Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 223 (2009) 552–566
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

On theoretical pricing of options with fuzzy estimators

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Received 9 October 2007; received in revised form 24 November 2007

Abstract

In this paper we present an application of a new method of constructing fuzzy estimators for the parameters of a given probability distribution function, using statistical data. This application belongs to the financial field and especially to the section of financial engineering. In financial markets there are great fluctuations, thus the element of vagueness and uncertainty is frequent. This application concerns Theoretical Pricing of Options and in particular the Black and Scholes Options Pricing formula. We make use of fuzzy estimators for the volatility of stock returns and we consider the stock price as a symmetric triangular fuzzy number. Furthermore we apply the Black and Scholes formula by using adaptive fuzzy numbers introduced by Thiagarajah et al. [K. Thiagarajah, S.S. Appadoo, A. Thavaneswaran, Option valuation model with adaptive fuzzy numbers, Computers and Mathematics with Applications 53 (2007) 831–841] for the stock price and the volatility and we replace the fuzzy volatility and the fuzzy stock price by possibilistic mean value. We refer to both cases of call and put option prices according to the Black & Scholes model and also analyze the results to Greek parameters. Finally, a numerical example is presented for both methods and a comparison is realized based on the results.

Keywords: Fuzzy estimators; Adaptive fuzzy numbers; Fuzzy volatility; Possibilistic mean; Black–Scholes option pricing formula

1. Introduction

Many stochastic models present uncertainty and they are usually solved by probability theory and fuzzy sets theory. In finance there are many models presenting uncertainty due to the fluctuation of financial markets. One of the most popular models in finance that have had the element of uncertainty models is the Black–Scholes option pricing formula. Since Black and Scholes derived the options pricing formula many methodologies have been developed. In [5] Cherubini determined the price of a corporate debt contract and provided a fuzzified version of the Black and Scholes model by means of a special class of fuzzy measures. On the other hand, Ghaziri et al. [9] introduced the artificial intelligence approach to price the options, using neural networks and fuzzy logic. Since the Black–Scholes option pricing formula is only approximate, it leads to considerable errors, Trenev [17] obtained a refined formula for pricing options. In [19] Wu applied fuzzy approach to the Black–Scholes formula. Zmeskal [20] applied the Black–Scholes methodology of appraising equity as a European call option. Carlson and Fuller [4] used the possibility theory for fuzzy real option valuation. Applications of fuzzy sets theory to volatility models have been studied by

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.12.006

Thavaneswaran et al. [14] and Thiagarajah and Thavaneswaran [16]. In [14] Thavaneswaran et al. studied stochastic volatility in the pricing of options and in [16] introduced a class of FRC (fuzzy random coefficient) volatility models and studied their moment properties (for this analysis see [14]).

In statistics, as we know, there is the point estimation of a parameter. But this is not enough for us to derive safe conclusions. That is why statistics introduces confidence intervals. The disadvantage of confidence intervals is that we have to choose the probability so that the parameter for estimation should be in this interval. With this methodology and by making use of the tool of fuzzy numbers we define fuzzy estimators for any estimated parameter, using the confidence intervals. The fuzzy number that results is considered to be the statistical estimator that expresses a degree of an unbiased estimation. The motivation is the following: We wonder if the confidence intervals for the mean μ are the α-cuts of a fuzzy number A.

The Black–Scholes model depends on five parameters. Of these, the exercise price and time to expiry are specified by the option contract. The risk-free rate of interest is the yield on a notional default-free bond of duration similar to the life of the option. There is some degree of choice involved here, although the formula is not particularly sensitive to the rate of interest and a reasonable value for this parameter can easily be chosen [1]. The parameters which are the most difficult ones to estimate are the volatility and the price of the underlying stock. The volatility is calculated either by historical data, known as historical volatility, or by equating the theoretical call price from the Black–Scholes formula with the market price, known as implied volatility. Volatility changes though time randomly thus it is a random variable as well as a fuzzy number.

In this paper we use the fuzzy estimators based on confidence intervals introduced by Papadopoulos and Sfiris in [12] in order to estimate the volatility of stock returns having sample data (historical volatility) and also a symmetric triangular fuzzy number in order to model the uncertainty of the stock price in this model. Furthermore we apply a method proposed by Thiagarajah, Appadoo, Thavaneswaran [15], which models the uncertainty of the characteristics such as stock price and volatility using adaptive fuzzy numbers and replaces the fuzzy stock price and the fuzzy volatility by the possibilistic mean value (see Carlsson, Fuller [3]) in the fuzzy Black–Scholes formula. Another advantage is that by using fuzzy numbers we can realize operations of the form $[f(A, B)][a] = f(A[a], B[a])$, $\forall a \in [0, 1]$ (see Nguyen [11]).

2. Fuzzy sets theory

Here we need the following definitions and propositions which are mentioned in [10]:

Definition 2.1. If A is a function from X into the interval [0, 1] then A is called a fuzzy set. A is convex iff or every $t \in [0, 1]$ and $x_1, x_2 \in X$ we have

$$A(tx_1 + (1 - t)x_2) \geq \min\{A(x_1), A(x_2)\}. \tag{2.1}$$

A is normalized if there exists $x \in X$, such that $A(x) = 1$.

Definition 2.2. If A is a fuzzy set, by a-cuts $a \in [0, 1]$ we mean the sets $A[a] = \{x \in X : A(x) \geq a\}$.

It is known that the a-cuts determine the fuzzy set A. Let now A and B denote fuzzy numbers and let * denote any of the four basic arithmetic operations. Then, we define a fuzzy set on $\Re$, $A * B$, by defining it as an a-cut, $(A * B)[a]$ as

$$(A * B)[a] = A[a] * B[a] \tag{2.2}$$

for any $a \in (0, 1]$.

Definition 2.3. We say that A is a fuzzy number if the following conditions hold:

1. A is normal,
2. A is a convex fuzzy set,
3. A is upper semicontinuous,
4. The support of A, $\bigcup_{a\in(0,1]} A[a] = \{x : A(x) > 0\}$ is compact.

Then the a-cuts of A are closed intervals. We also know that if $A[a] = B[a]$ $\forall a \in [0, 1]$ for arbitrary fuzzy sets A and B then $A = B$.

For the realization of the operations we use Nguyen's propositions [11].

