*}
\]
we need to prove that \{\varepsilon_n\} has a convergent subsequence. By the standard argument, it suffices to show \{\varepsilon_n\} is bounded in \(E\). If not, without loss of generality, we suppose that \(\|\varepsilon_n\|_E \to \infty\) as \(n \to \infty\). From this, we will reach a contradiction no matter whether there is resonance. Set \((\bar{u}_n, \bar{v}_n) = (\frac{u_n}{\|\varepsilon_n\|_E}, \frac{v_n}{\|\varepsilon_n\|_E})\), one has

\[
\begin{align*}
\bar{u}_n & \to \bar{u}, \quad \bar{v}_n \to \bar{v} \quad \text{in } H_0^1(\Omega), \\
\bar{u}_n & \to \bar{u}, \quad \bar{v}_n \to \bar{v} \quad \text{in } L^q(\Omega) \text{ with } q \in (1, 2^*), \\
\bar{u}_n(x) & \to \bar{u}(x), \quad \bar{v}_n(x) \to \bar{v}(x) \quad \text{for a.e. } x \in \Omega,
\end{align*}
\]

(7)

where \(2^* := \frac{2N}{N-2}\) if \(N \geq 3\) or \(2^* := +\infty\) if \(N = 1, 2\). In addition, it follows from \((g_\varepsilon)\) and \((g_\infty)\) that for every \(\varepsilon > 0\) there exists \(\bar{\rho}_\varepsilon > 0\) such that

\[
|g_\varepsilon(x, t)| \leq \varepsilon|t| + L\bar{\rho}_\varepsilon(x)
\]

for all \(t \in \mathbb{R}\) and a.e. \(x \in \Omega, i = 1, 2\). Then we have

\[
\frac{1}{\|\varepsilon_n\|_E^2} \int_\Omega g_1(x, u_n)u_n \, dx \leq \frac{1}{\|\varepsilon_n\|_E^2} \int_\Omega \|g_1(x, u_n)\|_{\varepsilon_n} \, dx
\]

\[
\leq \frac{1}{\|\varepsilon_n\|_E^2} \left( \varepsilon \|u_n\|_{L^2(\Omega)}^2 + \|L\bar{\rho}_\varepsilon\|_{L^2(\Omega)} \cdot \|u_n\|_{L^2(\Omega)} \right)
\]

\[
\leq \frac{\varepsilon}{2\lambda_1} + \frac{L\bar{\rho}_\varepsilon}{\sqrt{\lambda_1}\|\varepsilon_n\|_E}
\]

let \(n \to \infty\) in this expression, one has

\[
\lim_{n \to \infty} \frac{1}{\|\varepsilon_n\|_E^2} \int_\Omega g_1(x, u_n)u_n \, dx \leq \frac{\varepsilon}{2\lambda_1}.
\]

(9)

by the arbitrariness of \(\varepsilon\), we get

\[
\lim_{n \to \infty} \frac{1}{\|\varepsilon_n\|_E^2} \int_\Omega g_1(x, u_n)u_n \, dx = 0,
\]

(10)

Similarly, one has

\[
\lim_{n \to \infty} \frac{1}{\|\varepsilon_n\|_E^2} \int_\Omega g_2(x, v_n) \, dx = 0,
\]

(11)

\[
\lim_{n \to \infty} \frac{1}{\|\varepsilon_n\|_E^2} \int_\Omega g_1(x, u_n) \, dx = 0,
\]

(12)

where \((\phi, \psi) \in E\). Then, by (6), (7), (9) and (10), passing to the limit in

\[
\left( \frac{J'(u_n, \varepsilon_n), (u_n, -v_n)}{\|\varepsilon_n\|_E^2} \right) = 1 + \int_\Omega \left( A\left( \bar{u}_n, \bar{v}_n \right) \cdot \left( \bar{u}_n, -\bar{v}_n \right) \right) \, dx - \int_\Omega \left( \mu \left( \bar{u}_n, \bar{v}_n \right) \cdot \left( \bar{u}_n, -\bar{v}_n \right) \right) \, dx - \int_\Omega \frac{g_1(x, u_n)u_n}{\|\varepsilon_n\|_E^2} \, dx
\]

\[
- \int_\Omega \frac{g_2(x, v_n)(-v_n)}{\|\varepsilon_n\|_E^2} \, dx + \int_\Omega \frac{h_1u_n}{\|\varepsilon_n\|_E^2} \, dx + \int_\Omega \frac{h_2(-v_n)}{\|\varepsilon_n\|_E^2} \, dx
\]

gives
Meanwhile, for every \((\phi, \psi) \in E\), by (6), (7), (11) and (12), passing to the limit in

\[
\left( \frac{f'(u_n, v_n)}{\|z_n\|_E}, (\phi, \psi) \right) \rightarrow \int \left( \mathcal{R} \left( -\Delta u_n + \Delta v_n \right), (\phi, \psi) \right) \, dx + \int \left( A \left( \frac{u_n}{v_n} \right), (\phi, \psi) \right) \, dx - \int \left( \mu \left( \frac{u_n}{v_n} \right), (\phi, \psi) \right) \, dx
\]

which implies that

\[
(\bar{u}, \bar{v}) \neq (0, 0).
\]

Meanwhile, for every \((\phi, \psi) \in E\), by (6), (7), (11) and (12), passing to the limit in

\[
\left( \frac{f'(u_n, v_n)}{\|z_n\|_E}, (\phi, \psi) \right) \rightarrow \int \left( \mathcal{R} \left( -\Delta u_n + \Delta v_n \right), (\phi, \psi) \right) \, dx + \int \left( A \left( \frac{u_n}{v_n} \right), (\phi, \psi) \right) \, dx - \int \left( \mu \left( \frac{u_n}{v_n} \right), (\phi, \psi) \right) \, dx
\]

gives

\[
\int \left( \mathcal{R} \left( -\Delta \frac{u_n}{v_n} \right), (\phi, \psi) \right) \, dx + \int \left( A \left( \frac{u_n}{v_n} \right), (\phi, \psi) \right) \, dx = \int \left( \mu \left( \frac{u_n}{v_n} \right), (\phi, \psi) \right) \, dx
\]

by the arbitrariness of \((\phi, \psi) \in E\), we have

\[
-\Delta \left( \frac{u_n}{v_n} \right) + \mathcal{R} A \left( \frac{u_n}{v_n} \right) = \mu \mathcal{R} \left( \frac{u_n}{v_n} \right).
\]

In the nonresonant case, this contradicts (13).

