31. Normal subspaces of \( L^\sim \) and their carriers in \( L \) (continued)

In Lemma 27.16 of Note VIII [1] the case was considered that \( \varphi \) is a strictly positive linear functional on the \( \sigma \)-Dedekind complete Riesz space \( L \). It was proved that (i) \( L \) is super Dedekind complete, (ii) if \( \varphi \) is a strictly positive integral, then \( \varphi \) is normal and \( L \) has the Egoroff property.

If we drop now the condition that \( L \) is \( \sigma \)-Dedekind complete, it can be observed first that \( L \) is still Archimedean since \( L \) carries a Riesz norm \( q \), namely \( q(i)=q(|i|) \). It turns out that the result (i) holds now in a weaker form and (ii) continues to hold without any change. The following theorem provides the details.

Theorem 31.11. Let \( \varphi \) be a strictly positive linear functional on the Riesz space \( L \). Then the following holds.

(i) If \( 0<u_r \uparrow u \), then \( 0<u_{m_i} \uparrow u \) for some sequence \( \{u_{m_i}\} \subset \{u_r\} \).

(ii) If \( \varphi \) is a strictly positive integral on \( L \), then \( \varphi \) is a normal integral, and \( L \) has the Egoroff property.

Proof. (i) Let \( 0<u_r \uparrow u \), and set \( \alpha=\sup \varphi(u_r) \). Then there exists a sequence \( \{v_n=u_{r_n}\} \subset \{u_r\} \) such that \( v_n \uparrow \) and \( \varphi(v_n) \uparrow \alpha \). If \( u \) is not the least upper bound of the sequence \( \{v_n\} \), there exists \( u_0<u \) such that \( v_n<u_0 \) for all \( n \). Then, since \( u_r \uparrow u \) and \( u_0<u \), there exists \( r_0 \) such that \( \sup (u_{r_0}, u_0)-u_0=w_0>0 \), and so \( \sup (u_{r_0}, v_n)-v_n\geqslant w_0>0 \) for all \( n \). It follows that \( \varphi(\sup (u_{r_0}, v_n)) > \alpha \) for some \( n_0 \). But \( \{u_r\} \) is directed upwards, so there exists \( u_{r_0}\sup (u_{r_0}, v_{n_0}) \), and hence \( \varphi(u_{r_0}) > \alpha \). Contradiction.

(ii) It follows immediately from part (i) that \( \varphi \) is a normal integral. In order to prove that \( L \) has the Egoroff property, assume that \( 0<u \in L \) and \( 0<u_{nk} \uparrow_k u \) for \( n=1,2, \ldots \). For every pair of indices \( (m,n) \) we determine an index \( j(m,n) \) satisfying

\[
\varphi(u-u_{n,j(m,n)}) < m^{-1} \cdot 2^{-n},
\]

and evidently we may assume that \( j(m,n) \) is increasing as \( m \) increases.

For \( m \) fixed and \( \sigma=(n_1, \ldots, n_p) \), let \( u_\sigma \) be the infimum of the \( u_{n,j(m,n)} \) with \( n=n_1, \ldots, n_p \), and let \( \alpha_m=\inf \varphi(u_\sigma) \). Then \( \alpha_m \uparrow \), and since

\[
0<\varphi(u)-\varphi(u_\sigma)=\varphi(u-u_\sigma) \leqslant \sum_n \varphi(u-u_{n,j(m,n)}) < m^{-1}
\]

1) Work on this paper was supported in part by the National Science Foundation of the U.S.A. under grant NSF–G 19914 to the California Institute of Technology.
for every \( \sigma \), we have \( 0 < \varphi(u) - \alpha_m < m^{-1} \), so \( \alpha_m \uparrow \varphi(u) \) as \( m \to \infty \). Now, let \( A_m = \{ v : 0 < v < u_\sigma \text{ for all } \sigma \} \) and \( \beta_m = \sup (\varphi(v) : v \in A_m) \). Evidently \( \beta_m < \alpha_m \). We select a sequence \( w_{m,l} \uparrow t \) from \( A_m \) such that \( \varphi(w_{m,l}) \uparrow t \beta_m \). The collection \( u_\sigma - w_{m,l} \) (\( \sigma \) and \( l \) variable, \( m \) fixed) is directed downwards, and if there would exist \( w > 0 \) such that \( u_\sigma - w_{m,l} > w > 0 \), then \( w_{m,l} + w \in A_m \) for all \( l \), and this would contradict the fact that

\[
\sup_l \varphi(w_{m,l}) = \sup (\varphi(v) : v \in A_m).
\]