Proposition 2.1. *Let $f : XxY \to Z$, and $A \in P(X)$, $B \in P(Y)$ then:*

$$f(A, B) = \int_0^1 af(A[a], B[a])\mathrm{d}a. \tag{2.3}$$

Proposition 2.2. *With the notation of Proposition 2.1 and if $f : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is continuous then, $\forall A, B \in P(\mathfrak{R}, S, K)$ and we have:*

$$[f(A, B)][a] = f(A[a], B[a]) \quad \forall a \in [0, 1] \tag{2.4}$$

if $\forall z \in Z$, $\sup_{(x,y)\in f^{-1}(x)}[\mu_{\mathrm{A}}(x) \wedge \mu_A(y)]$ is attained.

We also mention that the a-cut of a fuzzy number A can be written as an interval of this form:

$$A[a] = [A_1(a), A_2(a)].$$

3. Arithmetic operations on intervals

In [10] we can see the fundamentals of fuzzy arithmetic which is based on two properties of fuzzy numbers:

Property 1. *Each fuzzy set and thus also each fuzzy number, can fully and uniquely be represented by its a-cuts.*

Property 2. *The a-cuts of each fuzzy number are closed intervals of real numbers for all $a \in (0, 1]$.*

These properties enable us to define arithmetic operations on fuzzy numbers, in terms of arithmetic operations, on their a-cuts. The latter operations are a subject of interval analysis, a well-established area of classical mathematics.

Let $*$ denote any of the four arithmetic operations on closed intervals: addition $+$, subtraction $-$, multiplication $\cdot$, division $/$. Then:

$$[a, b] * [d, e] = [f * g | a \le f \le b, d \le g \le e]$$

is a general property of all arithmetic operations on closed intervals, except that $[a, b]/[d, e]$ is not defined when $0 \in [d, e]$. That is, the result of an arithmetic operation on closed intervals is again a closed interval.

The four arithmetic operations on closed intervals are defined as follows ([10] and [5]):

- $[a, b] + [d, e] = [a + d, b + e]$
- $[a, b] - [d, e] = [a - e, b - d]$
- $[a, b] \cdot [d, e] = [\min\{ad, ae, bd, be\}, \max\{ad, ae, bd, be\}]$, and provided that $0 \notin [d, e]$. If $a, b, c, d > 0$ then $[a, b] \cdot [d, e] = [ad, be]$
- $[a, b]/[d, e] = [a, b] \cdot [\frac{1}{d}, \frac{1}{e}] = [\min\{\frac{a}{d}, \frac{a}{e}, \frac{b}{d}, \frac{b}{e}\}, \max\{\frac{a}{d}, \frac{a}{e}, \frac{b}{d}, \frac{b}{e}\}]$ and provided that $0 \notin [d, e]$.

Note that a real number r may also be regarded as a special (degenerated) interval $[r, r]$. When one of the above intervals is degenerated, we obtain special operations; when both of them are degenerated we obtain the standard arithmetic of real numbers.

Arithmetic operations on closed intervals satisfy some useful properties. To overview them, let:

$$A = [a_1, a_2]$$
$$B = [b_1, b_2]$$
$$C = [c_1, c_2]$$
$$0 = [0, 0]$$
$$1 = [1, 1].$$

Using these symbols, the properties are formulated as follows:

Property	Name
$A+B=B+A$ $A\cdot B=B\cdot A$	(Commutativity)
$A+(B+C)=(A+B)+C$ $A\cdot(B\cdot C)=(A\cdot B)\cdot C$	(Associativity)
$A=A+0=0+A$ $A=1\cdot A=A\cdot 1$	(Identity)
$A\cdot(B+C)\subseteq A\cdot B+A\cdot C$	(Subdistributivity)
If $A\subseteq E$ and $B\subseteq F$ then:	
$A+B\subseteq E+F$ $A-B\subseteq E-F$ $A\cdot B\subseteq E\cdot F$ $A/B\subseteq E/F$	(Inclusion Monotonicity)

Moreover:

- If $b\cdot c\geq 0$ for every $b\in B, c\in C$ then $A\cdot(B+C)=A\cdot B+A\cdot C$
- Furthermore if $A=[a,a]$ then $a\cdot(B+C)=a\cdot B+a\cdot C$
- $0\in A-A, 1\in A/A$.

4. Estimate fuzzy volatility from fuzzy estimators of σ^2 — fuzzy estimators based on confidence intervals

In [12] Papadopoulos and Sfiris proved the next proposition:

Proposition 4.1. *Let $X_1, X_2, \ldots, X_n$ be a random sample and let $x_1, x_2, \ldots x_n$ be sample values assumed by the sample. Let also $\beta\in[0,1)$. If the sample size is large enough, then*

$$M(x)=\begin{cases}\dfrac{2-\beta}{1-\beta}-\dfrac{2}{1-\beta}\Phi\left(\sqrt{\dfrac{n-1}{2}}\left(\dfrac{s^2}{x}-1\right)\right) & \text{if } \dfrac{s^2}{1+\Phi^{-1}\left(1-\frac{\beta}{2}\right)\sqrt{\frac{2}{n-1}}}\leq x\leq s^2\\[2ex] \dfrac{2-\beta}{1-\beta}-\dfrac{2}{1-\beta}\Phi\left(\sqrt{\dfrac{n-1}{2}}\left(1-\dfrac{s^2}{x}\right)\right) & \text{if } s^2\leq x\leq\dfrac{s^2}{1-\Phi^{-1}\left(1-\frac{\beta}{2}\right)\sqrt{\frac{2}{n-1}}}\end{cases}\tag{4.1}$$

is a fuzzy number, the base of which is exactly the $(1-\beta)$ confidence interval for s^2 and the α-cuts of this fuzzy number are the closed intervals:

$$^{a}M=\left[\frac{s^2}{1+z_{g(a)}\sqrt{\frac{2}{n-1}}},\frac{s^2}{1-z_{g(a)}\sqrt{\frac{2}{n-1}}}\right]\tag{4.2}$$

which is exactly the $(1-\alpha)(1-\beta)$ confidence interval for s^2.

Here

$$g(a)=\left(\frac{1}{2}-\frac{\beta}{2}\right)\alpha+\frac{\beta}{2},\qquad\left(g:[0,1]\to\left[\frac{\beta}{2},0.5\right]\right)$$

and

$$z_{g(\alpha)}=\Phi^{-1}\left(1-g(\alpha)\right).$$

5. Weighted possibilistic mean values and possibilistic mean value of fuzzy numbers

Fuller and Majlender [8] referred to the next example for the weighted possibilistic mean of a fuzzy number:

Let $S = \{c, k, d\}$ be a symmetric triangular fuzzy number with center k, left-width $c > 0$ and right-width $d > 0$, and let $f(a) = (n+1)a^n$, $n \geq 0$ then an a-level of S is computed by

$$S[a] = [c - (1-a)k, c + (1-a)d] \quad \forall a \in [0, 1] \tag{5.1}$$

then the power weighted possibilistic mean of fuzzy S is:

$$M_f(S) = \int_0^1 a[c - (1-a)k + c + (1-a)d]\mathrm{d}a = c + \frac{d-k}{2(n+2)}. \tag{5.2}$$

When S is a symmetric triangular fuzzy number then the equation $M_f(S) = E(S)$ holds for any weighting function of f.