In the resonance case, it follows from (13) and (14) that \((\bar{u}, \bar{v}) \in \mathcal{N}_\mu \setminus \{(0, 0)\}\). For any \(\epsilon > 0\), set

\[
C_i^\epsilon(x) := F_i(x, -\infty) - \epsilon, \quad D_i^\epsilon(x) := F_i(x, +\infty) + \epsilon,
\]

\(i = 1, 2\), then there exists \(\rho_\epsilon > 0\) such that

\[
F_i(x, t) \begin{cases} 
\geq C_i^\epsilon(x), & \text{for } t \leq -\rho_\epsilon \text{ and a.e. } x \in \Omega, \\
\leq D_i^\epsilon(x), & \text{for } t \geq \rho_\epsilon \text{ and a.e. } x \in \Omega,
\end{cases}
\]

\(i = 1, 2\), which implies that

\[
F_i(x, t) t \leq \begin{cases} 
C_i^\epsilon(x) t, & \text{for } t \leq -\rho_\epsilon \text{ and a.e. } x \in \Omega, \\
D_i^\epsilon(x), & \text{for } t \geq \rho_\epsilon \text{ and a.e. } x \in \Omega,
\end{cases}
\]

\(i = 1, 2\). It follows from this and \((g_\epsilon)\) that

\[
\left( \frac{f'(z_n) - 2f(z_n)}{\|z_n\|_E} \right) \rightarrow \int \frac{F_1(x, u_n)}{\|z_n\|_E} \, dx + \int \frac{F_2(x, v_n)}{\|z_n\|_E} \, dx - \int \frac{h_1 u_n}{\|z_n\|_E} \, dx - \int \frac{h_2 v_n}{\|z_n\|_E} \, dx
\]

\[
\leq - \int \frac{C_1^\epsilon(x) u_n^-}{\|z_n\|_E} \, dx + \int \left( 3L_{\rho_\epsilon} \epsilon u_n^2 - C_1^\epsilon(x) u_n \right) \, dx
\]

\[
+ \int \frac{D_1^\epsilon(x) u_n^+}{\|z_n\|_E} \, dx + \int \left( 3L_{\rho_\epsilon} \epsilon u_n^2 - D_1^\epsilon(x) u_n \right) \, dx
\]

\[
- \int \frac{C_2^\epsilon(x) v_n^-}{\|z_n\|_E} \, dx + \int \left( 3L_{\rho_\epsilon} \epsilon v_n^2 - C_2^\epsilon(x) v_n \right) \, dx
\]

\[
+ \int \frac{D_2^\epsilon(x) v_n^+}{\|z_n\|_E} \, dx + \int \left( 3L_{\rho_\epsilon} \epsilon v_n^2 - D_2^\epsilon(x) v_n \right) \, dx
\]

\[
- \int \frac{h_1 u_n}{\|z_n\|_E} \, dx - \int \frac{h_2 v_n}{\|z_n\|_E} \, dx.
\]

Meanwhile, using Hölder inequality and (7), we have
\[
\left| \int_{\Omega} C_1^e(x) \frac{u_n}{\|z_n\|_E} \, dx - \int_{\Omega} C_1^e(x) \bar{u} \, dx \right| \\
\leq \left| \int_{\Omega} C_1^e(x) \frac{v_n}{\|z_n\|_E} \, dx \right| - \int_{\Omega} C_1^e(x) \bar{v} \, dx \\
\leq \left\| C_1^e \right\|_{L^2(\Omega)} \left\| \frac{u_n}{\|z_n\|_E} - \bar{u} \right\|_{L^2(\Omega)} \\
\rightarrow 0, \tag{17}
\]

as \( n \rightarrow \infty \). Similarly, one has
\[
\left| \int_{\Omega} D_1^e(x) \frac{u_n}{\|z_n\|_E} \, dx - \int_{\Omega} D_1^e(x) \bar{u} \, dx \right| \rightarrow 0, \tag{18}
\]
\[
\left| \int_{\Omega} C_2^e(x) \frac{v_n}{\|z_n\|_E} \, dx - \int_{\Omega} C_2^e(x) \bar{v} \, dx \right| \rightarrow 0, \tag{19}
\]
and
\[
\left| \int_{\Omega} D_2^e(x) \frac{v_n}{\|z_n\|_E} \, dx - \int_{\Omega} D_2^e(x) \bar{v} \, dx \right| \rightarrow 0, \tag{20}
\]
as \( n \rightarrow \infty \). By (6), (7), (17)–(20), passing to limit in (16) gives
\[
\left| \int_{\Omega} h_1 \bar{u} \, dx + \int_{\Omega} h_2 \bar{v} \, dx \right| \\
\leq \left| \int_{\Omega} C_1^e(x) \bar{u} \, dx \right| + \left| \int_{\Omega} D_1^e(x) \bar{u} \, dx \right| + \left| \int_{\Omega} C_1^e(x) \bar{v} \, dx \right| + \left| \int_{\Omega} D_1^e(x) \bar{v} \, dx \right|,
\]
by the arbitrariness of \( \varepsilon \), one has
\[
\left| \int_{\Omega} h_1(-\bar{u}) \, dx + \int_{\Omega} h_2(-\bar{v}) \, dx \right| \\
\geq \left| \int_{\Omega} F_1(x, -\infty)(\bar{u})^+ \, dx \right| - \left| \int_{\Omega} F_1(x, +\infty)(\bar{u})^- \, dx \right| \\
+ \left| \int_{\Omega} F_2(x, -\infty)(\bar{v})^+ \, dx \right| - \left| \int_{\Omega} F_2(x, +\infty)(\bar{v})^- \, dx \right|,
\]
which is in contradiction with (3) because \((-\sigma, -\bar{v}) \in N_{\mu} \setminus \{(0, 0)\}).

To sum up, \( \{z_n\} \) is bounded in \( E \) no matter whether there is resonance for system (1). Our proof is completed. \( \Box \)

