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Solution of an Enumerative Problem Connected with Lattice Paths

G. KREWERAS AND H. NIEDERHAUSEN

In the integer plane $\mathbb{Z}\times\mathbb{Z}$, we call *horizontal step* a step from some (x, y) to $(x+1, y)$ and *vertical step* a step from some (x, y) to $(x, y-1)$, and we consider the *paths* starting from $(0, q)$ and reaching $(p, 0)$ with p horizontal and q vertical steps. It is clear that these paths form a set $\mathscr{C}$ of cardinality $(p+q)!/p!q!$, which may be (partially) ordered by "dominance": a path C dominates C' if C' lies entirely between C and the path along the coordinate axes ("between" is meant in the broad sense).

Let $w(C)$ be the number of distinct paths dominated by C. Then it has been proved by one of the authors [1] that

$$\sum_{C\in\mathscr{C}} w(C)=\frac{(p+q)!(p+q+1)!}{p!(p+1)!q!(q+1)!} \tag{1}$$

(this result is in fact a particular case of a more general result appearing in [2, vol. 2, p. 242]).

The aim of the present paper is to prove a formula that has been a conjecture for several years, namely

$$\sum_{C\in\mathscr{C}} [w(C)]^2=\frac{(p+q+1)!(2p+2q+1)!}{(p+1)!(2p+1)!(q+1)!(2q+1)!}. \tag{2}$$

(It is obvious by symmetry that the words "dominated by" in the definition of $w(C)$ could be replaced by "dominating" without change of validity of (1) or (2).)

In the sequel we shall call $[a, b]$ the set of integers z such that $a\leqslant z\leqslant b$, and note $|S|$ the cardinality of any set S.

Since each path in $\mathscr{C}$ can be defined by the sequence of the p ordinates of its horizontal steps, it is clear that the first member of (2) is also equal to $|E(p, q)|$, where the general element of $E(p, q)$ is defined as a sequence of p triples (u_i, v_i, w_i) having the following properties:

(i) $u_i\in[0, p]$, $v_i\in[0, p]$, $w_i\in[0, p]$, $\quad i\in[1, q]$

(ii) $u_i\leqslant v_i$ and $u_i\leqslant w_i$, $\quad i\in[1, q]$

(iii) $u_i\leqslant u_{i+1}$, $v_i\leqslant v_{i+1}$, $w_i\leqslant w_{i+1}$, $\quad i\in[1, q-1]$.

Another way to describe $E(p, q)$ is to consider $e\in E(p, q)$ as a *sequence* of $3q$ integers, $e=(u_1, \ldots, u_q, v_1, \ldots, v_q, w_1, \ldots, w_q)$, which meet conditions (i), (ii), (iii).
The following remarks are obvious:

(A) $|E(p, q)|=|E(q, p)|$ (symmetric roles of the coordinates),

(B) $|E(0, q)|=|E(p, 0)|=1$ (the only element of $E(0, q)$ is the sequence of $3q$ zeros, while the only element of $E(p, 0)$ is the "empty sequence").

In order to formulate two further essential remarks about $E(p, q)$, we shall also have to consider a set $F(q)$, which is by definition the part of $E(3q, q)$ consisting of *permutations* of

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$[1, 3q]$. It was proved in [1] (unfortunately by a far-fetched method) that

$$|F(q)| = \frac{2^{2q}(3q)!}{(q+1)!(2q+1)!}. \tag{3}$$

With each sequence $e \in E(p, q)$ a permutation $f \in F(q)$ can be associated in the following way:
number from left to right, with $3q$ increasing integers starting from 1, first of all the 0's of e (if there are any), then all the 1's (if there are any), and so on, finishing by the p's (if there are any). The permutation f obtained from e will be called the *numbering* of e; each of the $3q$ terms of e will have its own *number*.

We can now define, for any $e \in E(p, q)$ with $p \geq 1$, an integer $\omega(e)$ which we shall call its *depth*: the depth of e is by definition the smallest k such that we obtain an element $e' \in E(p-1, q)$ with the same numbering as e, if we decrease by one the k terms in e with largest numbers. $\omega(e) = 0$ if and only if there are no p's in e. If there are p's, the definition of numbering implies that we can list them by decreasing numbers. Then we list the $(p-1)$'s, if there is one at the right of the leftmost p, and so on. We stop as soon as we can find no $i-1$ at the right of the leftmost i; the depth is then the number of listed terms provided the last of them is ≥ 1. One possible case is that all the $3q$ terms are listed but the last of them (e_1) is 0; in this case there is no acceptable e' and we say, by natural convention, that the depth is $3q+1$.

