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# On a Khovanskii transformation for continued fractions

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*Abstract:* Some aspects of a transformation for continued fractions due to Khovanskii are studied. It is shown that the transformation does not in general establish an identity, but has many interesting properties. It permits to transform some difficult continued fractions into suitable ones. Its even part is identical to the even part of $S_n(-\frac{1}{2})$ modification. It may also give acceleration for some slow continued fractions.

*Keywords:* Khovanskii transformation, continued fractions.

## 1. Introduction

For simplicity we consider continued fractions of the form

$$C = b_0 + \frac{a_1|}{|1} + \frac{a_2|}{|1} + \cdots \tag{1.1}$$

with $b_0,\ a_n \in \mathbb{C} \setminus \{0\}$.

Let

$$C_n = b_0 + \frac{a_1|}{|1} + \cdots + \frac{a_n|}{|1} = \frac{A_n}{B_n}$$

be its $n$th convergent. Khovanskii [5, pp. 19–23] formally defines what he calls an identity as,

$$b_0 + \frac{1|}{|1} + \frac{a_2|}{|1} + \cdots = b_0 - 1 + \frac{1|}{|1} - \frac{a_1|}{|1 - a_1 - 2a_2} - \frac{a_2|}{|1} - \frac{a_3|}{|1 + a_3 + 2a_4} - \frac{a_4|}{|1} - \cdots - \frac{a_{2n-1}|}{|1 + 2a_{2n-1} + 2a_{2n}} - \frac{a_{2n}|}{|1} + \cdots . \tag{1.2}$$

The continued fraction on the right-hand side will be called the Khovanskii's transform. Its successive convergents will be denoted by $K_n$.

With respect to this transformation, the only remark made by Khovanskii is "Identity (1.2) frequently permits the transformation of one continued fraction into an other in which all coefficients are positive. This sometimes enables us to apply convergence criteria relating to continued fractions with positive coefficients to continued fractions with arbitrary coefficients".

In Section 2 we shall express the successive convergents of the Khovanskii's transform in terms of the successive convergents of the initial continued fraction. This will, in particular, enable us to show that, in general, we have no identity.

In Section 3 we shall show that the Khovanskii transformation may transform a continued fraction (1.1) with $|a_n| \to \infty$ into

$$d_0 = \frac{d_1|}{|1} + \cdots$$

in which $d_n \to -\frac{1}{4}$ or $d_{2n} \to d^{(1)}$ and $d_{2n+1} \to d^{(2)}$. Through an example we shall show that such a bridge is useful to accelerate some continued fractions.

A similar bridge was obtained by Jacobsen and Waadeland [3] where they consider a transformation given by a contraction

In Section 4 we shall prove that, in the linear case, the sequence $(K_n)$ of successive convergents of the Khovanskii's transform converges like $(C_n)$ and

$$\lim(K_{n+1} - C)/(K_n - C) = -\lim(C_{n+1} - C)/(C_n - C).$$

On the other hand, we shall prove that

$$S_{2n}(-\tfrac{1}{2}) = K_{2n} \quad \text{where } S_n(-\tfrac{1}{2}) = \frac{(A_n - \frac{1}{2}A_{n-1})}{(B_n - \frac{1}{2}B_{n-1})} \quad [9].$$

## 2. Properties

### 2.1. The Khovanskii's transform

We shall now use an equivalence transformation to bring the Khovanskii's transform under the form

$$d_0 + \frac{d_1|}{|1} + \frac{d_2|}{|1} + \cdots.$$

With the same notations as in Section 1, we assume that

$$1 + a_1 + 2a_2 \neq 0, \quad a_{2n-1} + a_{2n} \neq -\tfrac{1}{2}, \quad n = 2, 3, \ldots \quad \text{and} \quad a_n \neq 0 \quad \text{for } n = 1, \ldots .$$

Let

$$d_0 = b_0 - 1, \qquad d_1 = 1, \qquad d_2 = -\frac{a_1}{1 + a_1 + 2a_2}, \qquad d_3 = \frac{a_2}{1 + a_1 + 2a_2},$$

$$d_{2n} = -\frac{a_{2n-1}}{1 + 2a_{2n-1} + 2a_{2n}}, \qquad d_{2n+1} = -\frac{a_{2n}}{1 + 2a_{2n-1} + 2a_{2n}}.$$

With the above hypothesis on $a_n$ we get the following property.