**Proof of Theorem 1.** By Lemma 1, \( J \) satisfies the (P.S.) condition. To apply Theorem 5 via a geometry structure in Remark 3, conditions \((J_1)\) and \((J_2)\) are trivially verified, so we only need to check condition \((J_3)\), it suffices to prove that there exists a splitting \( E = E_1 \oplus E_2 \) with \( E_2 = E_1^* \), such that
\[
J(u, v) \rightarrow +\infty \tag{21}
\]
as \( \|(u, v)\|_E \rightarrow \infty \) in \( E_1 \) and
\[
J(u, v) \rightarrow -\infty \tag{22}
\]
as \( \|(u, v)\|_E \rightarrow \infty \) in \( E_2 \).
In the nonresonant case, i.e. \( \mu_j \leq \mu < \mu_{j+1} \) for some \( j \in \mathbb{Z} \), let
\[
E_1 = \bigoplus_{i=j+1}^{\infty} N_{\mu_i}, \quad E_2 = \bigoplus_{-\infty}^{j} N_{\mu_i}.
\]
Then we have \( E = E_1 \oplus E_2 \) and \( E_2 = E_1^* \). In addition, it follows from (8) that
\[
|G_i(x, t)| \leq \frac{\varepsilon}{2} |t|^2 + L_{\mu_{i}}(x)|t|
\]
for all \( t \in \mathbb{R} \) and \( x \in \Omega, i = 1, 2 \). On one hand, for \( z = (u, v) \in E_1 \), from (23) with \( \varepsilon \in (0, \frac{1}{\mu_{j+1}}) \) it follows that
exist a positive constant which implies that (22) holds with
\[ L_{\tilde{\rho}_{i}}(x) = \int \frac{1}{\sqrt{1 + \lambda}} \bigl( \| L_{\tilde{\rho}_{i}} \|_{L^{2}(\Omega)} + \| h_{1} \|_{L^{2}(\Omega)} + \| h_{2} \|_{L^{2}(\Omega)} \bigr) \| z \|_{E} \] 
which implies that (21) holds. On the other hand, for \( z := (u, v) \in E_{2} \), from (23) with \( \varepsilon \in (0, \frac{\nu \lambda}{2}) \) it follows that
\[ J(z) \leq I(z) - \frac{\mu_{j}}{2} \int_{\Omega} |z|^{2} dx - \frac{E}{2} \int_{\Omega} |z|^{2} dx - \int_{\Omega} L_{\tilde{\rho}_{i}}(x) (|u| + |v|) dx + \int h_{1} u dx + \int h_{2} v dx \]
\[ \leq \frac{-v_{j}}{2} \int_{\Omega} |z|^{2} dx + \frac{E}{2} \int_{\Omega} |z|^{2} dx + \frac{1}{\sqrt{1 + \lambda}} \bigl( \| L_{\tilde{\rho}_{i}} \|_{L^{2}(\Omega)} + \| h_{1} \|_{L^{2}(\Omega)} + \| h_{2} \|_{L^{2}(\Omega)} \bigr) \| z \|_{E} \]
which implies that (22) holds.
In the resonance case, i.e. \( \mu = \mu_{j} \) for some \( j \in \mathbb{Z} \), let
\[ W_{1} = \bigoplus_{i=j+1}^{\infty} N_{\mu_{j}}, \quad W_{0} = N_{\mu_{j}}, \quad W_{2} = \bigoplus_{i=-\infty}^{j-1} N_{\mu_{i}}, \]
it is clear that \( E = W_{1} \oplus W_{0} \oplus W_{2} \) and \( W_{2} = (W_{1} \oplus W_{0})^{-1} \). In a way similar to the proof of (9), from (23), we have
\[ \lim_{\|(u, v)\|_{E} \to \infty} \frac{1}{\|(u, v)\|_{E}^{2}} \int_{\Omega} G_{1}(x, u) dx = 0 \] 
(24)
and
\[ \lim_{\|(u, v)\|_{E} \to \infty} \frac{1}{\|(u, v)\|_{E}^{2}} \int_{\Omega} G_{2}(x, v) dx = 0. \] 
(25)
For \( z := (u, v) \in W_{2} \), by (24) and (25), passing to the limit \( \|z\|_{E} \to \infty \) in
\[ \frac{J(z)}{\|z\|_{E}^{2}} \leq \frac{I(z)}{\|z\|_{E}^{2}} - \frac{\mu_{j} \int_{\Omega} |z|^{2} dx}{2 \|z\|_{E}^{2}} - \int_{\Omega} \frac{G_{1}(x, u)}{\|z\|_{E}^{2}} dx + \int_{\Omega} \frac{G_{2}(x, v)}{\|z\|_{E}^{2}} dx + \int h_{1} u dx + \int h_{2} v dx \]
\[ \leq \frac{-v_{j}}{2} - \int_{\Omega} \frac{G_{1}(x, u)}{\|z\|_{E}^{2}} dx + \int_{\Omega} \frac{G_{2}(x, v)}{\|z\|_{E}^{2}} dx + \int h_{1} u dx + \int h_{2} v dx \]
(where \( v_{j} \) is a positive constant dependent of \( \mu_{j} \)) gives
\[ \limsup_{\|z\|_{E} \to \infty} \frac{J(z)}{\|z\|_{E}^{2}} \leq \frac{-v_{j}}{2}, \]
which implies that (22) holds with \( E_{2} = W_{2} \).
In the following, we check that (21) holds with \( E_{1} = W_{1} \oplus W_{0} \). If not, without loss of generality, we suppose that there exist a positive constant \( C \) and a sequence \( \{z_{n} \mid z_{n} := (u_{n}, v_{n}) \} \subset W_{1} \oplus W_{0} \) satisfying
\[ J(z_{n}) \leq C \quad \text{for all } n, \quad \|z_{n}\|_{E} \to \infty \quad \text{as } n \to \infty. \] 
(26)
Writing \( z_{n} \) in the form
\[ z_{n} = z_{1n} + z_{0n}, \]
where \( z_{1n} := (u_{1n}, v_{1n}) \in W_{1}, z_{0n} := (u_{0n}, v_{0n}) \in W_{0}, \) it follows from (26), (24)–(25) that
0 \geq \liminf_{n \to \infty} \frac{C}{\|z_n\|^2_E} \geq \liminf_{n \to \infty} \frac{f(z_n)}{\|z_n\|^2_E} \\
\geq \frac{1}{2} \liminf_{n \to \infty} \frac{v_j}{\|z_n\|^2_E} \geq \limsup_{n \to \infty} \frac{1}{\|z_n\|^2_E} \int_\Omega (G_1(x, u_n) + G_2(x, v_n)) \, dx + \liminf_{n \to \infty} \frac{1}{\|z_n\|^2_E} \int_\Omega (h_1 u_n + h_2 v_n) \, dx \\
\geq \frac{v_j}{2} \liminf_{n \to \infty} \int_\Omega \|\nabla z_n\|^2 \, dx / \|z_n\|^2_E,

which implies that 

\frac{z_{1n}}{\|z_n\|^2_E} \to (0, 0) \quad \text{in } E, \quad (27)

as \( n \to \infty \). Since \( W_0 \) is finite-dimensional, with loss of generality we may suppose that 

\frac{z_{0n}}{\|z_n\|^2_E} \to (u_0, v_0) \quad \text{in } W_0, \quad (28)

as \( n \to \infty \). Combining (27) and (28), one has 

\frac{z_n}{\|z_n\|^2_E} \to (u_0, v_0) \quad \text{in } E, \quad (29)

as \( n \to \infty \), which shows that \( \|u_0, v_0\|_E = 1 \), i.e. \((u_0, v_0) \in W_0 \setminus (0, 0)\).