Hence \( \inf_{\alpha_m} (u_\sigma - w_{m,l}) = 0 \), and so \( \inf (u_\sigma - w_{m,l}) = 0 \) since \( \varphi \) is a normal integral. In other words, \( \alpha_m = \beta_m \), and so \( \beta_m \uparrow \varphi(u) \) as \( m \to \infty \). Now, let \( z_l = \sup (w_{1,l}, w_{2,l}, \ldots, w_{l,l}) \) for \( l = 1, 2, \ldots \). Since \( w_{m,l} \leq u_{n,j(m,n)} < u_{n,j(l,n)} \) for all \( n \) and for \( m = 1, \ldots, l \), we have \( z_l < u_{n,j(l,n)} \) for all \( n \). Furthermore, \( z_l \uparrow u \) and \( \sup \varphi(z_l) > \sup_l \varphi(w_{m,l}) = \beta_m \) for all \( m \), so \( \sup \varphi(z_l) > \varphi(u) \); hence, it follows easily that \( z_l \uparrow u \). This is the desired result.

The proofs of the next following Theorems 31.12 and 31.13 are similar to the proofs of Theorem 27.17 and Corollary 27.19 in Note VIII.

**Theorem 31.12.** If \( L \) is Archimedean, then the carrier \( L_1 \) of \( L_\sim \) is the Riesz direct sum of normal subspaces \( A_\sigma \) such that each \( A_\sigma \) possesses a strictly positive normal integral, and hence each \( A_\sigma \) has the Egoroff property.

**Theorem 31.13.** If \( L \) is Archimedean, then \( L_\sim \) is the Riesz direct sum of normal subspaces \( B_\sigma \) such that each \( B_\sigma \) is super Dedekind complete and has the Egoroff property. If, in addition, the carrier \( L_1 \) of \( L_\sim \) has a finite or countable order basis, then \( L_\sim \) is super Dedekind complete.

Instead of Corollary 27.18 (Note VIII) we have the following result.

**Corollary 31.14.** If \( L \) is Archimedean, and \( L_\sim \) has a finite or countable order basis, then the carrier \( L_1 \) of \( L_\sim \) has the property that \( 0 < u_\sigma \uparrow u \in L_1 \) implies that \( 0 < u_\tau \uparrow u \) for some sequence \( \{ u_\tau \} \subset \{ u_\sigma \} \).

**Proof.** Let \( \{ \varphi_k : k = 1, 2, \ldots \} \) be an at most countable order basis of \( L_\sim \) with \( \varphi_k > 0 \) for every \( k \), and let \( 0 < u_\sigma \uparrow u \in L_1 \). It follows easily that \( \{ u_\sigma \} \) contains a sequence \( u_\tau \uparrow u \) such that \( \varphi_k(u_\tau) \uparrow \varphi_k(u) \) for every \( k \). If \( 0 < u_\tau < v < u \) for every \( n \), then \( \varphi_k(u-v) = 0 \) for all \( k \), so \( u-v \) is in \( L_1 \) as well as in the null ideal of every \( \varphi_k \). It follows that \( u-v \) is disjoint to the carrier of every \( \varphi_k \), i.e., \( u-v \) is disjoint to \( L_1 \). Hence \( u-v = 0 \), and so \( 0 < u_\tau \uparrow u \).

In Theorem 27.5 of Note VIII it was proved that if \( L \) is Dedekind complete and \( \varphi \) is an integral on \( L \), then \( \varphi \) is a normal integral if and only if the null ideal \( N_\varphi \) is a normal subspace of \( L \). We finally present a generalization of this theorem.