Carlson and Fuller [3] referred to the next example for the possibilistic mean of a fuzzy number:

Let $S = \{c, k, d\}$ be a triangular fuzzy number with center k, left-width $c > 0$ and right-width $d > 0$, then an a-level of S is computed by

$$S[a] = [c - (1-a)k, c + (1-a)d] \quad \forall a \in [0, 1] \tag{5.3}$$

then the possibilistic mean of fuzzy S is:

$$E(S) = \int_0^1 a[c - (1-a)k + c + (1-a)d]\mathrm{d}a = c + \frac{d-k}{6}. \tag{5.4}$$

If we consider S as a symmetric triangular fuzzy number we can replace k with $\frac{c+d}{2}$ and as a result we take that:

$$S[a] = \left[c - (1-a)\frac{c+d}{2}, c + (1-a)d\right] \quad \forall a \in [0, 1]. \tag{5.5}$$

And also that:

$$E(S) = \int_0^1 a\left[c - (1-a)\frac{c+d}{2} + c + (1-a)d\right]\mathrm{d}a = c + \frac{d - \frac{c+d}{2}}{6}. \tag{5.6}$$

6. The Black and Scholes model (options pricing formula)

The Black–Scholes (1973) option pricing formula [2] prices European put or call options on a stock that does not pay a dividend or make other distributions. The formula assumes that the underlying stock price follows a geometric Brownian motion with constant volatility. It is historically significant as the original option pricing formula published by Black and Scholes in their landmark (1973) paper [2].

This formula is developed from the principle that options can completely eliminate market risk from a stock portfolio. Black and Scholes postulate that the ratio of options to stock in this hedged position is constantly modified at no commission cost in order to offset gains or losses on the stock by losses or gains on the options. Because the position is theoretically riskless, we would expect the hedge to earn the risk-free rate, somewhat analogous to the assumption invoked in deriving the capital asset pricing model. Given that the risk-free rate hedge should earn the risk-free rate, we infer that the option premium at which the hedge yields a return equal to the risk-free short-term interest rate is the fair value of the option. If the price is greater or lesser than the fair value, the return from a risk-free hedged position could be different from the risk-free interest rate. Because this is inconsistent with the equilibrium, we would expect the option price to adjust toward the fair value [6].

6.1. Call option pricing

Assuming that no dividends are payable on the stock prior to maturity of the option, the Black and Scholes call option price, C_t, is given by

$$C_t = S_t \Phi(h) - X\mathrm{e}^{-rt}\Phi(h - \sigma\sqrt{\tau}) \tag{6.1}$$

where:

$$h = \frac{\log\left(\frac{S}{X}\right) + r\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

S = current stock price
r = risk-free interest rate
X = exercise price
τ = time to expiry in years
σ = volatility, standard deviation of logarithmic stock returns per unit time

$$\Phi(\omega) = \int_{-\infty}^{\omega} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y \quad \text{i.e. } P(Y \prec \omega) \text{ given } Y \sim N(0, 1).$$

It should be noted that natural logarithms are always used. The lognormal distribution for stock prices assumes that logarithmic stock returns are normally distributed [7].

It will be noticed that the call price is a function of only five parameters; S_t, r, X, τ and σ. $\Phi(h)$ and $\Phi(h - \sigma\sqrt{\tau})$ are simply values from the cumulative standard normal distribution. They are probabilities and therefore lie in the range 0 to 1.

Additional insight can be obtained by examining how the call price reacts to changes in the underlying model parameters. This can be achieved intuitively, by thinking about what we would expect to happen when a model parameter changes, or mathematically, by looking at the sensitivities of the model. Sensitivity is simply the rate of change of the call price with respect to a change in one of the model parameters, keeping the rest of the parameters fixed. Mathematically speaking, this is the first partial derivative with respect to one of the parameters of interest. The sensitivities of this model are the following:

Name	Formula[a]	Sign
Delta, Hedge Ratio	$\frac{\partial C}{\partial S} = \Phi(h)$	+
	$\frac{\partial C}{\partial X} = \mathrm{e}^{-rt}\,\Phi(h - \sigma\sqrt{\tau})$	−
Theta	$\frac{\partial C}{\partial r} = \frac{S\sigma}{2\sqrt{\tau}}\,\Phi'(h) + Xr\mathrm{e}^{-rt}\,\Phi(h - \sigma\sqrt{\tau})$	+
Rho	$\frac{\partial C}{\partial r} = \tau Xr\mathrm{e}^{-rt}\,\Phi(h - \sigma\sqrt{\tau})$	+
Kappa, Epsilon, Vega	$\frac{\partial C}{\partial r} = S\sqrt{t}\,\Phi'(h)$	+
Gamma	$\frac{\partial^2 C}{\partial S^2} = \frac{1}{S\sigma\sqrt{\tau}}\,\Phi'(h)$	+

[a] $\Phi'(y) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}$ i.e. $P(Y = y)$ where $Y \sim N(0, 1)$.

The positive sign shows that as the stock price, interest rate or volatility increases the value of the call increases. The negative sign shows that as the exercise price rises, the value of the call falls. Thus analysis of the sensitivities of the Black–Scholes call formula reveals that the theoretical call price reacts to changes in the model parameters in directions which would be expected [1].

6.2. *Put option pricing*

Assuming that no dividends are payable on the stock prior to maturity of the option, the Black and Scholes put option price, P_t, is given by

$$P_t = X\mathrm{e}^{-rt}\,\Phi(-h + \sigma\sqrt{\tau}) - S_t\,\Phi(-h) \tag{6.2}$$

where the symbols have been previously defined.

The Black and Scholes put formula is similar to the call formula, except it appears to be written in reverse and the items in the brackets have been multiplied by −1. Like the call formula, the put formula is a function of only five parameters S_t, r, X, τ and σ.