Additional, let \( s, \rho_\varepsilon, C_i^\varepsilon(x) \) and \( D_i^\varepsilon(x) \) be as in the proof of Lemma 1, then from (15) one has 

\frac{d}{dt} \left( -G_i(x, t) - Ct_i^\varepsilon \right) = \frac{F_i(x, t)}{t^2} \geq \frac{C_i^\varepsilon(x)}{t^2} = \frac{d}{dt} \left( -C_i^\varepsilon(x) / t \right)

for \( t \leq -\rho_\varepsilon \) and a.e. \( x \in \Omega, \, i = 1, 2 \). Integrating both sides over \([s, t] \subset (-\infty, -\rho_\varepsilon]\) and applying \( g_\infty \), we obtain 

\frac{G_i(x, t)}{t^2} \geq \frac{C_i^\varepsilon(x)}{t}

for \( t \leq -\rho_\varepsilon \) and a.e. \( x \in \Omega, \, i = 1, 2 \). Hence we get 

\( G_i(x, t) \leq C_i^\varepsilon(x) t \) \quad (30)

for \( t \leq -\rho_\varepsilon \) and a.e. \( x \in \Omega, \, i = 1, 2 \). Similarly, one has 

\( G_i(x, t) \leq D_i^\varepsilon(x) t \) \quad (31)

for \( t \geq \rho_\varepsilon \) and a.e. \( x \in \Omega, \, i = 1, 2 \). Besides this, in a way similar to the proof of (17)–(20), we have 

\| C_i^\varepsilon(x) u_n^- \|_E \, dx - \int_\Omega C_i^\varepsilon(x) u_0^- \, dx \to 0, \quad (32)

\| D_i^\varepsilon(x) u_n^+ \|_E \, dx - \int_\Omega D_i^\varepsilon(x) u_0^+ \, dx \to 0, \quad (33)

\| C_i^\varepsilon(x) v_n^- \|_E \, dx - \int_\Omega C_i^\varepsilon(x) v_0^- \, dx \to 0, \quad (34)

and 

\| D_i^\varepsilon(x) v_n^+ \|_E \, dx - \int_\Omega D_i^\varepsilon(x) v_0^+ \, dx \to 0, \quad (35)

as \( n \to \infty \). From (26), (30) and (31), we have 

\[ C \geq f(z_n) \]

\[ \geq - \int_\Omega G_1(x, u_n) \, dx - \int_\Omega G_2(x, v_n) \, dx + \int_\Omega h_1 u_n \, dx + \int_\Omega h_2 v_n \, dx \]
It is easy to check that $\pi$ is a homeomorphism. We extend the map $\pi$ by the arbitrariness of $\rho_s$ divided both sides of above inequality by $\|\langle u_n, v_n \rangle\|_E$ and passing to the limit, it follows from (32)-(35) that
\[
\int_{\Omega} h_1 u_0 \, dx + \int_{\Omega} h_2 v_0 \, dx \leq - \int_{\Omega} C_1^2(x) \langle u_0 \rangle^+ \, dx + \int_{\Omega} C_1^2(x) \langle v_0 \rangle^+ \, dx + \int_{\Omega} D_2^n(x) \langle v_0 \rangle^- \, dx,
\]
by the arbitrariness of $\varepsilon$, one has
\[
\int_{\Omega} h_1(u_0) \, dx + \int_{\Omega} h_2(-v_0) \, dx \geq \int_{\Omega} F_1(x, -\infty)(-u_0)^+ \, dx - \int_{\Omega} F_1(x, +\infty)(-u_0)^- \, dx
\]
\[
+ \int_{\Omega} F_2(x, -\infty)(-v_0)^+ \, dx - \int_{\Omega} F_2(x, +\infty)(-v_0)^- \, dx,
\]
which is in contradiction with (3) because $(-u_0, -v_0) \in \mathcal{N}_{\mu_1}(0, 0)$. Our proof is completed. $\square$

Theorem 2 can be proved similarly. To prove Theorem 3, we need several auxiliary results.

**Lemma 2.** Let $W_1, W_0, W_2$ be as in the proof of Theorem 1. Set
\[
A = \{ w_1 \in W_1 \mid \|w_1\|_E \geq T_1 \} \cup \{ w_2 = w_0 + w_1 \mid w_0 \in W_0, w_1 \in W_1, \|w\|_E = T_1 \},
\]
\[
B_{W_0 \oplus W_2} = \{ w_2 = w_0 + w_2 \mid w_0 \in W_0, w_2 \in W_2, \|w\|_E \leq 1 \},
\]
\[
B = T_2 \cap B_{W_0 \oplus W_2} = \{ w_2 = w_0 + w_2 \mid w_0 \in W_0, w_2 \in W_2, \|w\|_E = T_2 \},
\]
where $T_1 < T_2$. Then $A$ and $B$ link in the sense of Definition 1.

**Remark 4.** In fact, this result is a generalization of Lemma 4.6 in [21] where $W_0 \oplus W_2$ is finite-dimensional.

**Proof.** Arbitrarily choose $\mathbf{w} \in W_0$ with $\|\mathbf{w}\|_E = 1$, set
\[
\mathbf{A} = \{ w_1 \in W_1 \mid \|w_1\|_E \geq T_1 \} \cup \{ w = s\mathbf{w} + w_1 \mid w_1 \in W_1, s \geq 0, \|w\|_E = T_1 \}.
\]
Clearly, it suffices to verify that $\mathbf{A}$ and $B$ link. Let $P_{W_1}$ and $P_{W_0 \oplus W_2}$ are respectively projectors on $W_1$ and $W_0 \oplus W_2$, then map
\[
\pi : \mathbf{A} \rightarrow W_1: w \in \mathbf{A} \mapsto P_{W_1}(w)
\]

is a homeomorphism. We extend the map $\pi$ on $E$ given by
\[
\tilde{\pi} : E \rightarrow E: w_1 + w_0 + w_2 \mapsto w_1 + w_0 + w_2 + (w_1 - \pi^{-1}(w_1)).
\]
It is easy to check that $\tilde{\pi}$ is also a homeomorphism and $\tilde{\pi}^{-1}(0) = 0 - \pi^{-1}(0) = -T_1 \mathbf{w}$. Then, for any $\sigma \in \mathcal{S} := \{ \sigma \in C([0, 1] \times E, E) \mid \sigma(0, u) = u \}$ and $P_{W_0 \oplus W_2}\sigma(t, u) = P_{W_0 \oplus W_2}u - K(t, u)$, where $K : [0, 1] \times E \rightarrow W_0 \oplus W_2$ is compact, for every $t \in [0, 1]$, we have
\[
\sigma(t, T_2B_{W_0 \oplus W_2}) \cap \mathbf{A} \neq \emptyset \iff \sigma(t, T_2B_{W_0 \oplus W_2}) \cap \pi^{-1}(W_1) \neq \emptyset
\]
\[
\iff \sigma(t, T_2B_{W_0 \oplus W_2}) \cap \tilde{\pi}^{-1}(W_1) \neq \emptyset
\]
\[
\iff \tilde{\pi} \circ \sigma(t, T_2B_{W_0 \oplus W_2}) \cap W_1 \neq \emptyset
\]
\[
\iff \text{there exists } z \in T_2B_{W_0 \oplus W_2} \text{ such that } P_{W_0 \oplus W_2} \circ \tilde{\pi} \circ \sigma(t, z) = 0.
\]
(36)
Similarly, for every $t \in [0, 1]$, we have

$$
\sigma(t, B) \cap \tilde{A} = \emptyset \iff P_{W_0 \oplus W_2} \circ \pi \circ \sigma(t, B) \neq 0.
$$

(37)

