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Combinatorics of Free Cumulants

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We derive a formula for expressing free cumulants whose entries are products of random variables in terms of the lattice structure of non-crossing partitions. We

trace property for the considered linear functionals.

INTRODUCTION

Free probability theory, due to Voiculescu [17, 18], is a non-commutative probability theory where the classical concept of "independence" is replaced by a non-commutative analogue, called "freeness." Originally this theory was introduced in an operator-algebraic context for dealing with questions on special von Neumann algebras. However, since these beginnings free probability theory has evolved into a theory with a lot of links to quite different fields. In particular, there exists a combinatorial facet: main aspects of free probability theory can be considered as the combinatorics of non-crossing partitions.

There are two main approaches to freeness:

- the original approach, due to Voiculescu, is analytical in nature and relies on special Fock space constructions for the considered distributions.

- the approach of Speicher [14–16] is combinatorial in nature and describes freeness in terms of so-called free cumulants—these objects are defined via a precise combinatorial description involving the lattice of non-crossing partitions; a lot of questions on freeness reduce in this approach finally to combinatorial problems on non-crossing partitions.

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The relation between these two approaches is given by the fact that the free cumulants appear as coefficients in the operators constructed in the Fock space approach. This connection was worked out by Nica [6].

Here, we will investigate one fundamental problem in the combinatorial approach and show that there is a beautiful combinatorial structure behind this.

In the combinatorial approach to freeness one defines, for a given linear functional φ on a unital algebra $\mathcal{A}$, so-called free cumulants k_n ($n \in \mathbf{N}$), where each k_n is a multi-linear functional on $\mathcal{A}$ in n arguments. The connection between φ and the k_n is given by a combinatorial formula involving the lattice of non-crossing partitions. (The name "cumulants" comes from classical probability theory; there exist analogous objects with that name, the only difference is that there all partitions instead of non-crossing partitions appear.) It seems that many problems on freeness are easier to handle in terms of these free cumulants than in terms of moments of φ. In particular, the definition of freeness itself becomes much handier for cumulants than for moments. Since cumulants are multi-linear objects this implies that for problems involving the linear structure of the algebra $\mathcal{A}$ cumulants are quite easily and effectively to use. For problems involving the multiplicative structure of $\mathcal{A}$, however, it is not so clear from the beginning that cumulants are a useful tool for such investigations. Nevertheless in a lot of examples it has turned out that this is indeed the case. In a sense, we will here present the unifying reason for these positive results. Namely, dealing with multiplicative problems reduces on the level of cumulants essentially to the problem of understanding the structure of cumulants whose arguments are products of variables. Here, in Section 2, we will show that this can be understood quite well and that there exists a nice and simple combinatorial description for such cumulants.

That this formula is also useful will be demonstrated in Section 3. We will reprove and generalize a lot of results around the multiplication of free random variables. In particular, we will consider an important special class of distributions, so-called R-diagonal elements. These were introduced by Nica and Speicher in [8]. However, the investigations and characterizations in [8, 9] were not always straightforward and used a lot of ad hoc combinatorics. Our approach here will be much more direct and conceptually clearer. Furthermore, we will get in the same spirit direct proofs of results of Haagerup and Larsen [2, 5] on powers of R-diagonal elements.

An important point to make is that all earlier investigations on R-diagonal elements were always restricted to a tracial frame, i.e., φ was assumed to satisfy the trace condition $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$. In contrast, our approach does not rely on this assumption, so all our results are also valid for non-tracial φ. Thus we do not only get simple proofs for known results but also generalizations of all these results to the general, non-tracial case. (That non-tracial R-diagonal elements appear quite naturally can, e.g., be seen in [13], where such elements arise in the polar decomposition of generalized circular elements).

Our Propositions 3.5 and 3.9 were inspired by and prove some conjectures of the recent work [10]. There the notion of R-diagonality is also treated in the non-tracial case and some of our results of Section 3 are proved there for the general case, too. However, the approach in [10] is quite different from the present one and relies on Fock space representations and freeness with amalgamation.

The paper is organized as follows. In Section 1, we give a short and self-contained summary of the relevant basic definitions and facts about free probability theory and non-crossing partitions. In Section 2, we state and prove our main combinatorial result on the structure of free cumulants whose arguments are products and, in Section 3, we apply this result to derive various statements about R-diagonal elements.

1. PRELIMINARIES

In this section we provide a short and self-contained summary of the basic definitions and facts needed for our later investigations.

1.1. *Non-commutative Probability Theory.* (1) We will always work in the frame of a *non-commutative probability space* $(\mathcal{A}, \varphi)$. This is, by definition, a pair consisting of a unital $*$-algebra $\mathcal{A}$ and a unital linear functional $\varphi: \mathcal{A} \to \mathbf{C}$. ($\varphi$ unital means that $\varphi(1)=1$.)

The elements $a \in \mathcal{A}$ are called *non-commutative random variables*, or just *random variables* in $(\mathcal{A}, \varphi)$.

Let $a_1, ..., a_n$ be random variables in a non-commutative probability space $(\mathcal{A}, \varphi)$. Let $\mathbf{C}\langle X_1, ..., X_n\rangle$ denote the algebra of polynomials in n non-commuting indeterminants, i.e., the algebra generated by n free generators. Then the linear functional

$$\mu_{a_1, ..., a_n}: \mathbf{C}\langle X_1, ..., X_n\rangle \to \mathbf{C}$$

given by linear extension of

$$X_{i(1)} \cdots X_{i(m)} \mapsto \varphi(a_{i(1)} \cdots a_{i(m)}) \qquad (m \in \mathbf{N},\ 1 \leqslant i(1), ..., i(m) \leqslant n)$$

is called the *joint distribution* of $a_1, ..., a_n$.

The joint distribution of a and a^* is also called the *$*$-distribution* of a. Consider random variables a_i and $b_i (1 \leqslant i \leqslant n)$ in $(\mathcal{A}, \varphi)$. Then $a_1, ..., a_n$

and $b_1, \dots, b_n$ have the same joint distribution, if the following equation holds for all $m \in \mathbf{N}$, $1 \leqslant i(1), \dots, i(m) \leqslant n$:

$$\varphi(a_{i(1)} \cdots a_{i(m)}) = \varphi(b_{i(1)} \cdots b_{i(m)}).$$

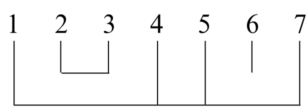
(2) Note that all our considerations will be on the algebraic (or combinatorial) level, thus we will not require that φ is a positive functional. However, it is well known that freeness—the crucial structure in our investigations—is compatible with positivity properties. The requirement that our probability space should be a $*$-algebra and not just an algebra is only for convenience, since, in Section 3, we will need the $*$ for dealing with Haar unitaries and R-diagonal elements. In all statements where no $*$ appears we could also replace the requirement "$*$-algebra" by "algebra."

(3) Most of the questions which we will investigate in Section 3 were up to now only considered for tracial linear functionals. We stress that all our considerations do not use the trace property; i.e., we will not use the equation $\varphi(ab) = \varphi(ba)$.

1.2. *Partitions.* (1) Fix $n \in \mathbf{N}$. We call $\pi = \{V_1, \dots, V_r\}$ a *partition* of $S = (1, \dots, n)$ if and only if the $V_i (1 \leqslant i \leqslant r)$ are pairwisely disjoint, non-void tuples such that $V_1 \cup \cdots \cup V_r = S$. We call the tuples $V_1, \dots, V_r$ the *blocks* of π. The number of components of a block V is denoted by $|V|$. Given two elements p und q with $1 \leqslant p, q \leqslant n$, we write $p \sim_\pi q$, if p and q belong to the same block of π.

We get a linear representation of a partition π by writing all elements $1, \dots, n$ in a line, supplying each with a vertical line under it and joining the vertical lines of the elements in the same block with a horizontal line.

Example. A partition of the tuple $S = (1, 2, 3, 4, 5, 6, 7)$ is

$\pi_1 = \{(1, 4, 5, 7), (2, 3), (6)\} \;\widehat{=}$ 

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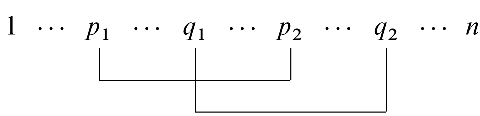
If we write a block V of a partition in the form $V = (v_1, \dots, v_p)$ then this shall always imply that $v_1 < v_2 < \cdots < v_p$.