**Property 2.1.** *The successive convergents of the Khovanskii's transform are given by*

$$K_0 = C_0 - 1,$$

$$K_{2n} = (C_{2n} - F_n C_{2n-2})/(1 - F_n), \quad \text{where } F_n = (C_{2n} - C_{2n-1})/(C_{2n-1} - C_{2n-2}),$$

$$K_{2n+1} = C_{2n}.$$

**Proof.** To prove $K_{2n+1} = C_{2n}$ it suffices to compare the even part of the initial continued fraction to the odd part of the Khovanskii's transform. Both of the parts exist [4, pp. 42–43].

To compute $K_{2n}$, it is easy to check that

$$a_n = -\frac{\Delta C_{n-1}/\Delta C_{n-2}}{(1+\Delta C_{n-1}/\Delta C_{n-2})(1+\Delta C_{n-2}/\Delta C_{n-3})},$$

and

$$d_n = -\frac{\Delta K_{n-1}/\Delta K_{n-2}}{(1+\Delta K_{n-1}/\Delta K_{n-2})(1+\Delta K_{n-2}/\Delta K_{n-3})}, \qquad n=3,\ldots .$$

Now,

$$d_{2n+1} = (a_{2n}/a_{2n-1})d_{2n}.$$

Replacing $a_{2n}$, $a_{2n-1}$, $d_{2n}$ and $d_{2n+1}$ by their expressions we get,

$$\frac{K_{2n}-C_{2n}}{K_{2n}-C_{2n-2}} = \frac{K_{2n-2}-C_{2n-4}}{K_{2n-2}-C_{2n-2}} \cdot \frac{C_{2n}-C_{2n-1}}{C_{2n-1}-C_{2n-2}} \cdot \frac{C_{2n-2}-C_{2n-3}}{C_{2n-3}-C_{2n-4}}.$$

Let

$$G_n = \frac{(K_{2n}-C_{2n})}{(K_{2n}-C_{2n-2})} \quad\text{and}\quad F_n = \frac{(C_{2n}-C_{2n-1})}{(C_{2n-1}-C_{2n-2})}.$$

Then

$$G_n = \frac{F_n F_{n-1}}{G_{n-1}} \quad\text{with } G_1 = \frac{C_2-C_1}{C_1-C_0}$$

Hence,

$$G_n = \frac{C_{2n}-C_{2n-1}}{C_{2n-1}-C_{2n-2}}, \quad\text{i.e. } G_n = F_n.$$

Whence $K_{2n}$. □

**Remarks 2.2.** (1) From Theorem 2.10 and 2.11 in [4, pp. 42–43] it follows that $K_{2n+1} = C_{2n}$ without any assumption on $a_n$.

(2) Assume $\lim_{n\to\infty} C_{2n} = C$ and $\lim_{n\to\infty} C_{2n+1} = C'$, $C$, $C' \in \mathbb{C}$ with $C \neq C'$. Then from property 2.1, it immediately follows that $\lim_{n\to\infty} K_n = C$. Hence the transformed way converge while the initial continued fraction diverges. This means that one has to be careful when using the transformation in the sense of the quotation cited in Section 1.

### *2.2. The reciprocal transformation*

The sequence $(K_n)$ is closely related to $(C_{2n})$, also, one would like to have at hand informations on $(C_{2n+1})$.

Let us determine a continued fraction

$$v_0 + \frac{v_1|}{|1} + \cdots$$

such that its transformed is,

$$\frac{b_0}{a_1} + \frac{1|}{|1} + \frac{a_2|}{|1} + \cdots .$$

To do this we assume

$$1 + a_2 + 2a_3 \neq 0 \quad \text{and} \quad a_{2n} + a_{2n+1} \neq -\tfrac{1}{2}, \quad n = 2, \ldots .$$

Straightforward computations leads to

$$v_0 = 1 + b_0/a_1, \qquad v_1 = \frac{-a_2}{1 + a_2 + 2a_3}, \qquad v_2 = -\frac{a_3}{1 + a_2 + 2a_3},$$

$$v_{2n-1} = -\frac{a_{2n}}{1 + 2a_{2n} + 2a_{2n+1}}, \qquad v_{2n} = -\frac{a_{2n+1}}{1 + 2a_{2n} + 2a_{2n+1}}.$$

The continued fraction defined by

$$b_0 + a_1 + \frac{a_1 v_1|}{|1} + \frac{v_2|}{|1} + \cdots$$

is called the reciprocal transform. Let $L_n$ be its $n$th convergent.