The concept of depth can be illustrated by the following example with $q = 5$, in which all the sequences correspond to the same numbering.

$$
\begin{array}{lllllllllllllll}
f=(1 & 2 & 5 & 6 & 8 & 3 & 4 & 11 & 12 & 13 & 7 & 9 & 10 & 14 & 15)
\end{array}
$$

$$
\begin{array}{llllllllllllllll}
(0 & 3 & 4 & 4 & 5 & 3 & 3 & 7 & 7 & 7 & 4 & 6: & 6 & 7 & 7) & \in E(7,5), \text{depth } 7\\
(0 & 3 & 4 & 4 & 5 & 3 & 3 & 6 & 6 & 6 & 4 & 5 & 5 & 6 & 6) & \in E(6,5), \text{depth } 14\\
(0 & 2 & 3 & 3 & 4 & 2 & 2 & 5 & 5 & 5 & 3 & 4 & 4 & 5 & 5) & \in E(5,5), \text{depth } 14\\
(0 & 1 & 2 & 2 & 3 & 1 & 1 & 4 & 4 & 4 & 2 & 3 & 3 & 4 & 4) & \in E(4,5), \text{depth } 14\\
(0 & 0 & 1 & 1 & 2 & 0 & 0 & 3 & 3 & 3 & 1 & 2 & 2 & 3 & 3) & \in E(3,5), \text{depth } 16
\end{array}
$$

A fundamental remark about depths is that for any $p \geq 1$ and $\omega \in [0, 3q]$ there are exactly as many elements in $E(p, q)$ with depth ω as in $E(p-1, q)$ with depth $\geq \omega$.

The reason is that from any element e' with depth $\geq \omega$ belonging to $E(p-1, q)$ we can form an element e of $E(p, q)$ by increasing by 1 the ω terms of e' with largest numbers; the depth of this e in $E(p, q)$ is then exactly ω and the numbering remains the same. It is easy to check that this correspondence is bijective. Thus, if we call $d_p^q(k)$ the number of elements in $E(p, q)$ with depth $\geq k$, we can write

(C) $\quad d_p^q(k) - d_p^q(k+1) = d_{p-1}^q(k) \qquad \forall k \in [0, 3q].$

A further important property of the depths will be expressed by the following equality:

(D) $\quad d_p^q(3q+1) = d_{2q-1-p}^q(3q+1).$

To justify (D), we have to remember that a sequence $e \in E(p, q)$ is of depth $3q+1$ if and only if, for any $i \in [1, p]$, the last $i-1$ appears in e after the first i. The corresponding numbering f of e has then exactly p *switchbacks* (i.e. opportunities in which some number k stands before $k-1$), since the number of the first i follows immediately the number of the last $i-1$. Furthermore, if a permutation with p switchbacks is given in $F(q)$, the corresponding e is uniquely determined in $E(p, q)$. Thus(D) will be proved if we can prove that there are, in $F(q)$, exactly as many permutations with p switchbacks as with $2q-1-p$ switchbacks.

The latter fact is a consequence of some general properties about finite partially ordered sets ("posets"), such as investigated by R. Stanley in [3]. For a poset M of m elements, there exists a set L of "linear extensions", i.e. of total orders on M which are *consistent with* the partial order. If a particular extension $a \in L$ is considered, the elements of M can then be re-labeled from 1 to m in such a way that a corresponds to the natural (increasing) order of the labels.

After this labeling, any linear extension of M may be viewed as a permutation of $[1, m]$, which has a certain number p of descents (i.e. opportunities in which some label i stands before any smaller label); p belongs always to $[0, m-h]$, and a sequence $l_0, l_1, \ldots, l_{m-h}$ is defined, in which l_p is the number of linear extensions with p descents ($l_0 + l_1 + \cdots + l_{m-h} = |L|$).

What Stanley's results imply is (of course with quite different notations):

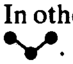
(1) that the sequence l_p is independent of the particular choice of a in L [3, p. 43],
(2) that if *all* the maximal chains of M consist *exactly* of h elements, the sequence l_p has the property $l_p = l_{m-h-p}$ (internal symmetry) [3, p. 71].

As M we may now take the following set M_q of $m = 3q$ points in $\mathbb{N}^3$:

$$(0, 0, z), (1, 0, z), (0, 1, z), \qquad \text{with } z \in [0, q-1],$$

the partial order being defined by

$$[(x', y', z') \leq (x, y, z)] \Leftrightarrow [x' \leq x, y' \leq y, z' \leq z].$$

In other words, M_q is the direct product of a q-element chain with the three-element poset .