**Theorem 31.15.** (i) If \( L \) is \( \sigma \)-Dedekind complete and if \( \varphi \in L_\sim \), then \( L = C_\varphi \oplus \{ N_\varphi \} \).
(ii) If \( L \) is \( \sigma \)-Dedekind complete and \( \varphi \) is an integral on \( L \), then \( \varphi \) is a normal integral on \( L \) if and only if \( N_\varphi \) is a normal subspace of \( L \), and hence we have \( L = C_\varphi \oplus N_\varphi \) in this case.

Proof. i) We may assume that \( 0 < \varphi \in L^\sim \). Since \( L \) is \( \sigma \)-Dedekind complete, and hence Archimedean, the ideal \( C_\varphi \oplus N_\varphi \) is order dense in \( L \). Hence, given \( 0 < u \in L \), we have

\[
u = \sup (v + w : 0 < v + w < u, \; v \in C_\varphi, \; w \in N_\varphi).
\]

Let \( \alpha = \sup (\varphi(v) : 0 < v < u, \; v \in C_\varphi) \). There exists a sequence \( 0 < v_n \uparrow < u \), \( v_n \in C_\varphi \), such that \( \varphi(v_n) \uparrow \alpha \). Since \( L \) is \( \sigma \)-Dedekind complete, the element \( v_0 = \sup v_n \) exists in \( C_\varphi \), \( v_0 < u \), and we shall prove that \( v_0 = \sup \{v : 0 < v < u, \; v \in C_\varphi\} \). If not, there exists \( 0 < v < u \) satisfying \( \sup (v_0, v) > v_0 \), so

\[
sup (v_n, v) - v_n > sup (v_0, v) - v_0 > 0,
\]

which implies

\[
\varphi(sup (v_n, v) - v_n) > \varphi(sup (v_0, v) - v_0) > 0.
\]

But the expression on the left tends to zero as \( n \to \infty \). Thus a contradiction is obtained, and so \( v_0 = \sup \{v : 0 < v < u, \; v \in C_\varphi\} \). Observe now that for \( 0 < v \in C_\varphi \) and \( 0 < w \in N_\varphi \), the condition \( v + w < u \) is equivalent to the separate conditions \( v < u \) and \( w < u \) (since \( v + w = \sup (v, w) \) on account of \( v \perp w \)). Hence, writing \( A = (v_0 + w : 0 < w < u, \; w \in N_\varphi) \), we have for any fixed \( w_1 \in N_\varphi \) with \( 0 < w_1 < u \) that

\[
v_0 + w_1 = \sup (v + w_1 : 0 < v < u, \; v \in C_\varphi) <
\]

\[
\sup (v + w : 0 < v < u, \; 0 < w < u, \; v \in C_\varphi, \; w \in N_\varphi) = u,
\]

and so \( u \) is an upper bound of the collection \( A \). Evidently \( u \) is the least upper bound of \( A \), so \( u - v_0 = \sup \{w : 0 < w < u, \; w \in N_\varphi\} \), and this shows that \( w_0 = u - v_0 \in \{N_\varphi\} \). Hence \( u = v_0 + w_0 \) with \( v_0 \in C_\varphi \) and \( w_0 \in \{N_\varphi\} \).

(ii) If \( \varphi \) is a normal integral on \( L \), then it is evident that \( N_\varphi \) is a normal subspace of \( L \). Conversely, assume that \( \varphi \) is an integral such that \( N_\varphi \) is a normal subspace of \( L \). It follows from part (i) that \( L = C_\varphi \oplus N_\varphi \), and from this point on the proof of Theorem 27.5 (Note VIII) can be repeated in order to prove that \( \varphi \) is a normal integral.