**Property 2.3.** *If* $1 + a_2 + 2a_3 \neq 0$ *and* $a_{2n} + a_{2n+1} \neq -\frac{1}{2}$ *for* $n = 2, \ldots$ *then,*

$$L_{2n-1} = (C_{2n+1} - G_n C_{2n-1})/(1 - G_n) \quad \textit{where } G_n = (C_{2n+1} - C_{2n})/(C_{2n} - C_{2n-1}),$$

$$L_{2n} = C_{2n+1}.$$

## 3. Connections

### *3.1. Case 1 : $a_n \to \infty$*

Let the continued fraction (1.1) be such that $\lim_{n\to\infty} |a_n| = \infty$. It is known [3] that if $\lim_{n\to\infty} a_n/a_{n+1} = \xi$ exists and $|\xi| \neq 1$ then the continued fraction diverges. For our purpose we assume that $\lim_{n\to\infty} a_{2n-1}/a_{2n} = \xi \neq -1$ exists. We further assume that $1 + a_1 + 2a_2 \neq 0$ and $a_{2n-1} + a_{2n} \neq -\frac{1}{2}$ for $n = 2, \ldots$ .

With these hypothesis, the elements $d_{n_1}$ in section 2.1 are such that

$$\lim_{n\to\infty} d_{2n} = -\frac{\xi}{2(1+\xi)} \quad \text{and} \quad \lim_{n\to\infty} d_{2n+1} = -\frac{1}{2(1+\xi)}.$$

The Khovanskii's transform is limit 2-periodic when $\xi \neq 1$ and its elements tend to $-\frac{1}{4}$ when $\xi = 1$. This property is important. It permits to apply known results relating to a given class of continued fractions to an other class. It is also of use for other purpose as we can see in the following example.

**Example 3.1.** Let

$$b_0 = -1 \qquad a_1 = 1, \qquad a_2 = -\frac{33}{47}, \qquad a_3 = -\frac{25}{47},$$

$$a_{2n} = -\frac{6 + \left(\frac{3}{2}\right)^{2n}}{20}, \qquad a_{2n+1} = -\frac{4 + \left(\frac{3}{2}\right)^{2n}}{20}.$$

Numerical computations show that the continued fraction converges to $C = 0.682152\ldots$ . We have

$$\lim_{n\to\infty} a_n = -\infty \qquad \lim_{n\to\infty} \frac{a_{2n-1}}{a_{2n}} = \tfrac{4}{9}, \qquad \lim_{n\to\infty} \frac{a_{2n}}{a_{2n+1}} = 1.$$

Note that $a_n \to \infty$, which makes useless many processes of acceleration. See [8] for example.

Applying Khovanskii's transformation we get

$$d_0 = -2, \qquad d_1 = 1, \qquad d_2 = -\frac{47}{28}, \qquad d_3 = \frac{33}{28}, \qquad d_4 = -\frac{188000}{413553},$$

$$d_5 = \frac{2773}{5866}, \qquad d_{2n} = -\frac{2}{13}\left[1 + 4\cdot\left(\tfrac{2}{3}\right)^{2n-2}\right],$$

$$d_{2n+1} = -\frac{9}{26}\left[1 + 4\cdot\left(\tfrac{2}{3}\right)^{2n-1}\right], \quad n = 3,\ldots .$$

The transformed is a limit 2-periodic continued fraction with $\lim d_{2n} = -\frac{2}{13}$ and $\lim d_{2n+1} = -\frac{9}{26}$ and processes to accelerate it are known, see [2,7] for example.

