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A Note on Spectral Approximation of Linear Operations

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Abstract—This work deals on sufficient conditions for the spectral convergence of a sequence of linear operators. The general context is a complex separable Banach space and the pointwise limit of the sequence is a continuous linear operator which is not supposed to be compact. By spectral convergence is meant the self-range-uniform convergence of the approximate spectral projections. This implies the gap convergence of the approximate maximal invariant subspaces to those of the limit operator corresponding to a nonzero isolated eigenvalue (or a subset of close nonzero isolated eigenvalues) with finite algebraic multiplicity. Neither the exact nor the approximate eigenvalues are supposed to be semisimple.

Keywords—Eigenvalues, Maximal invariant subspaces, Gap, Spectral convergence.

1. INTRODUCTION

The basic problems to be studied in this chapter are:

Under which conditions on a sequence T_n of linear operators are the maximal invariant subspaces of a continuous linear operator T well approximated by those of T_n, and in what sense does this approximation take place?

These questions are far from being elementary since, commonly in practice, to an eigenvalue λ of T there is associated a subset $\lambda_n = \{\lambda_{1,n}, \lambda_{2,n}, \dots, \lambda_{r_n,n}\}$ of r_n different eigenvalues of T_n which may be interpreted as the approximations of λ produced by T_n. To each $\lambda_{j,n}$ is associated a maximal invariant subspace $M_{j,n}$ of T_n. If M denotes the maximal invariant subspace of T corresponding to λ, we should expect $M_n = \bigoplus_{j=1}^{r_n} M_{j,n}$ to be an approximation to M. We would like, in particular, to have dim M_n= dim M for n large enough. This means that the sum of the r_n algebraic multiplicities corresponding to the eigenvalues in λ_n equals the algebraic multiplicity of the exact eigenvalue λ. At the same time, we expect the set $\bigcup_{n\in\mathbb{N}} \lambda_n$ to have λ in its adherence. We would also like to obtain error bounds for M_n as an approximation to M and for each $\lambda_{j,n}$ (or some function of the set λ_n) as an approximation to λ.

The natural framework of our research is a complex separable Banach space $(B, |\cdot|)$. Let M be a finite-dimensional linear subspace of B, N a closed linear subspace of B, and b a vector in B. We recall the following definitions (see [1]): $\operatorname{dist}(b, N) = \inf\{|b - v| : v \in N\}$, $\delta(M, N) = \sup\{\operatorname{dist}(u, N) : u \in M, |u| = 1\}$, and $\gamma(M, N) = \max\{\delta(M, N), \delta(N, M)\}$, called *the gap between M and N.*

We shall consider an operator $T \in \mathcal{L}(B)$, the Banach algebra of bounded linear operators in B. As an approximation to T we take a sequence T_n of linear operators defined in B. In order to deal with a more general situation than a single eigenvalue, λ will denote either a nonzero isolated

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eigenvalue of T with finite algebraic multiplicity m or a finite set of nonzero isolated eigenvalues of T with finite total algebraic multiplicity m. We recall that if Γ is any closed Jordan curve lying in the resolvent set $\mathrm{re}(T)$ of T isolating λ from 0 and from the rest of the spectrum $\mathrm{sp}(T)$ of T, and if I denotes the identity in B, then the integral $P = -\frac{1}{2\pi i}\int_\Gamma (T - zI)^{-1}\,dz$ defines the spectral projection of T associated to λ. For each $z \in \mathrm{re}(T)$, $R(z) = (T - zI)^{-1} \in \mathcal{L}(B)$ is the resolvent operator of T at z. It is natural to conceive $P_n = -\frac{1}{2\pi i}\int_\Gamma (T_n - zI)^{-1}\,dz$ as an approximation to P. But then, at least two questions should be answered before developing this idea, namely:

(a) Are the operators P_n well defined projections for n large enough? The answer is yes if we demonstrate the existence in $\mathcal{L}(B)$ of the approximate resolvents $R_n(z) = (T_n - zI)^{-1}$, at each $z \in \Gamma$, for n large enough.