We make one final remark. If every Archimedean Riesz space \( L \), which fails to have the property that \( \sup_n \{\inf (v, nu)\} \) exists for every pair \( 0 < u, \; v \in L \), would not possess any normal integral (i.e., if \( L_n^\sim = \{0\} \) for every such space \( L \), then the results in the present section would be devoid of any interest. Hence, it is of some importance to observe that by Example 29.11 there exists such a space \( L \) possessing sufficiently many normal integrals in order that \( \sigma (L_n^\sim) = \{0\} \), i.e., the carrier \( L_1 \) of \( L^\sim \) satisfies \( L_1 = L \).

32. Dedekind completion

Let \( L \) and \( L^1 \) be Riesz spaces.
Definition 32.1. The Riesz space $L^1$ is called a Dedekind completion of $L$ if

(i) $L^1$ is Dedekind complete,

(ii) $L$ is imbedded in $L^1$ as a Riesz subspace (more precisely, there is a one-one mapping of $L$ onto a Riesz subspace $L_1$ of $L^1$ preserving the algebraic and order relations; we shall think of $L$ and $L_1$ as identified),

(iii) for every $f^1 \in L^1$ we have

$$f^1 = \sup (g : g \in L, g < f^1) = \inf (h : h \in L, h > f^1).$$

Given the Dedekind completion $L^1$ of $L$ and the elements $f, g \in L$, the element $\sup (f, g)$, where the supremum is taken in $L^1$, is also in $L$ since $L$ is a Riesz subspace of $L^1$, and hence this element is also the supremum of $f$ and $g$ in $L$. It follows that the imbedding of $L$ in $L^1$ preserves finite suprema and infima. It will be shown now, first of all, that arbitrary suprema and infima are preserved.

Lemma 32.2. If $L^1$ is a Dedekind completion of $L$, the imbedding of $L$ into $L^1$ preserves arbitrary suprema and infima.

Proof. Given that $f = \sup f\tau$ in $L$, we have to show that $f = \sup f\tau$ in $L^1$. Assume that there exists $g^1 \in L^1$ such that, in $L^1$, we have $g^1 < f$ and $g^1 > f\tau$ for all $\tau$. Then, since $g^1 = \inf (h : h \in L, h > g^1)$, there exists $h_0 \in L$ such that $h_0 > g^1 > f\tau$ for all $\tau$, but not $h_0 > f$. Contradiction, since $h_0 > f\tau$ for all $\tau$ implies that $h_0 > f$ in $L$, and hence $h_0 > f$ in $L^1$.

The following lemma is useful for practical purposes.

Lemma 32.3. The condition (iii) in the definition of a Dedekind completion $L^1$ of $L$ can be replaced by the following condition.

(iv) For every $0 < f^1 \in L^1$ there exist $g, h \in L$ such that $0 < g < f^1 < h$.

Proof. Evidently, \{(i), (ii), (iii)\} implies \{(i), (ii), (iv)\}. Conversely, let (i), (ii), (iv) hold and let $0 < f^1 \in L^1$. It will be proved that $f^1 = \sup (g : g \in L, g < f^1)$; this supremum exists in $L^1$ since $L^1$ is Dedekind complete. Obviously, $0 < u^1 < f^1$, and we have to show that $u^1 = f^1$. If $f^1 - u^1 > 0$, then (iv) implies the existence of $v \in L$ such that $0 < v < f^1 - u^1$. But then $u^1 + v = \sup (g + v : g \in L, g < f^1) < u^1$ since $g \in L, g < f^1$ implies $g + v \in L$ and $g + v < u^1 + v < f^1$. Hence $u^1 + v < u^1$, so $v < 0$. Contradiction, and so $f^1 = \sup (g : g \in L, g < f^1)$ for every $f^1 > 0$.

If $f^1 \in L^1$ is arbitrary, we set $f^1 = (f^1)^+ - (f^1)^-$. There exists $h_0 \in L$ such that $(f^1)^- < -h_0$, so $h_0 < -(f^1)^- < f^1$. Then $f^1 - h_0 > 0$, so $f^1 - h_0 = \sup (g : g \in L, g < f^1 - h_0)$ by what has already been proved, which implies $f^1 = \sup (g : g \in L, g < f^1)$. But then also $-f^1 = \sup (g : g \in L, g < -f^1)$, so $f^1 = \inf (h : h \in L, h > f^1)$. It has thus been shown that (iii) holds for any $f^1 \in L^1$. 
Lemma 32.4. The condition (iii) or (iv) in the definition of a Dedekind completion \( L^1 \) of \( L \) can also be replaced by the following pair of conditions.