(2) A partition π is called *non-crossing*, if the following situation does not occur: There exist $1 \leqslant p_1 < q_1 < p_2 < q_2 \leqslant n$ such that $p_1 \sim_\pi p_2 \not\sim_\pi q_1 \sim_\pi q_2$:



The set of all non-crossing partitions of $(1, ..., n)$ is denoted by $NC(n)$. In the same way as for $(1, ..., n)$ one can introduce non-crossing partitions $NC(S)$ for each finite linearly ordered set S. Of course, $NC(S)$ depends only on the number of elements in S. In our investigations, non-crossing partitions will appear as partitions of the index set of products of random variables $a_1 \cdots a_n$. In such a case, we will also sometimes use the notation $NC(a_1, ..., a_n)$. (If some of the a_i are equal, this might make no rigorous sense, but there should arise no problems by this.)

If S is the union of two disjoint sets S_1 and S_2 then, for $\pi_1 \in NC(S_1)$ and $\pi_2 \in NC(S_2)$, we let $\pi_1 \cup \pi_2$ be that partition of S which has as blocks the blocks of π_1 and the blocks of π_2. Note that $\pi_1 \cup \pi_2$ is not automatically non-crossing.

(3) Let $\pi, \sigma \in NC(n)$ be two non-crossing partitions. We write $\sigma \leqslant \pi$, if every block of σ is completely included in a block of π. Hence, we obtain σ out of π by refining the block-structure. For example, we have

$$\{(1, 3), (2), (4, 5), (6, 8), (7)\} \leqslant \{(1, 3, 7), (2), (4, 5, 6, 8)\}.$$

The partial order $\leqslant$ induces a lattice structure on $NC(n)$. In particular, given two non-crossing partitions $\pi, \sigma \in NC(n)$, we have their join $\pi \vee \sigma$, which is the unique smallest $\tau \in NC(n)$ such that $\tau \geqslant \pi$ and $\tau \geqslant \sigma$.

The maximum of $NC(n)$—the partition which consists of one block with n components—is denoted by 1_n. The partition consisting of n blocks, each of which has one component, is the minimum of $NC(n)$ and denoted by 0_n.

(4) The lattice $NC(n)$ is self-dual and there exists an important anti-isomorphism $K: NC(n) \to NC(n)$ implementing this self-duality. This complementation map K is defined as follows: Let π be a non-crossing partition of the numbers $1, ..., n$. Furthermore, we consider numbers $\bar{1}, ..., \bar{n}$ with all numbers ordered like

$$1 \; \bar{1} \; 2 \; \bar{2} \; \cdots \; n \; \bar{n}.$$

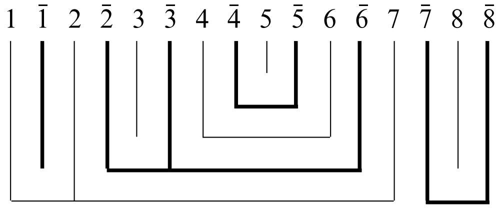
The *complement* $K(\pi)$ of $\pi \in NC(n)$ is defined to be the biggest $\sigma \in NC(\bar{1}, ..., \bar{n}) \triangleq NC(n)$ with

$$\pi \cup \sigma \in NC(1, \bar{1}, ..., n, \bar{n}).$$

Example. Consider the partition $\pi := \{(1, 2, 7), (3), (4, 6), (5), (8)\} \in NC(8)$. For the complement $K(\pi)$ we get

$$K(\pi) = \{(1), (2, 3, 6), (4, 5), (7, 8)\},$$

as can be seen from the graphical representation,



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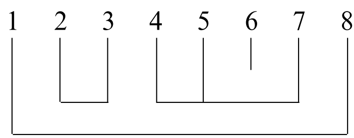
(5) Non-crossing partitions and the complementation map were introduced by Kreweras [4]; for further combinatorial investigations on that lattice, see, e.g., [1, 12].

(6) The main combinatorial ingredient of Theorem 2.2 will be joins with special partitions σ whose blocks consist of neighbouring elements, like $\pi \vee \{(1), (2), ..., (l, ..., l+k), ..., (n)\}$. This is given by uniting the blocks of π containing the elements $l, ..., l+k$, and we say that we obtain $\pi \vee \{(1), (2), ..., (l, ..., l+k), ..., (n)\}$ by *connecting* the elements $l, ..., l+k$.

EXAMPLE. Considering the partition

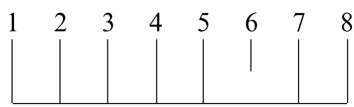
$\pi = \{(1, 8), (2, 3), (4, 5, 7), (6)\} \quad \hat{=}$



we have

$$\pi \vee \{(1, 2, 3, 4), (5), (6), (7), (8)\} = \{(1, 2, 3, 4, 5, 7, 8), (6)\}$$

$\hat{=}$



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1.3. *Free Cumulants.* Given a unital linear functional $\varphi: \mathscr{A} \to \mathbf{C}$ we define corresponding (*free*) *cumulants* $(k_n)_{n \in \mathbf{N}}$

$$\begin{aligned} k_n: \qquad & \mathscr{A}^n \to \mathbf{C}, \\ & (a_1, ..., a_n) \mapsto k_n(a_1, ..., a_n) \end{aligned}$$

indirectly by the system of equations

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} k_\pi[a_1, ..., a_n] \qquad (a_1, ..., a_n \in \mathscr{A}), \tag{1}$$

where k_π splits multiplicatively in a product of cumulants according to the block structure of π, i.e.,

$$k_\pi[a_1, ..., a_n] := \prod_{i=1}^{r} k_{|V_i|}(a_{i,1}, ..., a_{i,|V_i|}) \tag{2}$$

for a partition $\pi = \{V_1, ..., V_r\} \in NC(n)$ consisting of r blocks of the form $V_i = (a_{i,1}, ..., a_{i,|V_i|})$.

The defining relation (1) expresses the moment $\varphi(a_1 \cdots a_n)$ in terms of cumulants, but by induction this can also be resolved for giving the cumulants uniquely in terms of moments:

$$k_n(a_1, ..., a_n) = \varphi(a_1 \cdots a_n) - \sum_{\pi \in NC(n);\, \pi \neq 1_n} k_\pi[a_1, ..., a_n]. \tag{3}$$

Since, by induction, we know all cumulants of smaller order, i.e., all $k_\pi[a_1, ..., a_n]$ for $\pi \in NC(n)$ with $\pi \neq 1_n$, this leads to an expression for k_n in terms of moments. Abstractly, this is, of course, just the Moebius inversion of relation (1) and has the form

$$k_n(a_1, ..., a_n) = \sum_{\pi \in NC(n)} \mu(\pi, 1_n)\, \varphi_\pi[a_1, ..., a_n], \tag{4}$$

where μ is the Moebius function of the lattice of non-crossing partitions and where φ_π is defined in the same multiplicative way as k_π if we put $\varphi_n(a_1, ..., a_n) := \varphi(a_1 \cdots a_n)$.

EXAMPLES. Let us give the concrete form of $k_n(a_1, ..., a_n)$ for $n = 1, 2, 3$.

- $n = 1$.

$$k_1(a_1) = \varphi(a_1).$$

- $n = 2$. The only partition $\pi \in NC(2)$, $\pi \neq 1_2$ is $\vert\;\vert$. So we get

$$\begin{aligned} k_2(a_1, a_2) &= \varphi(a_1 a_2) - k_{\vert\,\vert}[a_1, a_2] \\ &= \varphi(a_1 a_2) - k_1(a_1)\, k_1(a_2) \\ &= \varphi(a_1 a_2) - \varphi(a_1)\, \varphi(a_2). \end{aligned}$$

Using the notation φ_π we can also write this as

$$k_2(a_1, a_2) = \varphi_{\sqcup}[a_1, a_2] - \varphi_{\vert\,\vert}[a_1, a_2].$$

- $n=3$. We have to take all partitions in $NC(3)$ except 1_3, i.e., the partitions

$$\vert\sqcup, \qquad \sqcup\vert, \qquad \sqcup\!\!\!\!\!\mid\,, \qquad \vert\vert\vert.$$

With this we obtain:

$$\begin{aligned}
k_3(a_1, a_2, a_3) &= \varphi(a_1 a_2 a_3) - k_{\vert\sqcup}[a_1, a_2, a_3] - k_{\sqcup\vert}[a_1, a_2, a_3] \\
&\quad - k_{\sqcup\!\!\!\!\!\mid}[a_1, a_2, a_3] - k_{\vert\vert\vert}[a_1, a_2, a_3] \\
&= \varphi(a_1 a_2 a_3) - k_1(a_1)\, k_2(a_2, a_3) - k_2(a_1, a_2)\, k_1(a_3) \\
&\quad - k_2(a_1, a_3)\, k_1(a_2) - k_1(a_1)\, k_1(a_2)\, k_1(a_3) \\
&= \varphi(a_1 a_2 a_3) - \varphi(a_1)\, \varphi(a_2 a_3) - \varphi(a_1 a_2)\, \varphi(a_3) \\
&\quad - \varphi(a_1 a_3)\, \varphi(a_2) + 2\varphi(a_1)\, \varphi(a_2)\, \varphi(a_3).
\end{aligned}$$

Again we can write this in the Moebius inverted form:

$$\begin{aligned}
k_3(a_1, a_2, a_3) &= \varphi_{\sqcup\!\sqcup}[a_1, a_2, a_3] - \varphi_{\vert\sqcup}[a_1, a_2, a_3] \\
&\quad - \varphi_{\sqcup\vert}[a_1, a_2, a_3] - \varphi_{\sqcup\!\!\!\!\!\mid}[a_1, a_2, a_3] \\
&\quad + 2\varphi_{\vert\vert\vert}[a_1, a_2, a_3].
\end{aligned}$$

1.4. *Freeness.* Freeness of subalgebras or random variables is the crucial concept in free probability theory; it is a non-commutative replacement for the classical concept of "independence."