(b) Is the image space P_nB equal to M_n? If P_n is well defined, then it equals the sum of all the spectral projections of T_n corresponding to the eigenvalues of T_n lying in the open bounded subset of $\mathbb{C}$ whose boundary is Γ. Hence, the image space of P_n will be equal to M_n if the set λ_n of approximations to λ is defined to be the subset of eigenvalues of T_n isolated by Γ.

2. ABOUT STRONGLY STABLE CONVERGENCE

We recall the notion of strongly stable convergence.

T_n is a *strongly stable approximation to T at λ* iff (see [2])

(SS1) T_n is pointwise convergent to T.

(SS2) Given any closed Jordan curve Γ lying in $\mathrm{re}(T)$, isolating λ from 0 and from the rest of the spectrum of T, given any $z \in \Gamma$ the approximate resolvents $R_n(z)$ belong to $\mathcal{L}(B)$ and are bounded with respect to n, for n large enough.

(SS3) Associated with the curve Γ, the approximate projections P_n are such that, for n large enough, $\dim P_nB = m$.

The following results are easy to prove:

PROPERTY 2.1. *(SS1) and (SS2) imply that for $z \in \Gamma$, $R_n(z)$ is pointwise convergent to $R(z)$ and, for n large enough, uniformly bounded in z for $z \in \Gamma$, and that P_n is pointwise convergent to P.*

We shall be concerned with the following notions of convergence:

$A_n \in \mathcal{L}(B)$ is a *collectively compact approximation to $A \in \mathcal{L}(B)$* iff (see [3])

(CC1) A_n is pointwise convergent to A.

(CC2) There exists $n_0 \in \mathbb{N}$ such that $\{A_nb : \ b \in B,\ |b| \le 1,\ n > n_0\}$ is a relatively compact set.

$A_n \in \mathcal{L}(B)$ is a *self-range-uniform approximation to $A \in \mathcal{L}(B)$* iff

(SRU1) A_n is pointwise convergent to A.

(SRU2) $(A_n - A)A_n$ converges in norm to 0.

PROPERTY 2.2. *Suppose the pointwise convergence of P_n to P. Then the following are equivalent:*

(i) *(SS3).*
(ii) *P_n is a collectively compact approximation to P.*
(iii) *P_n is a self-range-uniform approximation to P.*
(iv) *The gap between P_nB and PB tends to 0 as n tends to infinity.*

In the light of Property 2.2, we are led to propose the following notion of convergence, specially concerned with spectral approximation:

T_n is a *spectral approximation to T at λ* if given any closed Jordan curve Γ isolating λ from 0 and from the rest of $\mathrm{sp}(T)$, the sequence P_n is self-range-uniform convergent to the spectral projection P.

PROPERTY 2.3. *If T_n is a strongly stable approximation to T at λ, then T_n is a spectral approximation to T at λ.*

PROPERTY 2.4. *The strongly stable convergence of T_n to T at λ implies $\lim_{n\to\infty} \gamma(M_n, M) = 0$.*

3. OUR RESULTS

Here, ρ denotes the spectral radius.

THEOREM 3.1. *Under the hypotheses*

(H1) *T_n is pointwise convergent to T,*

(H2) *either $\lim_{n\to\infty} |(T_n - T)T_n| = 0$ or $\lim_{n\to\infty} |T_n(T - T_n)| = 0$,*

(H3) $\lim_{n\to\infty} \rho(T_n - T) = 0$,

the sequence of linear operators T_n is a spectral approximation to T at any subset λ of nonzero isolated eigenvalues of T with finite total algebraic multiplicity.