*3.2. Case 2: $a_n \to -\frac{1}{4}$*

In the case $\lim_{n\to\infty} a_n = -\frac{1}{4}$ we immediately get $\lim_{n\to\infty} |d_n| = \infty$. Summarizing we have the following theorem.

**Theorem 3.2.** *If $1 + a_1 + 2a_2 \neq 0$ and $a_{2n-1} + a_{2n} \neq -\frac{1}{2}$ the Khovanskii's transformed is such that*

(i) *if $\lim_{n\to\infty} |a_n| = \infty$ and $\lim_{n\to\infty} a_{2n-1}/a_{2n} = \xi \neq -1$, then*

$$\lim_{n\to\infty} d_{2n} = -\frac{\xi}{2(1+\xi)} \quad \text{and} \quad \lim_{n\to\infty} d_{2n+1} = -\frac{1}{2(1+\xi)},$$

(ii) *if $\lim_{n\to\infty} a_n = -1/4$ then, $\lim_{n\to\infty} |d_n| = \infty$.*

## 4. Convergence and acceleration

We assume $\lim_{n\to\infty} C_n = C \in \mathbb{C}$, $\lim_{n\to\infty} a_n = a \notin (-\infty, -\frac{1}{4})$ and $1 + a_1 + 2a_2 \neq 0$, $a_{2n-1} + a_{2n} \neq -\frac{1}{2}$. It is known that if $a \neq -\frac{1}{4}$ then $\exists r \in \mathbb{C}$, $|r| < 1$ such that $r = \lim_{n\to\infty}(C_{n+1} - C)/(C_n - C)$ [1].

Under the above hypothesis we have the following theorem.

**Theorem 4.1**

(1) *If $a \neq -\frac{1}{4}$ then*

(i) $\lim_{n\to\infty} K_n = C$,

(ii) $\lim_{n\to\infty}(K_{n+1} - C)/(K_n - C) = q$, *where* $q = -r$,

(iii) $\lim_{n\to\infty}(K_{2n} - C)/(C_{2n-2} - C) = q$.

(2) *If $a = -\frac{1}{4}$ and $\lim_{n\to\infty} \Delta C_{n+1}/\Delta C_n = \lim_{n\to\infty}(e_{n+1})/e_n$ exists then $\forall p \in \mathbb{N}$,*

$$\lim_{n\to\infty} (K_{2n} - C)/(C_{2n+p} - C) = 0 \quad \text{iff} \quad \lim_{n\to\infty} \frac{1 - \phi_{2n}/\phi_{2n-1}}{\phi_{2n-1}} = 1.$$

*with $e_{n+1}/e_n = 1 - \phi_{n+1}$ and $e_n = C_n - C$.*

**Proof.** (1) (i) Since $a \notin (-\infty, -\frac{1}{4}]$ we have $\lim_{n\to\infty} d_n = a/(1+4a) = d \notin (-\infty, -\frac{1}{4})$. Hence $\lim_{n\to\infty} K_n = K$ exists in $\mathbb{C} \cup \{\infty\}$ [5]. Since $\lim_{n\to\infty} K_{2n+1} = C$ then $K = C$.

(1) (ii) Let $x_1$ be the zero of smallest modulus of $ax^2 + x - 1 = 0$. We have $r = x_1 - 1$ [1,6]. Now, $\lim_{n\to\infty}(K_{n+1} - C)/(K_n - C) = q$ where $1 + q$ is the zero of smallest modulus of $\mathrm{d}y^2 + y - 1 = 0$; whence $q = -r$.

(1) (iii) Obvious since $C_{2n-2} = K_{2n-1}$.

(2) It is easy to check that if $\lim_{n\to\infty} \Delta C_{n+1}/\Delta C_n = \rho$ exists then $\rho = 1$. Hence $\lim_{n\to\infty} \phi_n = 0$. Now, since $\lim_{n\to\infty} e_{n+1}/e_n = 1$ it suffices to consider $(K_{2n} - C)/e_{2n-2}$. We have

$$\frac{K_{2n} - C}{e_{2n-2}} = 1 - \frac{1 - e_{2n}/e_{2n-2}}{i - F_n}. \qquad (*)$$

On the other hand,

$$\frac{C_{2n} - C_{2n-1}}{C_{2n-1} - C_{2n-2}} = \frac{(1 - \phi_{2n-1})\phi_{2n}}{\phi_{2n-1}} = \frac{\phi_{2n}}{\phi_{2n-1}} - \phi_{2n}.$$

Since $\lim_{n\to\infty} \Delta C_{n+1}/\Delta C_n = 1$, then $\lim_{n\to\infty} \phi_{2n}/\phi_{2n-1} = 1$.