In addition, for $\sigma \in S$, we have

$$
\sigma(t, u) = P_{W_1} \circ \sigma(t, u) + P_{W_0 \oplus W_2} \circ \sigma(t, u) = P_{W_1} \circ \sigma(t, u) + P_{W_0 \oplus W_2} u - K(t, u),
$$

then one has

$$
P_{W_0 \oplus W_2} \circ \pi \circ \sigma(t, u) = P_{W_0 \oplus W_2} u - K(t, u) - P_{W_0 \oplus W_2} \circ \pi^{-1} \circ P_{W_1} \circ \sigma(t, u),
$$

where $P_{W_0 \oplus W_2} \circ \pi^{-1} \circ P_{W_1} \circ \sigma([0, 1], E) \subset S$ and $S \in [0, T_1]$, which implies that $P_{W_0 \oplus W_2} \circ \pi^{-1} \circ P_{W_1} \circ \sigma : [0, 1] \times E \to W_2 \oplus W_0$ is compact. Hence, for $\sigma \in S$ such that $\sigma(t, B) \cap \tilde{A} = \emptyset$ for all $t \in [0, 1]$, from (37) it follows that $\text{deg}_{L-S}(P_{W_0 \oplus W_2} \circ \pi \circ \sigma(t, \cdot), T_2 B_{W_0 \oplus W_2}, 0)$ is well defined for all $t \in [0, 1]$, where $\text{deg}_{L-S}$ is Leray–Schauder degree. By homotopy invariance of Leray–Schauder degree, for every $t \in [0, 1]$, we have

$$
\text{deg}_{L-S}(P_{W_0 \oplus W_2} \circ \pi \circ \sigma(t, \cdot), T_2 B_{W_0 \oplus W_2}, 0) = \text{deg}_{L-S}(P_{W_0 \oplus W_2} \circ \pi \circ \sigma(0, \cdot), T_2 B_{W_0 \oplus W_2}, 0) = 1.
$$

Hence, from (36) it follows that

$$
\sigma(t, T_2 B_{W_0 \oplus W_2}) \cap \tilde{A} \neq \emptyset
$$

for all $t \in [0, 1]$. Therefore, $\tilde{A}$ and $B$ link in the sense of Definition 1. □

**Lemma 3.** Let $X$ be a Hilbert space which is the topological direct sum of subspaces $X_1$ and $X_2$. Let $P_1, P_2$ be the projectors of $X$ onto $X_1, X_2$. Suppose that $J \in C^1(X, \mathbb{R})$ with $J(u) = \frac{1}{2} \langle Lu, u \rangle + \Psi(u)$, where $Lu = L_1 P_1 u + L_2 P_2 u$, $L_i : X_i \to X_i$ is bounded and selfadjoint, $i = 1, 2$, and $\Psi'$ is compact. Moreover, suppose that there exist $r_1, r_2 > 0$ such that

$$
\sup_{z \in \sigma(z) \in \mathbb{B}_{X_2}} J(z) < \alpha := \sup_{z \in \mathbb{R} \cap B_{X_1}} J(z) < \inf_{z \in \sigma(z) \in \mathbb{B}_{X_2}} J(z).
$$

(38)

If $(P.S.)_c$ holds for any $c \in [\alpha, \beta]$, there exists at least one critical point $z_0$ of $J$ such that $\alpha \leq J(z_0) \leq \beta$.

**Remark 5.** In fact, this result is a generalization of Theorem 8.1 of [15] in which A. Marino, A.M. Micheletti and A. Pistoi obtained the same result when either $X_1$ or $X_2$ is finite-dimensional.

**Proof.** By negation, suppose that every $c \in [\alpha, \beta]$ is a regular value. Then by Proposition A.18 in [22], there exists $\epsilon > 0$ small enough and a deformation $\zeta : [0, 1] \times J^\beta \rightarrow J^\beta$ of the form

$$
\zeta(t, z) = e^{\theta(t, z) + K(t, z)},
$$

where $J^\tau = \{z \in X | J(z) \leq \tau\}$, $\tau \in \mathbb{R}$, $0 \leq \theta(t, z) \leq 1$ and $K : [0, 1] \times X \to X$ is compact, such that

$$
\zeta(0, z) = z, \quad \forall z \in J^\beta,
$$

$$
\zeta(t, z) = z, \quad \forall t \in [0, 1], \quad \forall z \in J^{\alpha-\epsilon},
$$

$$
\zeta(1, z) \in J^\beta, \quad \forall z \in J^\beta.
$$

Then, on one hand, by the definition of $\beta$, we have $\zeta(1, r_2 X_2) \subset J^{\alpha-\epsilon}$, which implies that

$$
\zeta(1, r_2 B_{X_2}) \cap r_1 B_{X_1} = \emptyset.
$$

(39)

On the other hand, writing $z \in X$ in the form $z = z_1 + z_2$, where $z_1 \in X_1$, $z_2 \in X_2$, one has

$$
\zeta(t, r_2 B_{X_2}) \cap r_1 B_{X_1} \neq \emptyset \iff \text{there is } z \in r_2 B_{X_2} \text{ such that } \|P_1 \circ \zeta(t, z)\|_X \leq r_1, \quad P_2 \circ \zeta(t, z) = 0,
$$

$$
\Rightarrow \text{there is } z \in r_2 B_{X_2} \text{ such that } \|P_1 \circ \zeta(t, z)\|_X \leq r_1, \quad z_2 = e^{-\theta(t, z)} K_2(t, z) = 0,
$$

(40)

where $K_2 := P_2 \circ K$. Define $\chi : [0, 1] \times [-1, 1] \times r_2 B_{X_2} \rightarrow \mathbb{R} \times X_2$ by
we denote by \( \chi(t, s, z) = \left( \frac{\|P_1 \circ \zeta(t, z)\|_\infty \cdot z_2 + e^{-\theta L_2} K_2(t, z)}{r_1} \right) + (s, 0) \) for every \( \in \mathbb{R} \times X_2 \) is compact. From (38) and nonincreasing of the deformation, we have

\[ \chi(t, t_2 \partial B_{X_2}) \cap r_1 B_{X_1} = \emptyset \quad \text{and} \quad \chi(t, t_2 B_{X_2}) \cap r_1 \partial B_{X_1} = \emptyset \]

for all \( t \in [0, 1] \), which respectively imply that

\[ (0, 0) \notin \chi([0, 1], [-1, 1] \times t_2 \partial B_{X_2}) \quad \text{and} \quad (0, 0) \notin \chi([0, 1], -1 \times t_2 B_{X_2}), \]

from these and the fact \((0, 0) \notin \chi([0, 1], 1 \times t_2 B_{X_2})\) it follows that

\[ (0, 0) \notin \chi([0, 1], \partial([-1, 1] \times t_2 B_{X_2})), \]

which implies that \( \operatorname{deg}_{L^\infty} \chi(t, \cdot), [-1, 1] \times t_2 B_{X_2}, (0, 0) \) is well defined for all \( t \in [0, 1] \). Then, by the homotopy invariance of Leray–Schauder degree, we have

\[ \operatorname{deg}_{L^\infty} \left( \chi(t, \cdot), [-1, 1] \times t_2 B_{X_2}, (0, 0) \right) = \operatorname{deg}_{L^\infty} \left( \chi(0, \cdot), [-1, 1] \times t_2 B_{X_2}, (0, 0) \right) \]

\[ = \operatorname{deg}_{L^\infty} \left( s, z_2 \right), [-1, 1] \times t_2 B_{X_2} \]

then for every \( t \in [0, 1] \), there exist \( s_t \in [-1, 1] \) and \( z_t \in t_2 B_{X_2} \) such that \( \chi(t, s_t, z_t) = (0, 0) \), i.e.