(1) Let $\mathcal{A}_1, ..., \mathcal{A}_m \subset \mathcal{A}$ be subalgebras with $1 \in \mathcal{A}_i$ $(i=1, ..., m)$. The subalgebras $\mathcal{A}_1, ..., \mathcal{A}_m$ are called *free*, if $\varphi(a_1 \cdots a_k)=0$ for all $k \in \mathbf{N}$ and $a_i \in \mathcal{A}_{j(i)}$ $(1 \leqslant j(i) \leqslant m)$ such that $\varphi(a_i)=0$ for all $i=1, ..., k$ and such that neighbouring elements are from different subalgebras, i.e., $j(1) \neq j(2) \neq \cdots \neq j(k)$.

(2) Let $\mathcal{X}_1, ..., \mathcal{X}_m \subset \mathcal{A}$ be subsets of $\mathcal{A}$. Then $\mathcal{X}_1, ..., \mathcal{X}_m$ are called free, if $\mathcal{A}_1, ..., \mathcal{A}_m$ are free, where, for $i=1, ..., m$, $\mathcal{A}_i := \text{alg}(1, \mathcal{X}_i)$ is the algebra generated by 1 and $\mathcal{X}_i$.

(3) In particular, if the algebras $\mathcal{A}_i := \text{alg}(1, a_i)$ $(i=1, ..., m)$ generated by the elements $a_1, ..., a_m \in \mathcal{A}$ are free, then $a_1, ..., a_m$ are called *free* random variables. If the *-algebras generated by the random variables $a_1, ..., a_m$ are free, then we call $a_1, ..., a_m$ **-free.*

(4) Freeness of random variables can be considered as a rule for expressing joint moments of free variables in terms of the moments of the

single variables. For example, if $\{a_1, a_2\}$ and b are free, then the following identity holds,

$$\varphi(a_1 b a_2) = \varphi(a_1 a_2)\, \varphi(b). \tag{5}$$

(5) The basic fact which shows the relevance of the free cumulants in connection with freeness is the following characterization of freeness in terms of cumulants. We will only use this characterization of freeness in our proofs. Thus, for the purpose of this paper, part (2) of the following proposition could also be used as the definition of freeness.

1.5. Proposition [15]. *Let $(\mathscr{A}, \varphi)$ be a non-commutative probability space and $\mathscr{A}_1, ..., \mathscr{A}_m \subset \mathscr{A}$ subalgebras. Then the following statements are equivalent*:

(1) *The subalgebras $\mathscr{A}_1, ..., \mathscr{A}_m$ are free.*

(2) *For all $n \geqslant 2$ and all $a_i \in \mathscr{A}_{j(i)}$ with $1 \leqslant j(1), ..., j(n) \leqslant m$ we have $k_n(a_1, ..., a_n) = 0$ whenever there are some $1 \leqslant l, k \leqslant n$ with $j(l) \neq j(k)$.*

2. MAIN COMBINATORIAL RESULT

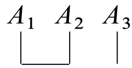
As mentioned in the Introduction we would like to understand the behaviour of free cumulants with respect to the multiplicative structure of our algebra. The crucial property in a multiplicative context is associativity. On the level of moments this just means that we can put brackets arbitrarily; for example, we have $\varphi((a_1 a_2)\, a_3) = \varphi(a_1 (a_2 a_3))$. But the corresponding statement on the level of cumulants is, of course, not true, i.e., $k_2(a_1 a_2, a_3) \neq k_2(a_1, a_2 a_3)$ in general. However, there is still a treatable and nice formula which allows to deal with free cumulants whose entries are products of random variables. This formula is the main combinatorial result of this paper and is presented in this section.

A special case of that theorem, where only one argument of the cumulant has the form of a product, appeared in [14]. However, although our theorem can be considered as an iteration of that special case, the structure of that iteration is not clear from the presentation in [14]. The main observation here is that this iteration really leads to a beautiful and useful combinatorial structure. Our proof will not rely on the special case from [14]. It is conceptually much clearer to prove the theorem directly in its general form than to do it by iteration.

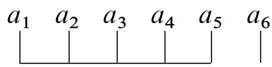
2.1. *Notation.* The general frame for our theorem is the following: Let an increasing sequence of integers be given, $1 \leqslant i_1 < i_2 < \cdots < i_m := n$ and let $a_1, ..., a_n$ be random variables. Then we define new random variables A_j

as products of the given a_i according to $A_j := a_{i_{j-1}+1} \cdots a_{i_j}$ (where $i_0 := 0$). We want to express a cumulant $k_\tau[A_1, ..., A_m]$ in terms of cumulants $k_\pi[a_1, ..., a_n]$. So let τ be a non-crossing partition of the m-tuple $(A_1, ..., A_m)$. Then we define $\hat{\tau} \in NC(a_1, ..., a_n)$ to be that partition which we get from τ by replacing each A_j by $a_{i_{j-1}+1}, ..., a_{i_j}$, i.e., for a_i being a factor in A_k and a_j being a factor in A_l we have $a_i \sim_{\hat{\tau}} a_j$ if and only if $A_k \sim_\tau A_l$.

For example, for $n = 6$ and $A_1 := a_1 a_2$, $A_2 := a_3 a_4 a_5$, $A_3 := a_6$ and

$$\tau = \{(A_1, A_2), (A_3)\} \quad \hat{=}$$ 

we get

$$\hat{\tau} = \{(a_1, a_2, a_3, a_4, a_5), (a_6)\} \quad \hat{=}$$ 

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Note also in particular, that $\hat{\tau} = 1_n$ if and only if $\tau = 1_m$.

2.2. THEOREM. *Let $m \in \mathbf{N}$ and $1 \leqslant i_1 < i_2 < \cdots < i_m := n$ be given. Consider random variables $a_1, ..., a_n$ and put $A_j := a_{i_{j-1}+1} \cdots a_{i_j}$ for $j = 1, ..., m$ (where $i_0 := 0$). Let τ be a partition in $NC(A_1, ..., A_m)$. Then the following equation holds*:

$$k_\tau[a_1 \cdots a_{i_1}, ..., a_{i_{m-1}+1} \cdots a_{i_m}] = \sum_{\pi \in NC(n);\, \pi \vee \sigma = \hat{\tau}} k_\pi[a_1, ..., a_n], \tag{6}$$

where $\sigma \in NC(n)$ is the partition $\sigma = \{(a_1, ..., a_{i_1}), ..., (a_{i_{m-1}+1}, ..., a_{i_m})\}$.

Before we give the proof of our theorem, we want to make clear the structure of the statement by an example.

For $A_1 := a_1 a_2$ and $A_2 := a_3$ we have $\sigma = \{(a_1, a_2), (a_3)\} \;\hat{=}\; \sqcup\, |$. Consider now $\tau = 1_2 = \{(A_1, A_2)\}$, implying that $\hat{\tau} = 1_3 = \{(a_1, a_2, a_3)\}$. Then the application of our theorem yields

$$\begin{aligned} k_2(a_1 a_2, a_3) &= \sum_{\pi \in NC(3);\, \pi \vee \sigma = 1_3} k_\pi[a_1, a_2, a_3] \\ &= k_{\sqcup\!\sqcup}[a_1, a_2, a_3] + k_{|\,\sqcup}[a_1, a_2, a_3] + k_{\sqcup\!\!|\,}[a_1, a_2, a_3] \\ &= k_3(a_1, a_2, a_3) + k_1(a_1)\, k_2(a_2, a_3) + k_2(a_1, a_3)\, k_1(a_2), \end{aligned}$$

which is easily seen to be indeed equal to $k_2(a_1 a_2, a_3) = \varphi(a_1 a_2 a_3) - \varphi(a_1 a_2)\, \varphi(a_3)$.

Proof. We show the assertion by induction over the number m of arguments of the cumulant k_τ.