As the exercise price rises or the underlying stock price falls, we would expect the put price to rise. As interest rate rises, the present value of exercise payments falls, resulting in a fall in the value of the put. As volatility rises, the probability of a favorable outcome rises and we would expect the value of the put to rise. As time to maturity changes, the effect on the value of the put is not clear since there are two effects working in opposite directions. As time to maturity falls, the present value of exercise payments rises, resulting in an increase in the value of the put. However, at the same time, there is a lower chance of a favorable outcome. The dominant effect is not immediately obvious and depends on a number of factors.

The sensitivities of the Black and Scholes put formula, derived mathematically, are shown in the table. The signs of the sensitivities are in agreement with those in the earlier discussion. The most frequently quoted among the sensitivities are the delta and gamma. The put delta is usually quoted as a positive number [1].

Name	Formula[a]	Sign
Delta, Hedge Ratio	$\frac{\partial C}{\partial S} = -\Phi(-h)$	$-$
	$\frac{\partial C}{\partial X} = \mathrm{e}^{-rt}\,\Phi(-h+\sigma\sqrt{\tau})$	$+$
Theta	$\frac{\partial C}{\partial r} = \frac{S\sigma}{2\sqrt{\tau}}\,\Phi'(h) - Xr\mathrm{e}^{-rt}\,\Phi(-h+\sigma\sqrt{\tau})$	$+$ or $-$
Rho	$\frac{\partial C}{\partial r} = -\tau Xr\mathrm{e}^{-rt}\,\Phi(-h+\sigma\sqrt{\tau})$	$-$
Kappa, Epsilon, Vega	$\frac{\partial C}{\partial r} = S\sqrt{t}\,\Phi'(h)$	$+$
Gamma	$\frac{\partial^2 C}{\partial S^2} = \frac{1}{S\sigma\sqrt{\tau}}\,\Phi'(h)$	$+$

[a] $\Phi'(y) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}$ i.e. $P(Y = y)$ where $Y \sim N(0, 1)$.

6.3. Sensitivities

The *Greeks* (see [18]) are a set of factor sensitivities used extensively by traders to quantify the exposures of portfolios that contain options. Each measures how the portfolio's market value should respond to a change in some variable, an underlier, volatility, interest rate or time. There are five Greeks:

- The delta measures sensitivity to price. The Δ of an instrument is the mathematical derivative of the value function with respect to the underlying price.
- The gamma measures second order sensitivity to price. The Γ is the second derivative of the value function with respect to the underlying price.
- The vega, which is not a Greek letter (ν, nu is used instead), measures sensitivity to volatility. The vega is the derivative of the option value with respect to the volatility of the underlying price. The term kappa, κ, is sometimes used instead of vega, and some trading firms have also used the term tau, τ.
- The theta measures sensitivity to the passage of time. Θ is the negative of the derivative of the option value with respect to the amount of time to expiry of the option.
- The rho measures sensitivity to the applicable interest rate. The ρ is the derivative of the option value with respect to the risk-free rate.

They are called the Greeks because four out of the five are named after letters of the Greek alphabet. Vega is the exception. For reasons unknown, it is named after the brightest star in the constellation Lyra. At times, vega has been called kappa, but the name vega is now well established. Four of the five are risk metrics. Theta is not because the passage of time is certain and it entails no risk. Theta is akin to the accrual of interest on a bond. The Greeks are defined as first, and in the case of gamma, second partial derivatives [13].

7. Calculations of formulas and final algorithm for the fuzzy estimators for the volatility and fuzzy stock price method

We have the following five steps in order to find the price of a call or of a put option.

1. We compute the fuzzy estimators for the variance.
2. We compute z and its sign.
3. We take the appropriate formulas according to the signs and we find $h[a]$ and $h_k[a]$.
4. We compute $\Phi(h[a])$and $\Phi(h_k[a])$.
5. We use the Black & Scholes formula for the option we want to price and for its sensitivities.

After the computation of the fuzzy estimators for the variance, which can be realized in the way we described in Section 4, we search h and then $\Phi(h)$, for this fuzzy σ. For h we have the expression:

$$h = \frac{\log\left(\frac{S}{X}\right) + r\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

The numerator can be either positive or negative. The denominator is always positive since $\sigma, \sqrt{\tau} > 0$. So we have two cases: The first when $h > 0$ and the second when $h < 0$. For convenience we consider:

$$h = \frac{\log\left(\frac{S}{X}\right) + r\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \quad \text{where } \log\left(\frac{S}{X}\right) + r\tau = z.$$

Thus

$$h = \frac{z + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

Let us consider h for the fuzzy estimator σ, $[\sigma_1(a), \sigma_2(a)]$ and the symmetric triangular fuzzy number $S = \{c, \frac{c+d}{2}, d\}$. We note that the a-level of the symmetric triangular fuzzy number S is:

$$S[a] = \left[\frac{a(d-c)+2c}{2}, \frac{a(c-d)+2d}{2}\right]. \tag{7.1}$$

As a consequence we have the fuzzy number h with its a-cuts:

$$[h_1(a), h_2(a)] = \frac{[z_1, z_2] + \left[\frac{1}{2}\sigma_1^2(a), \frac{1}{2}\sigma_2^2(a)\right][\tau, \tau]}{[\sigma_1(a), \sigma_2(a)][\sqrt{\tau}, \sqrt{\tau}]}. \tag{7.2}$$

Depending on the sign of z we have the following cases (we will sign $h - \sigma\sqrt{\tau}$ as h_k):

1. For the case where $z_1, z_2 > 0$ we have that,

$$\begin{aligned} &[h_1(a), h_2(a)] = \left[\frac{z_1 + \frac{1}{2}\sigma_1^2(a)\tau}{\sigma_2(a)\sqrt{\tau}}, \frac{z_2 + \frac{1}{2}\sigma_2^2(a)\tau}{\sigma_1(a)\sqrt{\tau}}\right] \\ &[h_{k1}(a), h_{k2}(a)] = [h_1(a), h_2(a)] - \sigma\sqrt{\tau}[a]. \end{aligned} \tag{7.3}$$

2. For the case where $z_1, z_2 < 0$ we have that,

$$\begin{aligned} &[h_1(a), h_2(a)] = \frac{\left[\frac{1}{2}\sigma_1^2(a)\tau - z_2, \frac{1}{2}\sigma_2^2(a)\tau - z_1\right]}{\left[\sigma_1(a)\sqrt{\tau}, \sigma_2(a)\sqrt{\tau}\right]} \\ &[h_{k1}(a), h_{k2}(a)] = [h_1(a), h_2(a)] - \sigma\sqrt{\tau}[a]. \end{aligned} \tag{7.4}$$

For this case we have three different cases:

(a) If $\frac{1}{2}\sigma_1^2(a)\tau - z_2 < 0$ and $\frac{1}{2}\sigma_2^2(a)\tau - z_1 < 0$ then

$$\begin{aligned} &[h_1(a), h_2(a)] = \left[\frac{-z_1 + \frac{1}{2}\sigma_2^2(a)\tau}{\sigma_1(a)\sqrt{\tau}}, \frac{-z_2 + \frac{1}{2}\sigma_1^2(a)\tau}{\sigma_2(a)\sqrt{\tau}}\right] \quad \text{(both tails negative)} \\ &[h_{k1}(a), h_{k2}(a)] = [h_1(a), h_2(a)] - \sigma\sqrt{\tau}[a]. \end{aligned} \tag{7.5}$$

(b) If $\frac{1}{2}\sigma_1^2(a)\tau - z_2 < 0$ and $\frac{1}{2}\sigma_2^2(a)\tau - z_1 > 0$ then

$$[h_1(a), h_2(a)] = \left[\frac{-z_2 + \frac{1}{2}\sigma_1^2(a)\tau}{\sigma_2(a)\sqrt{\tau}}, \frac{-z_1 + \frac{1}{2}\sigma_2^2(a)\tau}{\sigma_1(a)\sqrt{\tau}}\right] \quad \text{(left tail negative, right tail positive)}$$
$$[h_{k1}(a), h_{k2}(a)] = [h_1(a), h_2(a)] - \sigma\sqrt{\tau}[a]. \tag{7.6}$$

(c) If $\frac{1}{2}\sigma_1^2(a)\tau - z_2 > 0$ and $\frac{1}{2}\sigma_2^2(a)\tau - z_1 > 0$ then

$$[h_1(a), h_2(a)] = \left[\frac{-z_2 + \frac{1}{2}\sigma_1^2(a)\tau}{\sigma_2(a)\sqrt{\tau}}, \frac{-z_1 + \frac{1}{2}\sigma_2^2(a)\tau}{\sigma_1(a)\sqrt{\tau}}\right] \quad \text{(both tails positive)}$$
$$[h_{k1}(a), h_{k2}(a)] = [h_1(a), h_2(a)] - \sigma\sqrt{\tau}[a]. \tag{7.7}$$

3. For the case where $z_1 < 0$ and $z_2 > 0$ we have that,

(a) If $\frac{1}{2}\sigma_1^2(a)\tau - z_1 < 0$ and $\frac{1}{2}\sigma_2^2(a)\tau - z_2 > 0$ then

$$[h_1(a), h_2(a)] = \left[\frac{-z_1 + \frac{1}{2}\sigma_2^2(a)\tau}{\sigma_1(a)\sqrt{\tau}}, \frac{z_2 + \frac{1}{2}\sigma_1^2(a)\tau}{\sigma_1(a)\sqrt{\tau}}\right] \quad \text{(left tail negative, right tail positive)}$$
$$[h_{k1}(a), h_{k2}(a)] = [h_1(a), h_2(a)] - \sigma\sqrt{\tau}[a]. \tag{7.8}$$

(b) If $\frac{1}{2}\sigma_1^2(a)\tau - z_1 > 0$ and $\frac{1}{2}\sigma_2^2(a)\tau - z_2 > 0$ then

$$[h_1(a), h_2(a)] = \left[\frac{-z_1 + \frac{1}{2}\sigma_2^2(a)\tau}{\sigma_2(a)\sqrt{\tau}}, \frac{z_2 + \frac{1}{2}\sigma_1^2(a)\tau}{\sigma_1(a)\sqrt{\tau}}\right] \quad \text{(both tails positive)}$$
$$[h_{k1}(a), h_{k2}(a)] = [h_1(a), h_2(a)] - \sigma\sqrt{\tau}[a]. \tag{7.9}$$

We have found h for all possible cases, with fuzzy estimator for σ.

The formulas for the call and put options are respectively,

$$[C_{t1}(a), C_{t2}(a)] = [S_{t1}(a)\Phi(h_1(a)) - X\mathrm{e}^{-r\tau}\Phi(h_{k2}(a)), S_{t2}(a)\Phi(h_2(a)) - X\mathrm{e}^{-r\tau}\Phi(h_{k1}(a))] \tag{7.10}$$

and

$$[P_{t1}(a), P_{t2}(a)] = \left[X\mathrm{e}^{-rt}\Phi((-h_k)_1(a)) - S_{t2}(a)\Phi((-h)_2(a)), X\mathrm{e}^{-rt}\Phi((-h_k)_2(a)) - S_{t1}(a)\Phi((-h)_1(a))\right]. \tag{7.11}$$

As concerning the formulas for the sensitivities we have:

- These are the a-cuts of sensitivities for *call options*:

$$\text{Delta}[a] = [\Phi(h_1(a)), \Phi(h_2(a))],$$
$$\text{Hedge Ratio}[a] = [-\mathrm{e}^{-rt}\Phi((h_k)_1(a)), -\mathrm{e}^{-rt}\Phi(h_{k2}(a))]$$
$$\text{Theta}[a] = \left[\frac{S_{t1}(a)\sigma_1(a)\mathrm{e}^{-\frac{h^2}{2}}}{2\sqrt{2\pi\tau}} + Xr\mathrm{e}^{-rt}\Phi((h_k)_1(a)), \frac{S_{t2}(a)\sigma_2(a)\mathrm{e}^{-\frac{h^2}{2}}}{2\sqrt{2\pi\tau}} + Xr\mathrm{e}^{-rt}\Phi((h_k)_2(a))\right]$$
$$\text{Rho}[a] = [\tau X\mathrm{e}^{-rt}\Phi((h_k)_1(a)), \tau X\mathrm{e}^{-rt}\Phi(h_{k2}(a))]$$
$$\text{Kappa}[a] = \left[\frac{S_{t1}(a)\sqrt{\tau}\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\pi}}, \frac{S_{t2}(a)\sqrt{\tau}\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\pi}}\right]$$
$$\text{Gamma}[a] = \left[\frac{\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\tau\pi}S_{t2}(a)\sigma_2(a)}, \frac{\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\tau\pi}S_{t1}(a)\sigma_1(a)}\right].$$