\[ \|P_{X_1} \circ \zeta(t, z_t)\|_\infty = s_t r_1 \quad \text{and} \quad z_t + e^{-\theta L_2} K_2(t, z_t) = 0, \]

from this and (40) it follows that

\[ \chi(t, t_2 B_{X_2}) \cap r_1 B_{X_1} \neq \emptyset \quad \text{for all} \quad t \in [0, 1], \]

which is in contradiction with (39). Our proof is completed. \( \square \)

**Proof of Theorem 3.** By Lemma 1, \( J_\mu \) satisfies the (P.S.) condition in the case of near resonance, where and in the following \( J_\mu \) denote functional \( J \) with parameter \( \mu \). Let \( W_1, W_0, W_2 \) be as in the proof of Theorem 1, when \( \mu > \mu_j \) is sufficiently close to \( \mu_j \), we will find two different critical points of \( J_\mu \) in two steps.

**Step 1.** Construction of the first critical point.

For \( \mu > \mu_j \) and \( z = (u, v) \in W_2 \), it follows from (23) with \( \varepsilon \in (0, \frac{\nu_j L_2}{2}) \) that

\[ J_\mu(z) \leq l(z) - \frac{H_j}{2} \int O |z|^2 \, dx + \frac{\nu_j}{2} \int |z|^2 \, dx + \int L_{\bar{\rho}_j}(x) \left( |u| + |v| \right) \, dx + \int h_1 u \, dx + \int h_2 v \, dx \]

\[ \leq \frac{-\nu_j}{2} \|z\|_E^2 + \frac{\nu_j}{2} \int |z|^2 \, dx \, dx + \frac{1}{\sqrt{\lambda_1}} \left( \|L_{\bar{\rho}_j} \|_{L^2(\Omega)} + \|h_1 \|_{L^2(\Omega)} + \|h_2 \|_{L^2(\Omega)} \|z\|_E \right) \]

\[ < \frac{-\nu_j}{4} \|z\|_E^2 + \frac{1}{\sqrt{\lambda_1}} \left( \|L_{\bar{\rho}_j} \|_{L^2(\Omega)} + \|h_1 \|_{L^2(\Omega)} + \|h_2 \|_{L^2(\Omega)} \|z\|_E \right), \]

which implies that

\[ J_\mu(z) \to -\infty \]  

uniformly for \( \mu > \mu_j \) as \( \|z\|_E \to \infty \) in \( W_2 \). Hence, functional \( J_\mu \) is bounded from above on \( W_2 \) uniformly for \( \mu > \mu_j \), and we denote by

\[ M := \sup_{\mu \in (\mu_j, \frac{\mu_j + 1}{2})} \sup_{z \in W_2} J_\mu(z). \]

Now we claim: there exist constants \( R_1 > 0 \) and \( \delta_1 \in (0, \min\{\frac{\nu_j}{2}, \frac{\mu_j + 1 - \mu_j}{2}\}) \) dependent of \( M \) such that for every \( \mu \in (\mu_j, \mu_j + \delta_1) \), we have

\[ J_\mu(z) > M + 1 \]
for \( z \in W_1 \) with \( \|z\|_E \geq R_1 \) or \( u \in W_1 \oplus W_0 \) with \( \|z\|_E = R_1 \). If this claim is true, from (41) it follows that for every \( \mu \in (\mu_j, \mu_j + \delta_1) \), there exists constant \( R^* > 0 \) such that
\[
\sup_{R^* B_{W_2}} J_{\mu} < \alpha := \inf_{R_1 B_{W_1} \oplus W_0} J_{\mu}.
\]
Sum up, for every \( \mu \in (\mu_j, \mu_j + \delta_1) \), one has
\[
\sup_{R^* B_{W_2}} J_{\mu} < \alpha = \inf_{R_1 B_{W_1} \oplus W_0} J_{\mu} \leq \beta := \sup_{R^* B_{W_2}} J_{\mu} \leq M + 1 \leq \inf_{R_1 B_{W_1} \oplus W_0} J_{\mu},
\]
then the first critical point of \( J_{\mu} \) with critical value \( c_1 \in [\alpha, \beta] \) by Lemma 3.

Now, it remains to prove the claim above. Since for every \( z \in E \), \( J_{\mu}(z) \) is decreasing related to \( \mu \), it suffices to verify that there exist constants \( R_1 > 0 \) and \( \delta_1 \in (0, \min\{\frac{\nu_j\lambda_1}{2}, \frac{\mu_j + 1 - \mu_j}{2}\}) \) dependent of \( M \) such that
\[
J_{\mu, j + \delta_1}(z) > M + 1
\]
for \( z \in W_1 \) with \( \|z\|_E \geq R_1 \) or \( z \in W_1 \oplus W_0 \) with \( \|z\|_E = R_1 \).

On one hand, for \( \mu \in (\mu_j, \mu_j + \min\{\frac{\nu_j\lambda_1}{2}, \frac{\mu_j + 1 - \mu_j}{2}\}) \) and \( (u, v) \in W_1 \), from (23) with \( \varepsilon \in (0, \frac{\nu_j\lambda_1}{4}) \) it follows that
\[
J_{\mu}(z) \geq I(z) - \frac{\mu}{2} \int_{\Omega} |z|^2 \, dx - \frac{\varepsilon}{2} \int_{\Omega} |z|^2 \, dx - \int_{\Omega} L_{\tilde{\nu}_1}(\alpha)(|u| + |v|) \, dx + \int_{\Omega} h_1 u \, dx + \int_{\Omega} h_2 v \, dx
\]
\[
\geq \frac{\nu_j}{2} \|z\|_E^2 - \frac{\mu_j - \mu - \varepsilon}{2} \int_{\Omega} |z|^2 \, dx - \frac{1}{\sqrt{\lambda_1}} \left( \|L_{\tilde{\nu}_1}\|_{L^2(\Omega)} + \|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\Omega)} \right) \|z\|_E
\]
\[
\geq \frac{\nu_j}{8} \|z\|_E^2 - \frac{1}{\sqrt{\lambda_1}} \left( \|L_{\tilde{\nu}_1}\|_{L^2(\Omega)} + \|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\Omega)} \right) \|z\|_E,
\]
therefore, there exists a positive constant \( R_* \) dependent of \( M \) uniformly for \( \mu \in (\mu_j, \mu_j + \min\{\frac{\nu_j\lambda_1}{2}, \frac{\mu_j + 1 - \mu_j}{2}\}) \) such that
\[
J_{\mu}(z) > M + 1
\]
for \( z \in W_1 \) with \( \|z\|_E \geq R_* \).