Now

$$1 - e_{2n}/e_{2n-2} = \phi_{2n-1} + \phi_{2n} - \phi_{2n-1}\phi_{2n} \quad \text{and} \quad 1 - F_n = 1 - \phi_{2n}/\phi_{2n-1} + \phi_{2n},$$

$$\lim_{n\to\infty} \frac{K_{2n} - C}{e_{2n-2}} = 0 \quad \text{iff} \quad \lim_{n\to\infty} \frac{1 + \phi_{2n}/\phi_{2n-1} - \phi_{2n}}{(1 - \phi_{2n}/\phi_{2n-1})/\phi_{2n/1} + \phi_{2n}/\phi_{2n-1}} = 1.$$

that is iff

$$\lim_{n\to\infty} (1 - \phi_{2n}/\phi_{2n-1})/\phi_{2n-1} = 1. \qquad \square$$

**Remarks** (1) Similar results may be obtained if one uses the reciprocal transformation.

(2) To insure $\lim_{n\to\infty} K_n = C$ the existence of $r$ is not needed. We have

$$\lim_{n\to\infty} K_n = C \quad \text{iff} \quad \frac{(C_{2n} - C_{2n-1})(C_{2n} - C_{2n-1})}{(C_{2n} - C_{2n-1}) - (C_{2n-1} - C_{2n-2})} \to 0,$$

Assertion (2) in Theorem 4.1, shows that in the case $a = -\frac{1}{4}$ and under some hypothesis, $(K_{2n})$ improves $(C_{2n})$.

On the other hand, it is well known that when $a = -\frac{1}{4}$, $S_n(-\frac{1}{2})$ may accelerate the convergence, see [9] for example. Then $(K_n)$ and $(L_n)$ need to be compared to $(S_n(-\frac{1}{2}))$. To do this, we assume that $(K_n)$ and $(L_n)$ have meaning, see Section 2. We have the following property.

**Property 4.1.**

$$K_{2n} = S_{2n}(-\tfrac{1}{2}) \quad \text{and} \quad L_{2n-1} = S_{2n+1}(-\tfrac{1}{2}) \quad \text{for } n = 1, 2, \ldots .$$

**Proof**. We show that $K_{2n} = S_{2n}(-\frac{1}{2})$ for $n = 1, 2, \ldots$. $L_{2n-1} = S_{2n+1}(-\frac{1}{2})$ may be obtained in the same way.

We have

$$S_{2n}(-\tfrac{1}{2}) = \frac{2A_{2n} - A_{2n-1}}{2B_{2n} - B_{2n-1}}.$$

Let

$$g_n = 1 + \frac{C_n - C_{n-1}}{C_{n-1} - C_{n-2}}$$

for $n = 2, 3, \ldots$ and $g_1 = 1$. Then $g_n = B_{n-1}/B_n$ [6] and $S_{2n}(-\frac{1}{2}) = (2\ C_{2n} - g_{2n-1}C_{2n-1})/(2 - g_n)$.

Now, from Property 2.1 we have

$$K_{2n} = \frac{C_{2n} - C_{2n-2}\dfrac{C_{2n} - C_{2n-1}}{C_{2n-1} - C_{2n-2}}}{1 - \dfrac{C_{2n} - C_{2n-1}}{C_{2n-1} - C_{2n-2}}}$$

$$= \frac{\left(C_{2n} - C_{2n-1}\dfrac{(C_{2n} - C_{2n-1})}{(C_{2n-1} - C_{2n-2})} + C_{2n} - C_{2n-1}\right)}{(2 - g_{2n})}$$

$$= S_{2n}(-\tfrac{1}{2}) \quad \text{for } n = 1, 2, \ldots . \qquad \square$$

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