- These are the a-cuts of sensitivities for *put options*:

$$\text{Delta}[a] = [\Phi(h_1(a)) - 1, \Phi(h_2(a)) - 1],$$
$$\text{Hedge Ratio}[a] = [\mathrm{e}^{-rt}(1 - (\Phi(h_k)_2(a))), \mathrm{e}^{-rt}(1 - (\Phi(hk)_1(a)))]$$

$$\text{Theta}[a] = \left[\frac{S_{t1}(a)\sigma_1(a)\mathrm{e}^{-\frac{h^2}{2}}}{2\sqrt{2\pi\tau}} - Xr\mathrm{e}^{-rt}(1-\Phi((h_k)_1(a))), \right.$$
$$\left. \frac{S_{t2}(a)\sigma_2(a)\mathrm{e}^{-\frac{h^2}{2}}}{2\sqrt{2\pi\tau}} - Xr\mathrm{e}^{-rt}(1-\Phi((h_k)_2(a)))\right]$$
$$\text{Rho}[a] = [-\tau X\mathrm{e}^{-rt}(1-\Phi((h_k)_2(a))), -\tau X\mathrm{e}^{-rt}(1-\Phi((h_k)_1(a)))]$$
$$\text{Kappa}[a] = \left[\frac{S_{t1}(a)\sqrt{\tau}\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\pi}}, \frac{S_{t2}(a)\sqrt{\tau}\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\pi}}\right]$$
$$\text{Gamma}[a] = \left[\frac{\mathrm{e}^{-\frac{h^3}{2}}}{\sqrt{2\tau\pi}S\sigma_2(a)}, \frac{\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\tau\pi}S\sigma_1(a)}\right].$$

8. Calculations of formulas and final algorithm for the possibilistic mean of fuzzy volatility and fuzzy stock price (AFN with PM method — adaptive fuzzy numbers with possibilistic mean)

We have the following five steps in order to find the price of a call or of a put option.

1. We compute the a-level of the adaptive fuzzy number S, $S[\alpha]$.
2. We compute the possibilistic mean values $E(S)$, $E(\sigma)$, $E(\sigma^2)$.
3. We compute h and h_k.
4. We compute $\Phi(h)$ and $\Phi(h_k)$.
5. We use the Black & Scholes formula for the option we want to price and for its sensitivities.

Here we consider S as an adaptive fuzzy number. For the volatility we use the same fuzzy number that we used before but now we replace it in the final formula with its possibilistic mean value. From Section 5 we can easily see that if we consider $S = \{c, \frac{c+d}{2}, d\}$ as a symmetric triangular fuzzy number we take that:

$$S[a] = \left[c - (1-a)\frac{c+d}{2}, c + (1-a)d\right] \quad \forall a \in [0, 1]. \tag{8.1}$$

And also for the possibilistic mean value that:

$$E(S) = \int_0^1 a\left[c - (1-a)\frac{c+d}{2} + c + (1-a)d\right]\mathrm{d}a = c + \frac{d - \frac{c+d}{2}}{6}. \tag{8.2}$$

Here we must mention the fact that $S > 0$ so we have that $a > \frac{c-d}{d+c}$. Thus we can compute only for membership functions for $a > \frac{c-d}{d+c}$.

Furthermore from Section 4 we consider the fuzzy volatility as:

$$\sigma^2(x) = \begin{cases} \dfrac{2-\beta}{1-\beta} - \dfrac{2}{1-\beta}\Phi\left(\sqrt{\dfrac{n-1}{2}}\left(\dfrac{s^2}{x} - 1\right)\right) & \text{if } \dfrac{s^2}{1+\Phi^{-1}\left(1-\frac{\beta}{2}\right)\sqrt{\frac{2}{n-1}}} \le x \le s^2 \\ \dfrac{2-\beta}{1-\beta} - \dfrac{2}{1-\beta}\Phi\left(\sqrt{\dfrac{n-1}{2}}\left(1 - \dfrac{s^2}{x}\right)\right) & \text{if } s^2 \le x \le \dfrac{s^2}{1-\Phi^{-1}\left(1-\frac{\beta}{2}\right)\sqrt{\frac{2}{n-1}}}. \end{cases}$$

Its a-cut is the following:

$$\sigma^2[a] = \left[\frac{s^2}{1+z_{g(a)}\sqrt{\frac{2}{n-1}}}, \frac{s^2}{1-z_{g(a)}\sqrt{\frac{2}{n-1}}}\right].$$

In the same way as above we can calculate the possibilistic mean values $E(\sigma)$ and $E(\sigma^2)$.

$$E(\sigma)=\int_0^1 a\left(\sqrt{\frac{s^2}{1+z_{g(a)}\sqrt{\frac{2}{n-1}}}}+\sqrt{\frac{s^2}{1-z_{g(a)}\sqrt{\frac{2}{n-1}}}}\right)\mathrm{d}a \tag{8.3}$$

$$E(\sigma^2)=\int_0^1 a\left(\frac{s^2}{1+z_{g(a)}\sqrt{\frac{2}{n-1}}}+\frac{s^2}{1-z_{g(a)}\sqrt{\frac{2}{n-1}}}\right)\mathrm{d}a. \tag{8.4}$$

For h we have that:

$$h=\frac{\log\left(\frac{E(S)}{X}\right)+r\tau+\frac{1}{2}E(\sigma^2)\tau}{E(\sigma)\sqrt{\tau}}. \tag{8.5}$$

Similarly we find that:

$$h_k=\frac{\log\left(\frac{E(S)}{X}\right)+r\tau+\frac{1}{2}E(\sigma^2)\tau}{E(\sigma)\sqrt{\tau}}-E(\sigma)\sqrt{\tau}. \tag{8.6}$$

The formulas for the call and put options are respectively,

$$[C_{t1}(a),C_{t2}(a)]=[S_{t1}(a)\Phi(h)-X\mathrm{e}^{-r\tau}\Phi(h_k),S_{t2}(a)\Phi(h)-X\mathrm{e}^{-r\tau}\Phi(h_k)] \tag{8.7}$$

and

$$[P_{t1}(a),P_{t2}(a)]=\left[X\mathrm{e}^{-rt}\Phi(-h_k)-S_{t2}(a)\Phi(-h),X\mathrm{e}^{-rt}\Phi(-h_k)-S_{t1}(a)\Phi(-h)\right]. \tag{8.8}$$

As concerning the formulas for the sensitivities we have:

- These are the a-cuts of sensitivities for *call options*:

$$\mathrm{Delta}[a]=\Phi(h),$$
$$\text{Hedge Ratio}[a]=-\mathrm{e}^{-rt}\Phi(h_k)$$
$$\mathrm{Theta}[a]=\left[\frac{S_{t1}(a)E(\sigma)\mathrm{e}^{-\frac{h^2}{2}}}{2\sqrt{2\pi\tau}}+Xr\mathrm{e}^{-rt}\Phi(h_k),\frac{S_{t2}(a)e(\sigma)\mathrm{e}^{-\frac{h^2}{2}}}{2\sqrt{2\pi\tau}}+Xr\mathrm{e}^{-rt}\Phi(h_k)\right]$$
$$\mathrm{Rho}[a]=\tau X\mathrm{e}^{-rt}\Phi(h_k)$$
$$\mathrm{Kappa}[a]=\left[\frac{S_{t1}(a)\sqrt{\tau}\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\pi}},\frac{S_{t2}(a)\sqrt{\tau}\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\pi}}\right]$$
$$\mathrm{Gamma}[a]=\left[\frac{\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\tau\pi}S_{t2}(a)E(\sigma)},\frac{\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\tau\pi}S_{t1}(a)E(\sigma)}\right].$$

- These are the a-cuts of sensitivities for *put options*:

$$\mathrm{Delta}[a]=\Phi(h)-1,$$
$$\text{Hedge Ratio}[a]=\mathrm{e}^{-rt}[1-\Phi(h_k)]$$
$$\mathrm{Theta}[a]=\left[\frac{S_{t1}(a)E(\sigma)\mathrm{e}^{-\frac{h^2}{2}}}{2\sqrt{2\pi\tau}}-Xr\mathrm{e}^{-rt}(1-\Phi[(h_k)]),\frac{S_{t2}(a)E(\sigma)\mathrm{e}^{-\frac{h^2}{2}}}{2\sqrt{2\pi\tau}}-Xr\mathrm{e}^{-rt}[1-\Phi(h_k)]\right]$$
$$\mathrm{Rho}[a]=-\tau X\mathrm{e}^{-rt}[1-\Phi(h_k)]$$
$$\mathrm{Kappa}[a]=\left[\frac{S_{t1}(a)\sqrt{\tau}\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\pi}},\frac{S_{t2}(a)\sqrt{\tau}\mathrm{e}^{-\frac{h^2}{2}}}{\sqrt{2\pi}}\right]$$

$$\mathrm{Gamma}[a] = \left[\frac{e^{-\frac{h^2}{2}}}{\sqrt{2\tau\pi} SE(\sigma)}, \frac{e^{-\frac{h}{2}}}{\sqrt{2\tau\pi} SE(\sigma)} \right].$$

9. Numerical example

Let us calculate the theoretical price of call option, using the Black–Scholes formula, with both methods presented above. The option is due to expire in 180 days, if there are no dividends from now until expiry and

- the underlying stock is price is the triangular symmetric fuzzy number (147, 148, 149)€
- the exercise price is 152€
- the risk-free interest rate is 10% per annum payable continuously.

We will calculate the associated sensitivities. Let consider the stock prices of the appendix for 180 trading days.

9.1. *Fuzzy estimators and fuzzy stock price method*

By realizing the appropriate operations by following the steps from 1 to 5 we take the following fuzzy number for the call option price, C_t and also for the associated sensitivities (see Fig. 9.1):

Table 1
Fuzzy Estimators method results

	$a = 0$		$a = 1$
Option value	5.148775	33.8883587	19.07906868
Volatility	0.376941624	0.4987986	0.42594535
Delta	0.559787	0.629139	0.589095
Hedge ratio	−0.500676	−0.402863	−0.447714
Theta	17.56004	25.429225	21.081312
Rho	30.193439	37.450696	33.585885
Kappa	29.602783	35.900994	33.10497
Gamma	0.005592	0.008722	0.007205

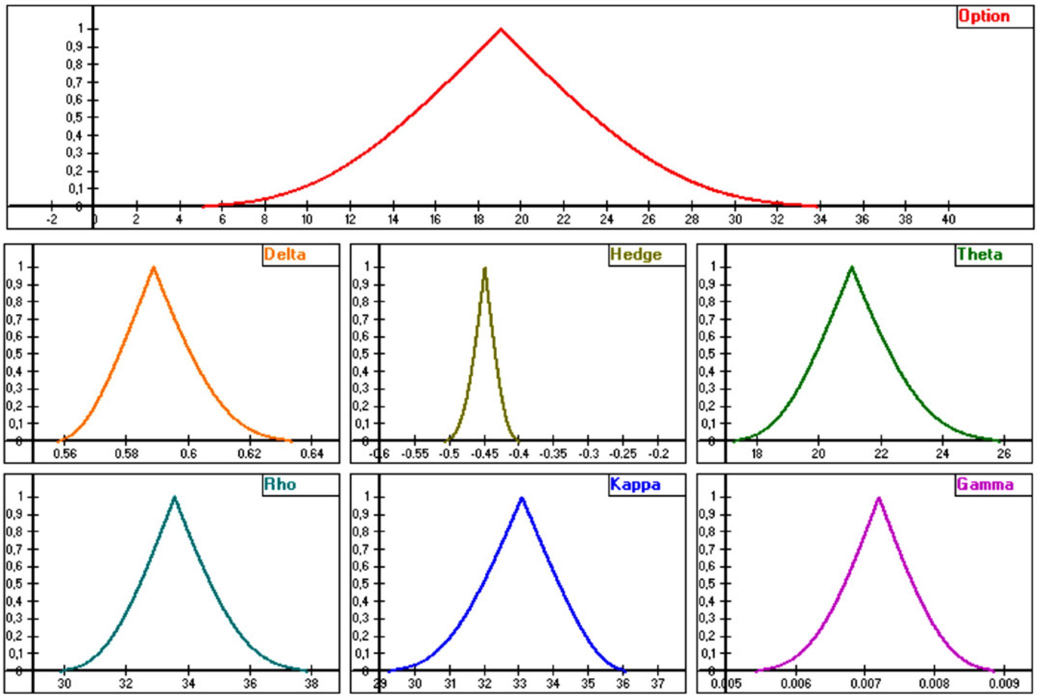

Fig. 9.1. Fuzzy option value and fuzzy sensitivities for fuzzy estimators method.