PROOF. (H1) corresponds to (SS1). To prove (SS2) we consider the following two identities:

$$\begin{aligned}(T_n - zI)\left(I - \frac{1}{z}(T - T_n)\right) &= (T - zI)\left(I - \frac{1}{z}R(z)T_n(T - T_n)\right) \\ \left(I - \frac{1}{z}(T - T_n)\right)(T_n - zI) &= (T - zI)\left(I - \frac{1}{z}R(z)(T - T_n)T_n\right).\end{aligned} \tag{3.1}$$

They prove that under (H2) and (H3), $T_n - zI$ has a bounded inverse $R_n(z)$, for n large enough. We remark that (H1) implies that the sequence T_n is bounded, by the Banach-Steinhaus theorem. Hence, depending on which hypothesis in (H2) takes place, one of the following two expressions for $R_n(z)$, obtained from (3.1), shows that it is uniformly bounded in n and z, for n large enough and $z \in \Gamma$,

$$R_n(z) = \begin{cases} \left(I - \frac{1}{z}(T - T_n)\right)\left(I - \frac{1}{z}R(z)T_n(T - T_n)\right)^{-1} R(z), \\ \left(I - \frac{1}{z}R(z)(T - T_n)T_n\right)^{-1} R(z)\left(I - \frac{1}{z}(T - T_n)\right). \end{cases} \tag{3.2}$$

Let Γ be a closed Jordan curve isolating λ from 0 and from the rest of the spectrum of T. Let λ_n be the subset of the spectrum of T_n in the bounded open subset of $\mathbb{C}$ with boundary Γ. The fact that P_n, T_n, and $R_n(z)$ commute, imply the bounds

$$|P_n(P_n - P)| \le c|P_n(T_n - T)|, \quad |(P - P_n)P_n| \le c|(T - T_n)P_n|,$$

where c is a generic positive constant. On the other hand, the identity $R_n(z) = \frac{1}{z}(R_n(z)T_n - I) = \frac{1}{z}(T_n R_n(z) - I)$ and the fact that $\int_\Gamma \frac{dz}{z} = 0$ imply the identities

$$P_n(T_n - T) = -\frac{1}{2\pi i}\int_\Gamma \frac{1}{z}R_n(z)T_n(T_n - T)\, dz,$$

$$(T - T_n)P_n = -\frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(T - T_n)T_n R_n(z)\, dz.$$

Since $\rho(P_n(P - P_n)) = \rho((P - P_n)P_n) \le \min\{|P_n(P - P_n)|, |(P - P_n)P_n|\}$, we conclude that $\lim_{n\to\infty} \rho(P_n(P_n - P)) = 0$, and hence, for n large enough, $\dim M_n \le \dim M$. But since P is compact, $(P_n - P)P$ converges in norm to 0. This implies that $\dim M_n \ge \dim M$. Hence, (SS3) follows and we apply Property 2.3. ∎

COROLLARY 3.2.

(i) *If T is compact and T_n a self-range-uniform approximation to T, then T_n is a spectral approximation to T at each nonzero isolated eigenvalue of finite algebraic multiplicity.*

(ii) *If T_n is compact, the conclusions of Theorem 3.1 remain true without (H3).*

THEOREM 3.3. *Let $\hat{\lambda}$ and $\hat{\lambda}_n$ be the arithmetic means (counting algebraic multiplicities) of the eigenvalues in λ and λ_n, respectively. Then $|\hat{\lambda}_n - \hat{\lambda}| \leq c|(T_nP_n - TP)P|$, where c is a positive constant.*

PROOF. For n large enough, given a uniformly bounded basis Φ_n of M_n, there exists a unique basis $\Phi^{(n)}$ of M such that $P_n \Phi^{(n)} = \Phi_n$. Let θ_n and $\theta^{(n)}$ be the $m \times m$ complex matrices representing the restriction of T_n to M_n in the basis Φ_n and that of T to M in the basis $\Phi^{(n)}$, respectively. We define the operator $K_n = P_n(T_nP_n - TP)P\,(P_n|_M)^{-1} \in \mathcal{L}(M_n)$. An easy computation shows that $\theta_n - \theta^{(n)}$ represents K_n in the basis Φ_n of M_n.

Hence, $|\hat{\lambda}_n - \hat{\lambda}| = \frac{1}{m}|\operatorname{tr}(\theta_n - \theta^{(n)})| \leq \rho(\theta_n - \theta^{(n)}) \leq c|(T_nP_n - TP)P|$, since P_n and $(P_n|_M)^{-1}$ are uniformly bounded in n for n large enough. ∎

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