To begin with, let us study the case when $m=1$. Then we have $\sigma=\{(a_1, ..., a_n)\}=1_n=\hat{\tau}$ and by the defining relation (1) for the free cumulants our assertion reduces to

$$\begin{aligned} k_1(a_1 \cdots a_n) &= \sum_{\pi \in NC(n);\, \pi \vee 1_n = 1_n} k_\pi[a_1, ..., a_n] \\ &= \sum_{\pi \in NC(n)} k_\pi[a_1, ..., a_n] \\ &= \varphi(a_1 \cdots a_n), \end{aligned}$$

which is true since $k_1=\varphi$.

Let us now make the induction hypothesis that for an integer $m \geqslant 1$ the theorem is true for all $m' \leqslant m$.

We want to show that it also holds for $m+1$. This means that for $\tau \in NC(m+1)$, a sequence $1 \leqslant i_1 < i_2 < \cdots < i_{m+1} =: n$, and random variables $a_1, ..., a_n$ we have to prove the validity of the equation

$$\begin{aligned} k_\tau[A_1, ..., A_{m+1}] &= k_\tau[a_1 \cdots a_{i_1}, ..., a_{i_m+1} \cdots a_{i_{m+1}}] \\ &= \sum_{\pi \in NC(n);\, \pi \vee \sigma = \hat{\tau}} k_\pi[a_1, ..., a_n], \end{aligned} \tag{7}$$

where $\sigma=\{(a_1, ..., a_{i_1}), ..., (a_{i_m+1}, ..., a_{i_{m+1}})\}$.

The proof is divided into two steps. The first one discusses the case where $\tau \in NC(m+1)$, $\tau \neq 1_{m+1}$ and the second one treats the case where $\tau = 1_{m+1}$.

Step 1°. The validity of relation (7) for all $\tau \in NC(m+1)$ except the partition 1_{m+1} is shown as follows: Each such τ has at least two blocks, so it can be written as $\tau=\tau_1 \cup \tau_2$ with τ_1 being a non-crossing partition of an s-tuple $(B_1, ..., B_s)$ and τ_2 being a non-crossing partition of a t-tuple $(C_1, ..., C_t)$ where $(B_1, ..., B_s) \cup (C_1, ..., C_t) = (A_1, ..., A_{m+1})$ and $s+t=m+1$. With these definitions, we have

$$k_\tau[A_1, ..., A_{m+1}] = k_{\tau_1}[B_1, ..., B_s]\, k_{\tau_2}[C_1, ..., C_t].$$

We will apply now the induction hypothesis on $k_{\tau_1}[B_1, ..., B_s]$ and on $k_{\tau_2}[C_1, ..., C_t]$. According to the definition of A_j, both $B_k (k=1, ..., s)$ and C_l $(l=1, ..., t)$ are products with factors from $(a_1, ..., a_n)$. Put $(b_1, ..., b_p)$ the tuple containing all factors of $(B_1, ..., B_s)$ and $(c_1, ..., c_q)$ the tuple consisting of all factors of $(C_1, ..., C_t)$; this means $(b_1, ..., b_p) \cup (c_1, ..., c_q) = (a_1, ..., a_n)$ (and $p+q=n$). We put $\sigma_1 := \sigma|_{(b_1, ..., b_p)}$ and $\sigma_2 := \sigma|_{(c_1, ..., c_q)}$, i.e., we have $\sigma=\sigma_1 \cup \sigma_2$. Note that $\hat{\tau}$ factorizes in the same way as $\hat{\tau}=\hat{\tau}_1 \cup \hat{\tau}_2$. Then we get with the help of our induction hypothesis

$$\begin{aligned}
k_\tau[A_1, ..., A_{m+1}] &= k_{\tau_1}[B_1, ..., B_s] \cdot k_{\tau_2}[C_1, ..., C_t] \\
&= \sum_{\pi_1 \in NC(p);\, \pi_1 \vee \sigma_1 = \hat{\tau}_1} k_{\pi_1}[b_1, ..., b_p] \\
&\quad \cdot \sum_{\pi_2 \in NC(q);\, \pi_2 \vee \sigma_2 = \hat{\tau}_2} k_{\pi_2}[c_1, ..., c_q] \\
&= \sum_{\pi_1 \in NC(p);\, \pi_1 \vee \sigma_1 = \hat{\tau}_1} \; \sum_{\pi_2 \in NC(q);\, \pi_2 \vee \sigma_2 = \hat{\tau}_2} k_{\pi_1 \cup \pi_2}[a_1, ..., a_n] \\
&= \sum_{\pi \in NC(n);\, \pi \vee \sigma = \hat{\tau}} k_\pi[a_1, ..., a_n].
\end{aligned}$$

Step 2°. It remains to prove that Eq. (7) is also valid for $\tau = 1_{m+1}$. With (3), we obtain

$$\begin{aligned}
k_{1_{m+1}}[A_1, ..., A_{m+1}] &= k_{m+1}(A_1, ..., A_{m+1}) \\
&= \varphi(A_1 \cdots A_{m+1}) - \sum_{\tau \in NC(m+1);\, \tau \neq 1_{m+1}} k_\tau[A_1, ..., A_{m+1}].
\end{aligned} \tag{8}$$

First we transform the sum in (8) with the result of step 1°,

$$\begin{aligned}
&\sum_{\tau \in NC(m+1);\, \tau \neq 1_{m+1}} k_\tau[A_1, ..., A_{m+1}] \\
&\qquad = \sum_{\tau \in NC(m+1);\, \tau \neq 1_{m+1}} \; \sum_{\pi \in NC(n);\, \pi \vee \sigma = \hat{\tau}} k_\pi[a_1, ..., a_n] \\
&\qquad = \sum_{\pi \in NC(n);\, \pi \vee \sigma \neq 1_n} k_\pi[a_1, ..., a_n],
\end{aligned}$$

where we used the fact that $\tau = 1_{m+1}$ is equivalent to $\hat{\tau} = 1_n$.

The moment in (8) can be written as

$$\varphi(A_1 \cdots A_{m+1}) = \varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} k_\pi[a_1, ..., a_n].$$

Alltogether, we get

$$\begin{aligned}
k_{m+1}[A_1, ..., A_{m+1}] &= \sum_{\pi \in NC(n)} k_\pi[a_1, ..., a_n] \\
&\quad - \sum_{\pi \in NC(n);\, \pi \vee \sigma \neq 1_n} k_\pi[a_1, ..., a_n] \\
&= \sum_{\pi \in NC(n);\, \pi \vee \sigma = 1_n} k_\pi[a_1, ..., a_n]. \quad \blacksquare
\end{aligned}$$

2.3. *Remark.* In all our applications we will only use the special case of Theorem 2.2 where $\tau = 1_m$. Then the statement of the theorem is the following: Consider $m \in \mathbf{N}$, an increasing sequence $1 \leqslant i_1 < i_2 < \cdots < i_m := n$ and random variables $a_1, ..., a_n$. Put $\sigma := \{(a_1, ..., a_{i_1}), ..., (a_{i_{m-1}+1}, ..., a_{i_m})\}$. Then we have

$$k_m[a_1 \cdots a_{i_1}, ..., a_{i_{m-1}+1} \cdots a_{i_m}] = \sum_{\pi \in NC(n);\, \pi \vee \sigma = 1_n} k_\pi[a_1, ..., a_n]. \tag{9}$$

The next proposition, which is from [7, Theorem 1.4], is the basic fact on the multiplication of free random variables. We want to indicate that our Theorem 2.2 can be used to give a straightforward and conceptually simple proof of that statement.

2.4. Proposition [7]. *For a positive integer n, let $a_1, ..., a_n, b_1, ..., b_n$ be random variables such that $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_n\}$ are free. Then the following equation holds,*

$$k_n(a_1 b_1, ..., a_n b_n) = \sum_{\pi \in NC(n)} k_\pi[a_1, ..., a_n]\, k_{K(\pi)}[b_1, ..., b_n]. \tag{10}$$

Proof. We only give a sketch of the proof.
Applying Theorem 2.2 in the form mentioned above in Eq. (9), we get

$$k_n(a_1 b_1, ..., a_n b_n) = \sum_\pi k_\pi[a_1, b_1, ..., a_n, b_n],$$

where we have to sum over

$$\pi \in NC(2n) \qquad \text{with} \quad \pi \vee \{(a_1, b_1), ..., (a_n, b_n)\} = 1_{2n}.$$

Because of the assumption "$\{a_1, ..., a_n\}$, $\{b_1, ..., b_n\}$ free" we obtain with Proposition 1.5 that all cumulants vanish with the exception of those which have only elements from $\{a_1, ..., a_n\}$ or only elements from $\{b_1, ..., b_n\}$ as arguments. This means that all partitions π contributing to the sum must have the form $\pi = \pi_a \cup \pi_b$ with π_a being in $NC(a_1, ..., a_n)$ and π_b being in $NC(b_1, ..., b_n)$. Obviously, for each such π we have

$$k_\pi[a_1, b_1, ..., a_n, b_n] = k_{\pi_a}[a_1, ..., a_n]\, k_{\pi_b}[b_1, ..., b_n].$$