9.2. Adaptive Fuzzy Numbers with possibilistic mean value method (AFN with PM)

By realizing the appropriate operations by following the steps from 1 to 5 we take the following fuzzy number for the call option price, C_t and also for the associated sensitivities (see Fig. 9.2):

Table 2
AFN with PM method results

	$a = 0.00675$		$a = 1$
Option value	0	98.415199	18.434658
Volatility	0.3769416	0.4987860	0.425971145
Delta			0.540566
Hedge ratio			−0.401504
Theta	6.153082	41.461232	23.807157
Rho			50.096277
Kappa	0.232604	81.823065	40.920476
Gamma	0.004175	2.451889	0.007753

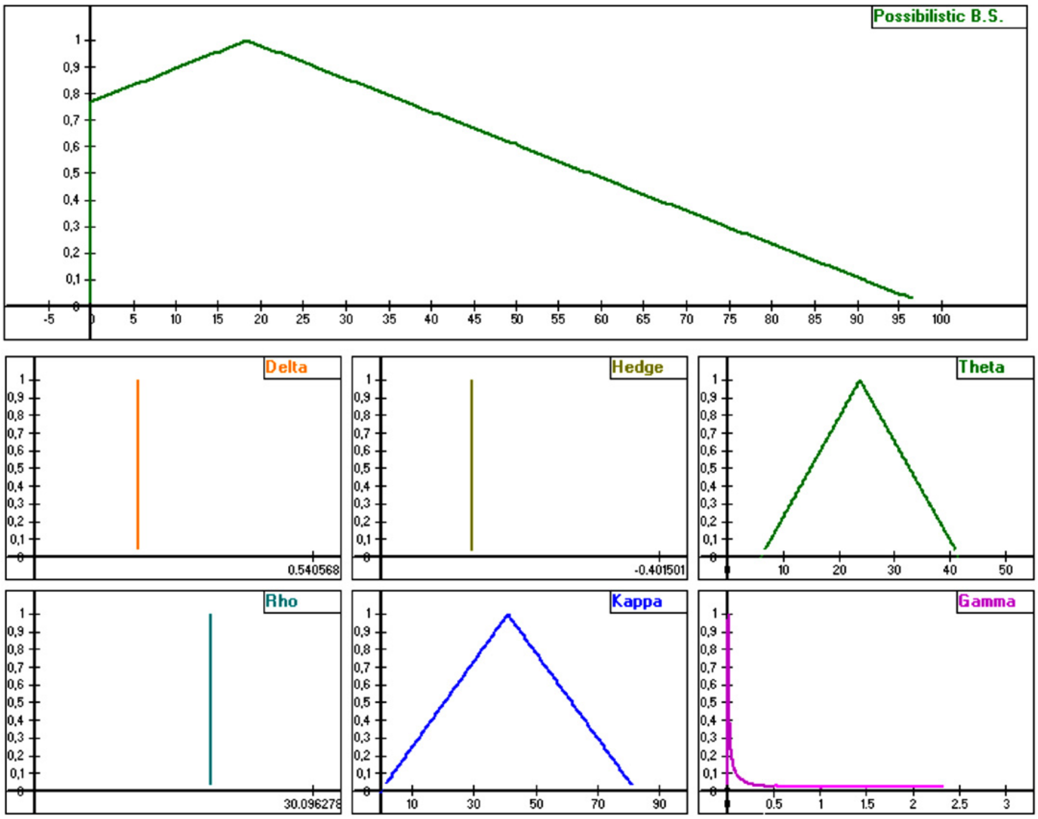

Fig. 9.2. Fuzzy option value and fuzzy sensitivities for adaptive fuzzy numbers and possibilistic mean method.

In Tables 1 and 2 are presented the results of these two methods, which are depicted on the figures above. As we can see fuzzy numbers are formed and they represent the option price and the price of the sensitivities of the model. As we notice, the intervals that are derived from the method of fuzzy estimators are obviously narrower than those we take from the other method.

It is well known that a financial analyst wants to choose a financial instrument or a derivative in order either to add it in a portfolio or for hedging reasons. That is why he should realize this choice according to the price of this instrument. With the method of fuzzy estimators we manage to give a narrower interval in which is moved the option price and also narrower intervals for the sensitivities in comparison with the other method. Something else that we notice is the fact that for three of the sensitivities of the Black and Scholes model the method of AFN with PM gives crisp and not fuzzy numbers.

10. Conclusions

With regard to the above findings we believe that we have managed to give an ability of a better approximation to the price of the option based on the Black and Scholes model so that the financial analyst can be accommodated in the pricing of financial instruments and in the choice of the appropriate one for his target. In general we could say that our method has applications in every branch of science and life where there is the need of estimation of parameters from statistical data. Suggestively we mention some branches of science, in which our method has an application (and is not limited to them), such as Biomedical Sciences, Medicine, Computer Sciences, Economics and Social Sciences. Applications also exist in every branch of life where estimation from statistical data is needed.

Acknowledgement

We would like to thank the reviewer for his valuable suggestions.

Appendix

Random Stock prices from 146€ to 156€
147.45, 151.29, 152.51, 154.95, 146.21, 146.08, 147.38, 155.13, 153.85, 151.18, 150.6, 152.47, 153.31, 150.84, 150, 151.64, 146.22, 153.29, 151.52, 152.9, 147.44, 147.13, 150.04, 149.76, 152.06, 150.19, 152.64, 149.15, 154.93, 151.01, 154.35, 154.03, 152.74, 152.69, 146.48, 148.62, 148.5, 149.32, 147.38, 146.01, 148.58, 154.89, 155.72, 148.29, 152.27, 154.69, 146.28, 149.29, 153.77, 149.2, 148.21, 149.65, 146.76, 150.51, 155.84, 154.52, 151.42, 151.26, 151.93, 153.7, 153.93, 155.98, 153.5, 154.34, 148.55, 155.05, 150.04, 146.02, 152.12, 151.06, 154.36, 148.46, 151.09, 147.78, 155.83, 151.67, 151.07, 151.5, 146.18, 148.59, 148.85, 154.97, 147.78, 150.27, 146.95, 153.82, 154.39, 153.5, 150.59, 155.42, 154.49, 150.97, 147.31, 150.33, 147.04, 153.96, 148.93, 146.19, 149.75, 147.45, 146.13, 154.13, 149.87, 147.8, 155.43, 147.65, 147.48, 152.91, 148.78, 154.08, 151.26, 154.86, 155.54, 148.44, 146.78, 155.09, 147.05, 149.85, 147.25, 152.02, 153.63, 150.46, 146.88, 151.26, 154.12, 146.86, 146.54, 148.2, 147.95, 153.93, 152.18, 153.23, 149.6, 151.33, 155.48, 149.34, 155.68, 152.32, 147.24, 152.32, 154.47, 155.25, 153.08, 153.4, 147.9, 147.4, 151.78, 154.4, 146.14, 151.42, 146, 155.4, 154.6, 154.21, 155.65, 154.88, 153.44, 148.38, 153.57, 152.38, 149.66, 146.08, 153.89, 153.66, 151.58, 155.47, 152.76, 148.39, 155.89, 149.6, 151.67, 150.22, 147.18, 147.82, 153.21, 149.54, 149.99, 149.68, 149.34, 148.54.

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