On the other hand, there exist constants \( R_1 \geq R_* \) and \( \delta_1 \in (0, \min\{\frac{\nu_j\lambda_1}{2}, \frac{\mu_j + 1 - \mu_j}{2}\}) \) dependent of \( M \) such that
\[
J_{\mu, j + \delta_1}(u, v) > M + 1
\]
for \( (u, v) \in W_1 \oplus W_0 \) with \( \|(u, v)\|_E = R_1 \). If not, for any two sequences \( R_n \geq R_* \) and \( \delta_n \in (0, \min\{\frac{\nu_j\lambda_1}{2}, \frac{\mu_j + 1 - \mu_j}{2}\}) \), there exist \( z_n := (u_n, v_n) \in W_1 \oplus W_0 \) with \( \|z_n\|_E = R_n \) such that
\[
M + 1 \geq J_{\mu, j + \delta_n}(z_n)
\]
for all \( n \). We select \( R_n, \delta_n \) satisfying
\[
R_n \to \infty, \quad \delta_n \to 0 \quad \text{and} \quad R_n \delta_n \to 0
\]
as \( n \to \infty \). Writing \( z_n \) in the form \( z_n = z_{1n} + z_{0n} \), where \( z_{1n} := (u_{1n}, v_{1n}) \in W_1 \), \( z_{0n} := (u_{0n}, v_{0n}) \in W_0 \), from (47), one has
\[
\lim_{n \to \infty} \frac{\|z_{0n}\|_E^2}{\|z_{0n}\|_E} = 0.
\]

Then from (24), (25) and (47), passing to the limit in
\[
\frac{M + 1}{\|z_n\|_E^2} \geq \frac{J_{\mu, j + \delta_n}(z_n)}{\|z_n\|_E^2}
\]
\[
\geq \left(\frac{\nu_j}{2} - \frac{\delta_n}{2\lambda_1}\right) \frac{\|z_{1n}\|_E^2}{\|z_n\|_E^2} - \frac{\delta_n}{2\lambda_1} \frac{\|z_{0n}\|_E^2}{\|z_n\|_E^2}
\]
\[
- \int_{\Omega} \frac{G_1(x, u_n)}{\|z_n\|_E^2} \, dx - \int_{\Omega} \frac{G_2(x, v_n)}{\|z_n\|_E^2} \, dx + \int_{\Omega} \frac{h_1 u_n}{\|z_n\|_E^2} \, dx + \int_{\Omega} \frac{h_2 v_n}{\|z_n\|_E^2} \, dx
\]
gives
which implies
\[
\frac{Z_{1n}}{\|z_n\|_E} \to 0 \quad \text{in } E,
\]
as \(n \to \infty\). Since \(W_0\) is finite-dimensional, with loss of generality, we may suppose that
\[
\frac{Z_{0n}}{\|z_n\|_E} \to (u_0, v_0) \quad \text{in } W_0,
\]
as \(n \to \infty\). Hence we have
\[
\frac{z_n}{\|z_n\|_E} \to (u_0, v_0) \quad \text{in } E,
\]
as \(n \to \infty\), which shows that \(\|(u_0, v_0)\|_E = 1\), i.e. \((u_0, v_0) \in W_0 \setminus \{(0, 0)\}\). Let \(\varepsilon\), \(\rho_\varepsilon\), \(C_1^f(x)\) and \(D_1^f(x)\) \((i = 1, 2)\) be as in the proof of Theorem 1, in a way similar to (17)–(20), one has
\[
\left| \int_{\Omega} C_1^f(x) \frac{u_n^-}{\|z_n\|_E} \, dx - \int_{\Omega} C_1^f(x) u_0^- \, dx \right| \to 0, \quad (49)
\]
\[
\left| \int_{\Omega} D_1^f(x) \frac{u_n^-}{\|z_n\|_E} \, dx - \int_{\Omega} D_1^f(x) u_0^- \, dx \right| \to 0, \quad (50)
\]
\[
\left| \int_{\Omega} C_2^f(x) \frac{v_n^-}{\|z_n\|_E} \, dx - \int_{\Omega} C_2^f(x) v_0^- \, dx \right| \to 0, \quad (51)
\]
and
\[
\left| \int_{\Omega} D_2^f(x) \frac{v_n^-}{\|z_n\|_E} \, dx - \int_{\Omega} D_2^f(x) v_0^- \, dx \right| \to 0, \quad (52)
\]
as \(n \to \infty\). From (46), (30) and (31), we have
\[
M + 1 \geq J_{\mu_j + \delta_n} (z_n)
\]
\[
\geq - \frac{\delta_n}{2\lambda_1} \|z_0\|_E^2 + \int_{\Omega} G_1(x, u_n) \, dx - \int_{\Omega} G_2(x, v_n) \, dx + \int_{\Omega} h_1 u_n \, dx + \int_{\Omega} h_2 v_n \, dx
\]
\[
\geq - \frac{\delta_n}{2\lambda_1} \|z_0\|_E^2 + \int_{\Omega} h_1 u_n \, dx + \int_{\Omega} h_2 v_n \, dx + \int_{\Omega} C_1^f(x) u_n^- \, dx - \int_{\Omega} D_1^f(x) u_n^+ \, dx
\]
\[
- \int_{-\rho_\varepsilon \leq u_n \leq 0} (L_{\rho_\varepsilon}(x)|u_n| - C_1^f(x) u_n) \, dx - \int_{0 \leq u_n \leq \rho_\varepsilon} (L_{\rho_\varepsilon}(x)|u_n| - D_1^f(x) u_n) \, dx
\]
\[
+ \int_{\Omega} C_2^f(x) v_n^- \, dx - \int_{\Omega} D_2^f(x) v_n^+ \, dx + \int_{\rho_\varepsilon \leq v_n \leq 0} (L_{\rho_\varepsilon}(x)|v_n| - C_2^f(x) v_n) \, dx
\]
\[
- \int_{0 \leq v_n \leq \rho_\varepsilon} (L_{\rho_\varepsilon}(x)|v_n| - D_2^f(x) v_n) \, dx,
\]
dividing both sides of above inequality by \(\|(u_n, v_n)\|_E\) and passing to the limit, it follows from (48)–(52) that
\[
\int_{\Omega} h_1 u_0 \, dx + \int_{\Omega} h_2 v_0 \, dx \leq - \int_{\Omega} C_1^f(x) u_0^- \, dx + \int_{\Omega} D_1^f(x) u_0^+ \, dx - \int_{\Omega} C_2^f(x) v_0^- \, dx + \int_{\Omega} D_2^f(x) v_0^+ \, dx,
\]
by the arbitrariness of \(\varepsilon\), one has
Step 2. Construction of the second critical point.