One can now convince oneself, that for each $\pi_a \in NC(a_1, ..., a_n)$ there exists exactly one $\pi_b \in NC(b_1, ..., b_n)$ such that $\pi = \pi_a \cup \pi_b$ fulfills the condition $\pi \vee \{(a_1, b_1), ..., (a_n, b_n)\} = 1_{2n}$ and that this π_b is nothing but the complement of π_a, i.e., we have to sum exactly over all $\pi = \pi_a \cup K(\pi_a)$ with $\pi_a \in NC(n)$. This is the assertion. ∎

2.5. *Remark.* In order to get an idea of the complications arising in the transition from the tracial to the general non-tracial case let us consider the following variant of the foregoing proposition. Let $\{a_1, ..., a_n\}$ be free from $\{b, c\}$ and consider the cumulant $k_n(ba_1c, ba_2c, ..., ba_nc)$. In the tracial case this is the same as $k_n(a_1cb, a_2cb, ..., a_ncb)$ and since $\{a_1, ..., a_n\}$ is free from cb our above proposition yields

$$k_n(ba_1c, ba_2c, ..., ba_nc) = \sum_{\pi \in NC(n)} k_\pi[a_1, ..., a_n]\, k_{K(\pi)}[cb, cb, ..., cb].$$

In the general situation the structure of the result—a summation over $\pi \in NC(n)$ and terms given by a product of cumulants corresponding to blocks of π and blocks of $K(\pi)$—is the same, but now not always cb appears as argument in the cumulants. Namely, a careful adaption of our above proof for Proposition 2.4 reveals that we have the following result.

2.6. Proposition. *For a positive integer n consider random variables $a_1, ..., a_n, b, c$ such that $\{a_1, ..., a_n\}$ and $\{b, c\}$ are free. Then we have*

$$\begin{aligned} &k_n(ba_1c, ba_2c, ..., ba_nc) \\ &\quad = \sum_{\pi \in NC(n)} k_\pi[a_1, ..., a_n]\, k_{|V_r|}(bc, bc, ..., bc) \\ &\quad\quad \times \prod_{i=1}^{r-1} k_{|V_i|}(\dashv c, bc, ..., bc, b \vdash), \end{aligned}$$

where, for $\pi \in NC(n)$, we have written $K(\pi) = \{V_1, ..., V_r\}$ such that V_r is the block of $K(\pi)$ containing the last element n. Thus the cumulant corresponding to the block of $K(\pi)$ containing n has only bc as entries, whereas all the other factors for $K(\pi)$ are of the form $k_m(\dashv c, bc, ..., bc, b \vdash)$, which is defined as

$$\begin{aligned} &k_m(\dashv c, b_1, ..., b_{m-1}, b \vdash) \\ &\quad := \sum_{\pi \in NC(m+1);\ \pi \vee \{(1, m+1), (2), (3), ..., (m)\} = 1_{m+1}} k_\pi[c, b_1, ..., b_{m-1}, b] \end{aligned}$$

for arbitrary random variables $c, b, b_1, ..., b_{m-1}$.

2.7. *Remarks.* (1) Note that the cumulant $k_m(\dashv c, b_1, ..., b_{m-1}, b \vdash)$ is a cumulant of order m; c and b are to be thought of as the factors of one argument. However, in the evaluation of the cumulant one has to take care of the positions of c and b. For example,

$$k_2(\dashv c, b_1, b \vdash) = \varphi(cb_1b) - \varphi(cb)\,\varphi(b_1).$$

(2) Proposition 2.6 suggests that one might consider also cumulants of the form

$$k^{\sigma}(a_1, ..., a_n) := \sum_{\pi \in NC(n);\, \pi \geqslant \sigma} \mu(\pi, 1_n)\, \varphi_{\pi}[a_1, ..., a_n] \tag{12}$$

for arbitrary $\sigma \in NC(n)$. Note that k^{σ} is not a product of cumulants like k_{π}, but a cumulant of order $|\sigma|$, where each block of σ corresponds to an argument given by multiplication of the corresponding variables a_i, but with respectation of the nested structure of the blocks. If σ is of the special form $\sigma = \{(1, ..., i_1), ..., (i_{m-1}+1, ..., i_m)\}$, as in Theorem 2.2, then k^{σ} is nothing but

$$k^{\{(1, ..., i_1), ..., (i_{m-1}+1, ..., i_m)\}}(a_1, ..., a_n) = k_m(a_1 \cdots a_{i_1}, ..., a_{i_{m-1}+1} \cdots a_{i_m}),$$

whereas $k_m(\dashv c, b_1, ..., b_{m-1}, b \vdash)$ from Proposition 2.6 reads now as

$$k_m(\dashv c, b_1, ..., b_{m-1}, b \vdash) = k^{\{(1, m+1), (2), (3), ..., (m)\}}(c, b_1, ..., b_{m-1}, b).$$

One should, however, note that the structure of the formula for k_{σ} in terms of moments does not only depend on $|\sigma|$, but on the concrete form of σ itself. For example, for $\sigma = \{(1, 3), (2), (4)\}$ we have

$$\begin{aligned} k^{\sigma}(a_1, b, a_2, c) = {} & \varphi(a_1 b a_2 c) - \varphi(a_1 b a_2)\, \varphi(c) \\ & - \varphi(a_1 a_2 c)\, \varphi(b) + \varphi(a_1 a_2)\, \varphi(b)\, \varphi(c), \end{aligned}$$

which should be compared with

$$\begin{aligned} k_3(a, b, c) = {} & \varphi(abc) - \varphi(ab)\, \varphi(c) - \varphi(ac)\, \varphi(b) - \varphi(a)\, \varphi(bc) \\ & + 2\varphi(a)\, \varphi(b)\, \varphi(c). \end{aligned}$$

One can generalize Theorem 2.2 for k^{σ} as follows: For $\sigma \in NC(n)$ and random variables $a_1, ..., a_n$ we have

$$k^{\sigma}(a_1, ..., a_n) = \sum_{\pi \in NC(n);\, \pi \vee \sigma = 1_n} k_{\pi}[a_1, ..., a_n]. \tag{13}$$

The proof of this statement goes along the same lines as our proof of Theorem 2.2. We will leave the details to the reader.

3. APPLICATIONS TO *R*-DIAGONAL ELEMENTS

3.1. *Notation* (*Alternating*). Let a be a random variable. A cumulant $k_{2r}(a_1, ..., a_{2r})$ with arguments from $\{a, a^*\}$ is said to have *alternating* *arguments*, if there does not exist any a_i $(1 \leqslant i \leqslant 2r-1)$ with $a_{i+1} = a_i$. We will also say that the cumulant k_{2r} $(a_1, ..., a_{2r})$ is *alternating*. Cumulants with an odd number of arguments will always be considered as not alternating.

EXAMPLE. The cumulant $k_6(a, a^*, a, a^*, a, a^*)$ is alternating, whereas $k_8(a, a^*, a^*, a, a, a^*, a, a^*)$ or $k_5(a, a^*, a, a^*, a)$ are not alternating.

3.2. DEFINITION (*R-diagonal*). A random variable a is called *R-diagonal* if for all $r \in \mathbf{N}$ we have that $k_r(a_1, ..., a_r) = 0$ whenever the arguments $a_1, ..., a_r \in \{a, a^*\}$ are not alternating in a and a^*.

3.3. DEFINITION (*Haar unitary*). We call an element u in a probability space $(\mathscr{A}, \varphi)$ *Haar unitary* if it has the following properties:

(1°) u is unitary, i.e., $uu^* = 1 = u^*u$.

(2°) $\varphi(u^k) = 0 = \varphi(u^{*k})$ for $k = 1, 2, 3, ...$.

3.4. *Remarks.* (1) Due to the relation (1) between moments and free cumulants, two tuples $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ of random variables have the same joint distribution if and only if all their cumulants are identical, i.e., if $k_m(a_{i(1)}, ..., a_{i(m)}) = k_m(b_{i(1)}, ..., b_{i(m)})$ for all $m \in \mathbf{N}$ and all $1 \leqslant i(1), ..., i(m) \leqslant n$. This implies, of course, that the property "R-diagonality" depends only on the $*$-distribution of a.

(2) It was proved in [16] that a Haar unitary is R-diagonal. Indeed, the examples of the Haar unitary and the circular element—which present the two most important non-selfadjoint distributions in free probability theory—provided the motivation for introducing the class of R-diagonal elements as a kind of interpolation between these two elements.