A particular $a \in L$ is the linear extension

$$(0,0,0), (0,0,1), \ldots, (0,0,q-1), (1,0,0), (1,0,1), \ldots, (1,0,q-1),$$
$$(0,1,0), (0,1,1), \ldots, (0,1,q-1).$$

After M_q is labeled corresponding to a, take any $b \in L$, now written as a permutation. Let $i(k)$ be the index of k in b, i.e. $b_{i(k)} = k$. Define $f(b) = (i(1), \ldots, i(3q))$. Then $f(b) \in F(q)$, and the number of descents in b equals the number of switchbacks in $f(b)$. But this correspondence between L and $F(q)$ is one-to-one; hence, (D) is proved if we can show that $l_p = l_{2q-1-p}$. This is true because all maximal chains of M_q start at $(0, 0, 0)$ and reach either $(1, 0, q-1)$ or $(0, 1, q-1)$, i.e. consist of exactly $h = q+1$ points.

(A) and (B) can of course be reformulated:

(A') $d_p^q(0) = d_q^p(0)$,

(B') $d_0^q(k) = 1 \quad \forall k \in [0, 3q+1]$ (since the only element of $E(0, q)$ is of depth $3q+1$).

It appears from (B') and (C), by recursion with respect to p, that the expression of $d_p^q(k)$ *as a function of* k will be a polynomial of degree p. Let us, for any integer $q \geq 1$, call $D_p^q(t)$ the corresponding polynomial of a variable $t \in \mathbb{Z}$; we have then the following four conditions:

(A'') $D_p^q(0) = D_q^p(0)$ for any integer $p \geq 1$,

(B'') $D_0^q(t) \equiv 1$,

(C'') $\Delta D_p^q(t) = -D_{p-1}^q(t)$ for any integer $p \geq 1$,

(D'') $D_p^q(3q+1) = D_{2q-1-p}^q(3q+1)$ ($D_n^q(t) = 0$ for any negative n, consistently with (B'') and (C'')).

This family of polynomials $D_p^q(t)$ is *unique*. To prove it, we first notice from (C″) that $D_n^q(t)$ is determined if $D_{n-1}^q(t)$ is given and one single initial value of $D_n^q(t)$ is prescribed. Of course, D_0^q is always obtained from (B″). Now assume that D_j^i is known for all $i = 1, \ldots, q-1$ and for all $j \in \mathbb{N}$, and that D_i^q is known for all $i = 0, \ldots, p-1$. If $p < q$, we get the initial value for D_p^q from (A″), and we get it from (D″) otherwise.

In order to prove (2) finally, we have to prove that the value of both sides of (A), or (A′), or (A″), is precisely equal to the right-hand side of (2), i.e. equal to

$$\frac{(p+q+1)!(2p+2q+1)!}{(p+1)!(2p+1)!(q+1)!(2q+1)!} = K(p, q) \text{ (by definition).}$$

The table below gives the first values of $K(p, q)$, which happen to be positive integers:

$p \backslash q$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	5	14	30	55	
2	1	14	84	330		
3	1	30	330			
4	1	55				
5	1					

Our task will be done by exhibiting a family of polynomials which meets the requirements (A″) to (D″), and thus coincides with $D_p^q(t)$, and by checking that $D_p^q(0) = K(p, q)$.

We begin by introducing a sequence of polynomials of a variable $n \in \mathbb{Z}$,

$$K_0(n), K_1(n), K_2(n), \ldots, K_q(n), \ldots,$$

defined by

$$K_q(n) = \frac{(n+q+1)_q(2n+2q+1)_{2q}}{(q+1)!(2q+1)!} \quad \text{(Vandermonde notation),}$$

where $(X)_q = X(X-1) \cdot \ldots \cdot (X-q+1)$ and $(X)_0 = 1$. $K_q(n)$ has following properties:

(a) $K_q(n)$ is of degree $3q$;

(b) if $n \in \mathbb{N}$, $K_q(n) = K(n, q) = K_n(q)$;

(c) if $q \geq 1$, $K_q(n)$ vanishes for $n \in [-q-1, -1]$;

(d) $K_q(-n-q-2) = (-1)^q K_q(n)$;

(e) $K_q(n)$ is integer-valued.

(a), (b), (c) and (d) are either obvious or easy to check; (e) can be proved by recursion with respect to q, because any polynomial of degree $3q$ of $n \in \mathbb{Z}$ is integer-valued provided it has $3q+1$ consecutive integer values. $K_0(n) = 1$,

$$\text{and } K_1(n) = \frac{(n+2)(n+1)(2n+3)}{6} \quad \text{is integer-valued since,}$$

for $n \geq 0$, it is known to be the sum of the squares of $1, 2, \ldots, n+1$. Once we know that $K_0(n), K_1(n), \ldots, K_{q-1}(n)$ are integer-valued, we can note that $K_q(n)$ is an integer for $0 \leq n \leq q-1$ as a consequence of (b), for $-(2q+1) \leq n \leq -(q+2)$ as a consequence of (d), and for $-(q+1) \leq n \leq -1$ as a consequence of (c), which yields $3q+1$ consecutive integer values and completes the proof of (e). In particular, all the numbers $K(p, q)$ are positive integers, which was not obvious from the outset.