In the case \( \mu \in (\mu_j, \mu_j + \delta_1) \), for \( z = (u, v) \in W_2 \oplus W_0 \), it follows from (23) with \( \varepsilon \in (0, \frac{\nu\lambda_1}{2}) \) that

\[
J_{\mu}(z) \leq I(z) - \frac{\mu_j}{2} \int_{\Omega} |z|^2 \, dx - \frac{\varepsilon}{2} \int_{\Omega} |z|^2 \, dx + \int \bar{L}_{\overline{p}}(x)(|u| + |v|) \, dx + \int h_1 u \, dx + \int_\Omega h_2 v \, dx
\]

\[
\leq -\frac{\nu}{2} \|z\|^2_E + \frac{\varepsilon}{2} \int_{\Omega} |z|^2 \, dx + \frac{1}{\sqrt{\lambda_1}} \left( \|L_{\overline{p}}\|_{L^2(\Omega)} + \|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\Omega)} \right) \|z\|_E
\]

\[
\leq -\frac{\nu}{4} \|z\|^2_E + \frac{1}{\sqrt{\lambda_1}} \left( \|L_{\overline{p}}\|_{L^2(\Omega)} + \|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\Omega)} \right) \|z\|_E
\]

which implies that

\[
J_{\mu}(z) \to -\infty
\]

as \( \|z\|_E \to \infty \) in \( W_2 \oplus W_0 \), then there exists a positive constant \( \hat{R} > R_1 \) such that

\[
J_{\mu}(z) \leq M, \quad \text{for } z \in W_2 \oplus W_0 \text{ with } \|z\|_E = \hat{R}.
\]

Let \( A := \{(u, v) \in W_1 \mid \|z\|_E \geq R_1\} \cup \{z \in W_1 \oplus W_0 \mid \|z\|_E = R_1\} \) and \( B := \{z \in W_2 \oplus W_0 \mid \|z\|_E = \hat{R}\} \), then \( A \) and \( B \) link by Lemma 2. So in the case \( \mu \in (\mu_j, \mu_j + \delta_1) \), the second critical point of \( J_{\mu} \) with critical value \( c_2 \geq M + 1 \) is obtained by Theorem 5.

Proof of Theorem 4. By Lemma 1, \( J_{\mu} \) satisfies the (P.S.) condition in the case of near resonance. Let \( W_1, W_0, W_2 \) be as the proof of Theorem 1. When \( \mu < \mu_j \) is sufficiently close to \( \mu_j \), we will obtain two different saddle points of \( J_{\mu} \) in two steps.

Step 1. Construction of the first saddle point.

For \( \mu < \mu_j \) and \( z = (u, v) \in W_1 \), from (23) with \( \varepsilon \in (0, \frac{\nu\lambda_1}{2}) \) it follows that

\[
J_{\mu}(z) \geq I(z) - \frac{\mu_j}{2} \int_{\Omega} |z|^2 \, dx - \frac{\varepsilon}{2} \int_{\Omega} |z|^2 \, dx + \int \bar{L}_{\overline{p}}(x)(|u| + |v|) \, dx + \int h_1 u \, dx + \int_\Omega h_2 v \, dx
\]

\[
\geq \frac{\nu_j}{2} \|z\|^2_E - \frac{\varepsilon}{2} \int_{\Omega} |z|^2 \, dx - \frac{1}{\sqrt{\lambda_1}} \left( \|L_{\overline{p}}\|_{L^2(\Omega)} + \|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\Omega)} \right) \|z\|_E
\]

\[
\geq \frac{\nu_j}{4} \|z\|^2_E - \frac{1}{\sqrt{\lambda_1}} \left( \|L_{\overline{p}}\|_{L^2(\Omega)} + \|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\Omega)} \right) \|z\|_E,
\]

then we get

\[
J_{\mu}(z) \to +\infty
\]

uniformly for \( \mu < \mu_j \) as \( \|z\|_E \to \infty \) in \( W_1 \). Hence, \( J_{\mu} \) is bounded from below on \( W_1 \) uniformly for \( \mu < \mu_j \), and denote by

\[
m := \inf_{\mu \in \left( \mu_j - \frac{\nu_j}{2}, \mu_j \right), z \in W_1} J_{\mu}(z).
\]

In a way similar to the proof of (42), it follows from assumptions \( (g_\rho), (g_\infty), (f_\rho) \) and (5) that there exist constants \( R_2 > 0 \) and \( \delta_2 \in (0, \min(\frac{\nu_j}{2}, \frac{\nu_j - \mu_j - 1}{2})) \) dependent of \( m \) such that for every \( \mu \in (\mu_j - \delta_2, \mu_j) \), we have

\[
J_{\mu}(z) < m - 1,
\]

(53)

for \( z \in W_2 \) with \( \|z\|_E \geq R_2 \) or \( z \in W_2 \oplus W_0 \) with \( \|z\|_E = R_2 \). In this case, the first saddle point of \( J_{\mu} \) is obtained via Remark 3 by Theorem 5 with the critical value
\[ c_1 := \inf_{\gamma \in \mathcal{I}_1} \sup_{z \in R_2 B_{W_2}} J_{\mu}(\gamma(z)), \]

where \( \mathcal{I}_1 := \{ \gamma \in C^0(R_2 B_{W_2}, E) \mid \gamma \big|_{R_2 B_{W_2}} = \text{id} \} \).

**Step 2.** Construction of the second saddle point.

In the case \( \mu \in (\mu_j - \delta_3, \mu_j) \), we have proved in the proof of Theorem 1

\[
J_{\mu}(z) \to +\infty,
\]
as \( \|z\|_E \to \infty \) in \( W_1 \oplus W_0 \) and

\[
J_{\mu}(z) \to -\infty,
\]
as \( \|z\|_E \to \infty \) in \( W_2 \). Hence, we can select \( R > R_2 \) such that

\[
\sup_{z \in R B_{W_2}} J_{\mu}(z) < \inf_{z \in W_1 \oplus W_0} J_{\mu}(z).
\]

Then the second saddle point of \( J_{\mu} \) is obtained via Remark 3 by Theorem 5 with critical value

\[
c_2 := \inf_{\gamma \in \mathcal{I}_2} \sup_{z \in R B_{W_2}} J_{\mu}(\gamma(z)),
\]

where \( \mathcal{I}_2 := \{ \gamma \in C^0(R B_{W_2}, E) \mid \gamma \big|_{R B_{W_2}} = \text{id} \} \). Moreover, we can assert \( c_1 < c_2 \). This assertion was proved in [21], we state it here for reader’s convenience. Arbitrarily chose a \( w_0 \in W_0 \) with \( \|w_0\|_E = 1 \), we can define a map \( \tilde{\gamma} : R B_{W_2} \to E \) as follows

\[
\tilde{\gamma}(z) = \begin{cases}
    z + \sqrt{R^2 - \|z\|^2} w_0, & z \in W_2 \text{ with } \|z\|_E \leq R_2, \\
    z, & z \in W_2 \text{ with } \|z\|_E \leq R.
\end{cases}
\]

It is difficult to check that \( \tilde{\gamma} \in \mathcal{I}_2 \), and from (53) it follows that \( \sup_{z \in R B_{W_2}} J_{\mu}(\tilde{\gamma}(z)) \leq m - 1 \) in the case \( \mu \in (\mu_j - \delta_2, \mu_j) \), which implies \( c_2 \leq m - 1 \). Then the assertion is true because \( c_1 \geq m \). Our proof is completed. \( \square \)

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**References**