(3) It is clear that all information on the $*$-distribution of an R-diagonal element a is contained in the two sequences of its alternating cumulants $\alpha_n := k_{2n}(a, a^*, a, a^*, ..., a, a^*)$ and $\beta_n := k_{2n}(a^*, a, a^*, a, ..., a^*, a)$. Another useful description of the $*$-distribution of a is given by the distributions of aa^* and a^*a. The next proposition connects these two descriptions of the *-distribution of a. The tracial case—in which $\alpha_n = \beta_n$ for all n—was treated in [8], whereas the result in the general case proves a conjecture, Eq. (5.7), from [10].

3.5. PROPOSITION. *Let a be an R-diagonal random variable in a non-commutative probability space $(\mathscr{A}, \varphi)$. Let*

$$\alpha_n := k_{2n}(a, a^*, a, a^*, ..., a, a^*),$$

$$\beta_n := k_{2n}(a^*, a, a^*, a, ..., a^*, a)$$

be the non-vanishing cumulants of a. Then we have

$$k_n(aa^*, ..., aa^*) = \sum_{\pi \in NC(n);\, \pi = \{V_1, ..., V_r\}} \alpha_{|V_1|} \beta_{|V_2|} \cdots \beta_{|V_r|}, \tag{14}$$

where V_1 denotes that block of $\pi \in NC(n)$ which contains the first element 1.

Proof. Applying Theorem 2.2 in the particular form of Eq. (9) yields

$$k_n(aa^*, ..., aa^*) = \sum_{\pi \in NC(2n);\, \pi \vee \sigma = 1_{2n}} k_\pi[a, a^*, ..., a, a^*] \tag{15}$$

with

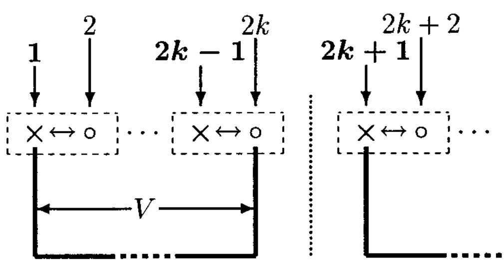
$$\sigma = \{(a, a^*), ..., (a, a^*)\} \triangleq \{(\mathbf{1}, 2), ..., (\mathbf{2n-1}, 2n)\}.$$

We claim now the following. The partitions π which fulfill the condition $\pi \vee \sigma = 1_{2n}$ are exactly those which have the following properties: the block of π which contains the element $\mathbf{1}$ contains also the element $2n$, and, for each $k = 1, ..., n-1$, the block of π which contains the element $2k$ contains also the element $\mathbf{2k+1}$.

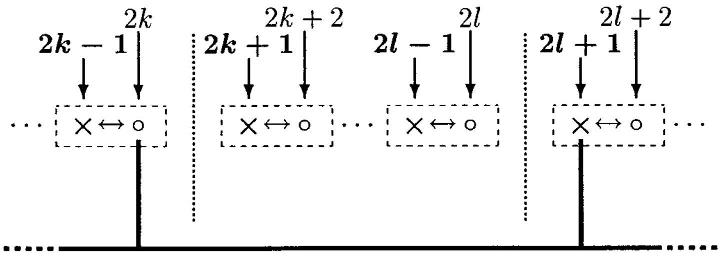
Since the set of those $\pi \in NC(2n)$ fulfilling the claimed condition is in canonical bijection with $NC(n)$ and since $k_\pi[a, a^*, ..., a, a^*]$ goes under this bijection to the product appearing in Eq. (14), this gives directly the assertion.

So it remains to prove the claim. It is clear that a partition which has the claimed property does also fulfill $\pi \vee \sigma = 1_{2n}$. So we only have to prove the other direction.

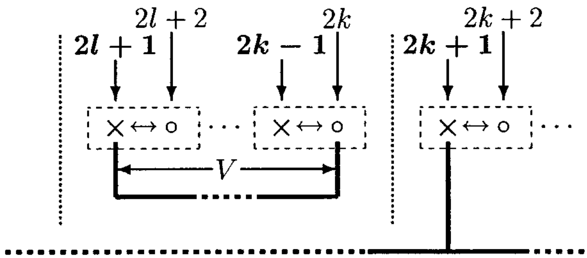
Let V be the block of π which contains the element $\mathbf{1}$. Since a is R-diagonal the last element of this block has to be an a^*, i.e., an even number, let us say $2k$. If this would not be $2n$ then this block V would in $\pi \vee \sigma$ not be connected to the block containing $\mathbf{2k+1}$, thus $\pi \vee \sigma$ would not give 1_{2n}. Hence $\pi \vee \sigma = 1_{2n}$ implies that the block containing the first element $\mathbf{1}$ contains also the last element $2n$.



Now fix a $k=1, ..., n-1$ and let V be the block of π containing the element $2k$. Assume that V does not contain the element $\mathbf{2k+1}$. Then there are two possibilities: Either $2k$ is not the last element in V, i.e. there exists a next element in V, which is necessarily of the form $\mathbf{2l+1}$ with $l>k$,



or $2k$ is the last element in V. In this case the first element of V is of the form $\mathbf{2l+1}$ with $0 \leqslant l \leqslant k-1$,



In both cases the block V gets not connected with $\mathbf{2k+1}$ in $\pi \vee \sigma$, thus this cannot give 1_{2n}. Hence the condition $\pi \vee \sigma = 1_{2n}$ forces $2k$ and $\mathbf{2k+1}$ to lie in the same block. This proves our claim and hence the assertion. ∎

We are now going to prove a fundamental characterization of R-diagonal elements as those random variables whose $*$-distribution remains invariant under the multiplication with a free Haar unitary. This theorem has been proven in [9] in the case when φ is a trace. The treatment there used some ad hoc combinatorics. In contrast to this, our approach here is more straightforward and conceptually clearer. Another proof of the general form of the theorem, relying on Fock space techniques, will appear in [10]. The main step in the proof of the theorem—the one in which we will use our combinatorial Theorem 2.2—is the following proposition. This appeared also, for the tracial case, in [8].

3.6. Proposition. *Let a and x be elements in a probability space $(\mathcal{A}, \varphi)$ with a being R-diagonal and such that $\{a, a^*\}$ and $\{x, x^*\}$ are free. Then ax is R-diagonal.*

Proof. We examine a cumulant $k_r(a_1a_2, ..., a_{2r-1}a_{2r})$ with $a_{2i-1}a_{2i} \in \{ax, x^*a^*\}$ for $i \in \{1, ..., r\}$.

According to the definition of R-diagonality we have to show that this cumulant vanishes in the following two cases:

(1°) r is odd.

(2°) There exists at least one s $(1 \leqslant s \leqslant r-1)$ such that $a_{2s-1}a_{2s} = a_{2s+1}a_{2s+2}$.

By Theorem 2.2, we have

$$k_r(a_1a_2, ..., a_{2r-1}a_{2r}) = \sum_{\pi \in NC(2r);\ \pi \vee \sigma = 1_{2r}} k_\pi[a_1, a_2, ..., a_{2r-1}, a_{2r}], \tag{16}$$

where $\sigma = \{(a_1, a_2), ..., (a_{2r-1}, a_{2r})\}$.

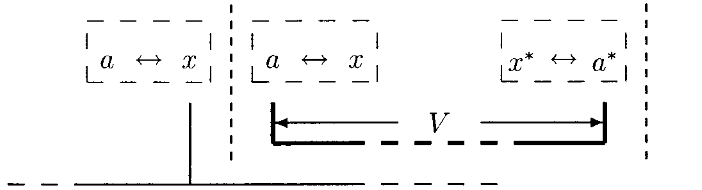
The fact that a and x are $*$-free implies, by Proposition 1.5, that only such partitions $\pi \in NC(2r)$ contribute to the sum each of whose blocks contains elements only from $\{a, a^*\}$ or only from $\{x, x^*\}$.

Case (1°). As there is at least one block of π containing a different number of elements a and a^*, k_π vanishes always. So there are no partitions π contributing to the sum in (16) which consequently vanishes.

Case (2°). We assume that there exists an $s \in \{1, ..., r-1\}$ such that $a_{2s-1}a_{2s} = a_{2s+1}a_{2s+2}$. Since with a also a^* is R-diagonal, it suffices to consider the case where $a_{2s-1}a_{2s} = a_{2s+1}a_{2s+2} = ax$, i.e., $a_{2s-1} = a_{2s+1} = a$ and $a_{2s} = a_{2s+2} = x$.