(Note that what is called here $K_q(n)$ is, in Stanley's terminology [3], the *order polynomial* $\Omega(M_q(n)$ of the poset M_q.)

Let us now define, for $q \geqslant 1$, a new set of integers $\alpha_0^q, \alpha_1^q, \ldots, \alpha_i^q, \ldots, \alpha_{2q-1}^q$, by

$$\alpha_n^q = K_q(n) - \binom{3q+1}{1} K_q(n-1) + \cdots + (-1)^n \binom{3q+1}{n} K_q(0). \qquad (4)$$

By (a), the $(3q+1)$th difference of $K_q(n)$ is identically 0. This remark, combined with (d) and the definition of α_n^q, shows the internal symmetry $\alpha_n^q = \alpha_{2q-1-n}^q$. It is also natural to set $\alpha_n^q = 0$ if $n \notin [0, 2q-1]$.

Furthermore, (4) can be inverted by writing

$$K_q(n) = \alpha_n^q + \binom{3q+1}{1} \alpha_{n-1}^q + \binom{3q+2}{2} \alpha_{n-2}^q + \cdots + \binom{3q+n}{n} \alpha_0^q; \qquad (5)$$

the inversion offers no difficulty by whatever method.

We can now assert that

$$D_p^q(t) = \sum_{n=0}^{p} (-1)^{p-n} \alpha_n^q . \binom{t-3q-1}{p-n}, \qquad (6)$$

which will be done by proving that the conditions (A″)–(D″) are met.

(A″): $D_p^q(0)$ is obtained by setting $t = 0$ in each term of the above sum. But

$$(-1)^{p-n} \binom{-3q-1}{p-n} = \binom{3q+p-n}{p-n}.$$

Thus

$$D_p^q(0) = \sum_{n=0}^{p} \binom{3q+p-n}{p-n} \alpha_n^q = K_q(p)$$

from (5), which proves (A″) since (b) is true.

(B″): Obvious.

(C″): $\Delta \binom{t-3q-1}{p-n} = \binom{t-3q-1}{p-n-1}$,

which, by the linear combination in (6), yields $-D_{p-1}^q(t)$.

(D″): For $t = 3q+1$, the only non-zero binomial on the right-hand side of (6) is obtained for $n = p$; hence

$$D_p^q(3q+1) = \alpha_p^q \qquad (7)$$

and (D″) results from $\alpha_p^q = \alpha_{2q-1-p}^q$.

This completes the proof of (2).

The above proof has several interesting by-products. First, (7) indicates that α_p^q is nothing else but the number of linear extensions of M_q with exactly p switchbacks. A table is given below.

$q=1$				1	1			
2			1	7	7	1		
3		1	20	75	75	20	1	
4	1	42	364	1001	1001	364	42	1
$p=0$	1	2	3	4	5	6	7	

These numbers appear in [1] with different notations, essentially as intermediate analytical tools to prove (3), without indication of their enumerative meaning.

Another by-product is to make it possible to prove (3) in a much simpler (although slightly less general) way than in [1]. Indeed $|F(q)|$ is the *total number* of linear extensions of M_q:

$$\text{i.e.}\quad |F(q)|=\alpha_0^q+\alpha_1^q+\cdots+\alpha_{2q-1}^q.$$

Applying (4), we easily find that this sum is equal to

$$K_q(2q-1)-\binom{3q}{1}K_q(2q-2)+\binom{3q}{2}K_q(2q-3)-\cdots-\binom{3q}{2q-1}K_q(0),$$

which we can complete formally by $q+1$ other terms with value 0, namely

$$\binom{3q}{2q}K_q(-1)-\binom{3q}{2q+1}K_q(-2)+\cdots+(-1)^qK_q(-q-1).$$

Consequently $|F(q)|$ is equal to the $(3q)$th difference of $K_q(n)$, which is a constant coming from the term of highest degree $(3q)$ of $K_q(n)$. This term is

$$\frac{n^q(2n)^{2q}}{(q+1)!(2q+1)!}=\frac{2^{2q}n^{3q}}{(q+1)!(2q+1)!}$$

and the $(3q)$th difference is seen to be

$$\frac{2^{2q}.(3q)!}{(q+1)!(2q+1)!};$$

hence (3).

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G. Kreweras
Institut de Statistique de l'Université de Paris,
9 quai St. Bernard, 75005 Paris, France

H. Niederhausen
Department of Mathematics, Arizona State University,
Tempe, Arizona 85281, U.S.A.