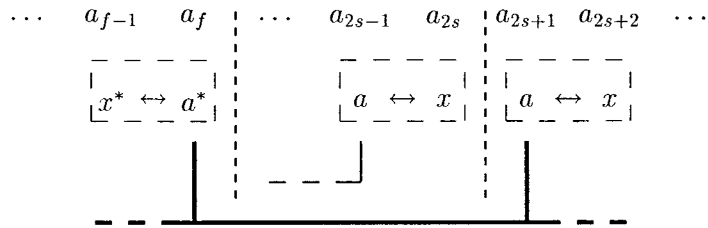
Let V be the block containing a_{2s+1}. We have to examine two situations:

(A) On the one hand, it might happen that a_{2s+1} is the first element in the block V. This can be sketched in the following way:



In this case the block V is not connected with a_{2s} in $\pi \vee \sigma$, thus the latter cannot be equal to 1_{2n}.

(B) On the other hand, it can happen that a_{2s+1} is not the first element of V. Because a is R-diagonal, the preceding element must be an a^*,



But then V will again not be connected to a_{2s} in $\pi \vee \sigma$. Thus again $\pi \vee \sigma$ cannot be equal to 1_{2n}.

As in both cases we do not find any partition contributing to the investigated sum in Eq. (16) this has to vanish. ∎

3.7. Theorem. *Let x be an element in a non-commutative probability space $(\mathscr{A}, \varphi)$. Furthermore, let u be a Haar unitary in $(\mathscr{A}, \varphi)$ such that $\{u, u^*\}$ and $\{x, x^*\}$ are free. Then x is R-diagonal if and only if (x, x^*) has the same joint distribution as (ux, x^*u^*):*

$$x \text{ } R\text{-diagonal} \Leftrightarrow \mu_{x, x^*} = \mu_{ux, x^*u^*}.$$

Proof. $(\Rightarrow)$ In order to show that the joint distributions of (x, x^*) and (ux, x^*u^*) are identical, we have to prove according to the Remark 3.4(1) that $k_m(b_1, ..., b_m) = k_m(c_1, ..., c_m)$ for all $m \in \mathbf{N}$, $b_i \in \{x, x^*\}$ and

$$c_i = \begin{cases} ux & \text{for} \quad b_i = x \\ x^*u^* & \text{for} \quad b_i = x^*. \end{cases}$$

In the cases when m is odd or when with even m the elements $b_1, ..., b_m$ do not alternate, the cumulant $k_m(b_1, ..., b_m)$ vanishes because of the R-diagonality of x. By Proposition 3.6 and the fact that u is R-diagonal, we get that ux is R-diagonal, too, and therefore $k_m(c_1, ..., c_m)$ also vanishes.

Hence we have to consider the case where the arguments $b_1, ..., b_m$ alternate (which implies alternating arguments $c_1, ..., c_m$).

We inductively show the validity of

$$k_{2r}(x, x^*, ..., x, x^*) = k_{2r}(ux, x^*u^*, ..., ux, x^*u^*)$$

and

$$k_{2r}(x^*, x, ..., x^*, x) = k_{2r}(x^*u^*, ux, ..., x^*u^*, ux)$$

for any natural r.

First, consider $r = 1$. On one hand, the equation

$$k_2(ux, x^*u^*) = \varphi(uxx^*u^*) - k_1(ux)\, k_1(x^*u^*)$$

holds by definition of k_2. With both cumulants $k_1(ux)$ and $k_1(x^*u^*)$ vanishing because of the R-diagonality of ux the second term of the sum is equal to zero.

Since $\{u, u^*\}$ and $\{x, x^*\}$ are assumed to be free, we can write the moment with the help of formula (5) as

$$\varphi(uxx^*u^*) = \varphi(uu^*)\,\varphi(xx^*) = \varphi(xx^*).$$

So we get $k_2(ux, x^*u^*) = \varphi(xx^*)$.

On the other hand, with x being R-diagonal we obtain

$$k_2(x, x^*) = \varphi(xx^*) - k_1(x)\,k_1(x^*) = \varphi(xx^*) = k_2(ux, x^*u^*).$$

Induction Hypothesis. Assume the following to be true for any $r' < r$ $(r \geqslant 2)$,

$$k_{2r'}(x, x^*, ..., x, x^*) = k_{2r'}(ux, x^*u^*, ..., ux, x^*u^*)$$
$$k_{2r'}(x^*, x, ..., x^*, x) = k_{2r'}(x^*u^*, ux, ..., x^*u^*, ux).$$

We have to show the validity of these equations for $r' = r$. It suffices to consider the first equation.

According to definition of the free cumulants we have

$$k_{2r}(ux, x^*u^*, ..., ux, x^*u^*) = \varphi(uxx^*u^* \cdots uxx^*u^*) - \sum_{\pi \in NC(2r);\, \pi \neq 1_{2r}} k_\pi[ux, x^*u^*, ..., ux, x^*u^*].$$

Because of the freeness of $\{u, u^*\}$ and $\{x, x^*\}$ and with the help of (5) we get

$$\varphi(uxx^*u^*ux \cdots x^*u^*uxx^*u^*) = \varphi(u[xx^*]^r\, u^*) = \varphi(uu^*)\,\varphi([xx^*]^r) = \varphi([xx^*]^r).$$

It follows that

$$k_{2r}(ux, x^*u^*, ..., ux, x^*u^*) = \varphi([xx^*]^r) - \sum_{\pi \in NC(2r);\, \pi \neq 1_{2r}} k_\pi[ux, x^*u^*, ..., ux, x^*u^*].$$

The only partitions $\pi \in NC(2r)$, $\pi \neq 1_{2r}$ contributing in the foregoing sum are those where all blocks are alternating in ux and x^*u^*. According to our induction hypothesis, we can then replace in all blocks the element ux by x and the element x^*u^* by x^*. So we finally obtain

$$k_{2r}(ux, x^*u^*, ..., ux, x^*u^*) = k_{2r}(x, x^*, ..., x, x^*).$$

($\Leftarrow$) We assume that $\mu_{x, x^*} = \mu_{ux, x^*u^*}$. As, by Proposition 3.6, ux is R-diagonal, x is R-diagonal, too. ∎

3.8. *Remark.* Proposition 3.6 implies in particular that the product of two free R-diagonal elements is R-diagonal again. This raises the question how the alternating cumulants of the product are given in terms of the alternating cumulants of the factors. This is answered in the next proposition. In the tracial case this reproduces a result of [8], whereas in the general case this proves the conjecture (5.8) from [10].

3.9. PROPOSITION. *Let a and b be R-diagonal random variables such that $\{a, a^*\}$ is free from $\{b, b^*\}$. Furthermore, put*

$$\alpha_n := k_{2n}(a, a^*, a, a^*, ..., a, a^*),$$

$$\beta_n := k_{2n}(a^*, a, a^*, a, ..., a^*, a),$$

$$\gamma_n := k_{2n}(b, b^*, b, b^*, ..., b, b^*).$$

Then ab is R-diagonal and the alternating cumulants of ab are given by

$$k_{2n}(ab, b^*a^*, ..., ab, b^*a^*)$$

$$= \sum_{\substack{\pi = \pi_a \cup \pi_b \in NC(2n) \\ \pi_a = \{V_1, ..., V_k\} \in NC(1, 3, ..., 2n-1) \\ \pi_b = \{V'_1, ..., V'_l\} \in NC(2, 4, ..., 2n)}} \alpha_{|V_1|}\, \beta_{|V_2|} \cdots \beta_{|V_k|}\, \gamma_{|V'_1|} \cdots \gamma_{|V'_l|}, \tag{17}$$

where V_1 is that block of π which contains the first element 1.

Proof. R-diagonality of ab is clear by Proposition 3.6. So we only have to prove Eq. (17).

By Theorem 2.2, we get

$$k_{2n}(ab, b^*a^*, ..., ab, b^*a^*)$$

$$= \sum_{\pi \in NC(4n);\, \pi \vee \sigma = 1_{4n}} k_\pi[a, b, b^*, a^*, ..., a, b, b^*, a^*], \tag{18}$$

where $\sigma = \{(a, b), (b^*, a^*), ..., (a, b), (b^*, a^*)\}$. Since $\{a, a^*\}$ and $\{b, b^*\}$ are assumed to be free, we also know, by Proposition 1.5, that for a contributing partition π each block has to contain components only from $\{a, a^*\}$ or only from $\{b, b^*\}$.

As in the proof of Proposition 3.5 one can show that the requirement $\pi \vee \sigma = 1_{4n}$ is equivalent to the following properties of π: The block containing 1 must also contain $4n$ and, for each $k = 1, ..., 2n-1$, the block containing $2k$ must also contain $2k+1$. (This couples always b with b^* and

a^* with a, so it is compatible with the $*$-freeness between a and b.) The set of partitions in $NC(4n)$ fulfilling these properties is in canonical bijection with $NC(2n)$. Furthermore we have to take care of the fact that each block of $\pi \in NC(4n)$ contains either only elements from $\{a, a^*\}$ or only elements from $\{b, b^*\}$. For the image of π in $NC(2n)$ this means that it splits into blocks living on the odd numbers and blocks living on the even numbers. Furthermore, under these identifications the quantity $k_\pi[a, b, b^*, a^*, ..., a, b, b^*, a^*]$ goes over to the expression as appearing in our assertion (17). ∎

3.10. *Remark.* According to Proposition 3.6 multiplication preserves R-diagonality if the factors are free. Haagerup and Larsen [2, 5] showed that, in the tracial case, the same statement is also true for the other extreme relation between the factors, namely if they are the same; i.e., powers of R-diagonal elements are also R-diagonal. The proof of Haagerup and Larsen relied on special realizations of R-diagonal elements. Here we will give a short combinatorial proof of that statement. In particular, our proof will—in comparison with the proof of Proposition 3.6—also illuminate the relation between the statements "$a_1, ..., a_r$ R-diagonal and free implies $a_1 \cdots a_r$ R-diagonal" and "a R-diagonal implies a^r R-diagonal." Furthermore, in contrast to the approach of [2, 5], our proof extends without problems to the non-tracial situation.

3.11. Proposition. *Let a be an R-diagonal element and let r be a positive integer. Then a^r is R-diagonal, too.*

Proof. For notational convenience we deal with the case $r = 3$. General r can be treated analogously.

The cumulants which we must have a look at are $k_n(b_1, ..., b_n)$ with arguments b_i from $\{a^3, (a^3)^*\}$ $(i = 1, ..., n)$. We write $b_i = b_{i,1} b_{i,2} b_{i,3}$ with $b_{i,1} = b_{i,2} = b_{i,3} \in \{a, a^*\}$. According to the definition of R-diagonality we have to show that for any $n \geqslant 1$ the cumulant $k_n(b_{1,1} b_{1,2} b_{1,3}, ..., b_{n,1} b_{n,2} b_{n,3})$ vanishes if (at least) one of the following things happens:

(1°) There exists an $s \in \{1, ..., n-1\}$ with $b_s = b_{s+1}$.

(2°) n is odd.

Theorem 2.2 yields

$$k_n(b_{1,1} b_{1,2} b_{1,3}, ..., b_{n,1} b_{n,2} b_{n,3}) = \sum_{\pi \in NC(3n);\, \pi \vee \sigma = 1_{3n}} k_\pi[b_{1,1}, b_{1,2}, b_{1,3}, ..., b_{n,1}, b_{n,2}, b_{n,3}],$$

where $\sigma := \{(b_{1,1}, b_{1,2}, b_{1,3}), ..., (b_{n,1}, b_{n,2}, b_{n,3})\}$. The R-diagonality of a implies that a partition π gives a non-vanishing contribution to the sum only if its blocks link the arguments alternatingly in a and a^*.

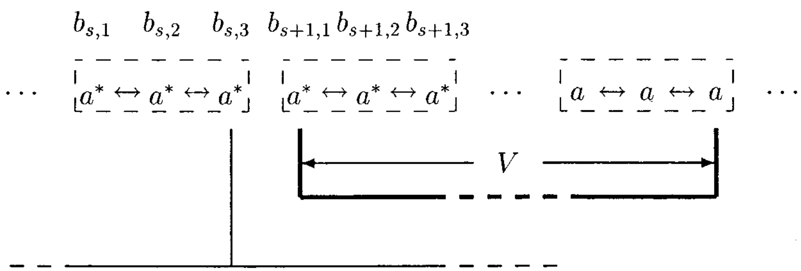
Case (1°). Without loss of generality, we consider the cumulant $k_n(..., b_s, b_{s+1}, ...)$ with $b_s = b_{s+1} = (a^3)^*$ for some s with $1 \leqslant s \leqslant n-1$. This means that we have to look at $k_n(..., a^*a^*a^*, a^*a^*a^*, ...)$. Theorem 2.2 yields in this case

$$k_n(..., a^*a^*a^*, a^*a^*a^*, ...)$$

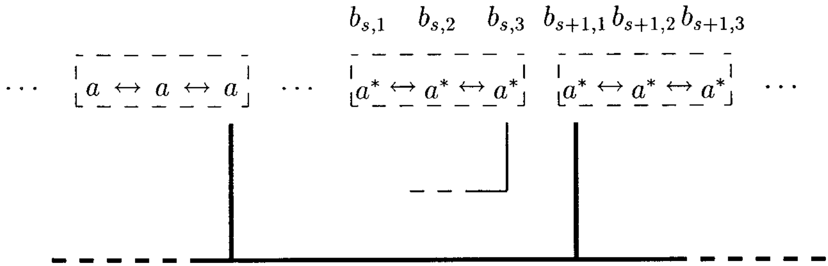
$$= \sum_{\pi \in NC(3n);\, \pi \vee \sigma = 1_{3n}} k_\pi[..., a^*, a^*, a^*, a^*, a^*, a^*, ...],$$

where $\sigma := \{..., (a^*, a^*, a^*), (a^*, a^*, a^*), ...\}$. In order to find out which partitions $\pi \in NC(3n)$ contribute to the sum we look at the structure of the block containing the element $b_{s+1,1} = a^*$; in the following we will call this block V.

There are two situations which can occur. The first possibility is that $b_{s+1,1}$ is the first component of V; in this case the last component of V must be an a and, since each block has to contain the same number of a and a^*, this a has to be the third a of an argument a^3. But then the block V gets in $\pi \vee \sigma$ not connected with the block containing $b_{s,3}$ and hence the requirement $\pi \vee \sigma = 1_{3n}$ cannot be fulfilled in such a situation,



The second situation that might happen is that $b_{s+1,1}$ is not the first component of V. Then the preceding element in this block must be an a and again it must be the third a of an argument a^3. But then the block containing $b_{s,3}$ is again not connected with V in $\pi \vee \sigma$. This possibility can be illustrated as



Thus, in any case there exists no π which fulfills the requirement $\pi \vee \sigma = 1_{3n}$ and hence $k_n(..., a^*a^*a^*, a^*a^*a^*, ...)$ vanishes in this case.

Case (2°). In the case n odd, the cumulant $k_\pi[b_{1,1}, b_{1,2}, b_{1,3}, ..., b_{n,1}, b_{n,2}, b_{n,3}]$ has a different number of a and a^* as arguments and hence at least one of the blocks of π cannot be alternating in a and a^*. Thus k_π vanishes by the R-diagonality of a.

As in both cases we do not find any partition giving a non-vanishing contribution, the sum vanishes and so do the cumulants $k_n(b_1, ..., b_n)$. ∎

3.12. *Remark.* Of course we are now left with the problem of describing the alternating cumulants of a^r in terms of the alternating cumulants of a. We will provide the solution to this question by showing that the similarity between $a_1 \cdots a_r$ and a^r goes even further as in the Remark 3.10. Namely, we will show that a^r has the same $*$-distribution as $a_1 \cdots a_r$ if all a_i $(i = 1, ..., r)$ have the same $*$-distribution as a. The distribution of a^r can then be calculated by an iteration of Proposition 3.9. In the case of a trace this reduces to a result of Haagerup and Larsen [2, 5]. The specical case of powers of a circular element was treated by Oravecz [11].

3.13. Proposition. *Let a be an R-diagonal element and r a positive integer. Then the $*$-distribution of a^r is the same as the $*$-distribution of $a_1 \cdots a_r$ where each a_i $(i = 1, ..., r)$ has the same $*$-distribution as a and where $a_1, ..., a_r$ are $*$-free.*

Proof. Since we know that both a^r and $a_1 \cdots a_r$ are R-diagonal we only have to see that the respective alternating cumulants coincide. By Theorem 2.2, we have

$$k_{2n}(a^r, a^{*r}, ..., a^r, a^{*r}) = \sum_{\pi \in NC(2nr);\, \pi \vee \sigma = 1_{2nr}} k_\pi[a, ..., a, a^*, ..., a^*, ..., a, ..., a, a^*, ..., a^*]$$

and

$$k_{2n}(a_1 \cdots a_r, a_r^* \cdots a_1^*, ..., a_1 \cdots a_r, a_r^* \cdots a_1^*) = \sum_{\pi \in NC(2nr);\, \pi \vee \sigma = 1_{2nr}} k_\pi[a_1, ..., a_r, a_r^*, ..., a_1^*, ..., a_1, ..., a_r, a_r^*, ..., a_1^*],$$

where in both cases $\sigma = \{(1, ..., r), (r+1, ..., 2r), ..., (2(n-1)r+1, ..., 2nr)\}$. The only difference between both cases is that in the second case we also have to take care of the freeness between the a_i which implies that only such π contribute which do not connect different a_i. But the R-diagonality of a implies that also in the first case only such π give a non-vanishing contribution, i.e., the freeness in the second case does not really give an extra condition. Thus both formulas give the same and the two distributions coincide. ∎

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