(v) For every \( 0 < f^1 \in L^1 \) there exists \( g \in L \) such that \( 0 < g < f^1 \).

(vi) The ideal in \( L^1 \) generated by \( L \) is \( L^1 \) itself.

Proof. Assume already that (i), (ii) hold, i.e., \( L \) is a Riesz subspace of the Dedekind complete space \( L^1 \). The ideal in \( L^1 \) generated by \( L \) is the set of all \( f^1 \in L^1 \) satisfying \(|f^1| < h \) for some \( h \in L^+ \). Indeed, it is easy to verify that this set is an ideal, and obviously it is the smallest ideal including \( L \).

Now, if in addition to (i) and (ii) one of (iii) and (iv) holds, then (iii) and (iv) hold simultaneously, so (v) holds by (iv). In order to show that (vi) holds, let \( f^1 \in L^1 \). Then \(|f^1| < h \) for some \( h \in L \) by (iv), so \( f^1 \) is in the ideal generated by \( L \). Hence, the ideal is the whole of \( L^1 \), i.e., (vi) holds.

Conversely, if in addition to (i) and (ii) the conditions (v) and (vi) hold, we have the left hand inequality in (iv) by (v). For the right hand inequality, assume that \( 0 < f^1 \in L^1 \). Then, by (vi), \( f^1 \) is in the ideal generated by \( L \), so there exists \( h \in L \) satisfying \( f^1 < h \).

Example 32.5. The condition (vi) in the last lemma cannot be replaced by the condition (vi') that the normal subspace generated by \( L \) is \( L^1 \) itself. Indeed, let \( L \) be the Riesz space of all real bounded functions \( f \) on \( X = [0, 1] \) such that \( f(x) \neq f(0) \) for at most countably many \( x \), and \( L^1 \) the space of all real functions on \( X \). Then (i), (ii), (v) and (vi') are satisfied, but (vi) is not satisfied since the ideal in \( L^1 \) generated by \( L \) is the space of all real bounded functions on \( X \). Hence, \( L^1 \) is not a Dedekind completion of \( L \).

Corollary 32.6. Let \( L \) be a Riesz subspace of the Dedekind complete space \( L^1 \), and let \( D \) be the ideal in \( L^1 \) generated by \( L \). If, for every \( 0 < f^1 \in D \), there exists \( g \in L \) such that \( 0 < g < f^1 \), then \( D \) is a Dedekind completion of \( L \).

If \( L^1 \) is a Dedekind completion of \( L \), then \( L^1 \) is Dedekind complete, so \( L^1 \) is surely Archimedean. Since any Riesz subspace of an Archimedean space is Archimedean, it follows that \( L \) is Archimedean. It can be proved (cf. H. Nakano [2], Theorems 30.2 and 30.3) that a Riesz space \( L \) has a Dedekind completion if and only if \( L \) is Archimedean, and that any two Dedekind completions of \( L \) are isomorphic.

Theorem 32.7. Let \( D \) be a Dedekind completion of \( L \). Then every \( 0 < \varphi \in L^\sim \) can be extended to a positive linear functional on \( D \), and \( 0 < \varphi \in L^\sim_n \) the extension is unique and normal which shows that \( L^\sim_n \) and \( D^\sim_n \) are isomorphic.

Proof. Let \( 0 < \varphi \in L^\sim \) be given. For every \( 0 < d \in D \) we set \( p_\varphi(d) = \inf (\varphi(u) : d < u \in L) \). Since every \( d \in D \) is majorized by some element of \( L \), the number \( p_\varphi(d) \) is finite. It is easy to verify that (i)
\( p_\varphi(u) = \varphi(u) \) for all \( 0 < u \in L \), (ii) \( p_\varphi(ad) = ap_\varphi(d) \) for all \( a > 0 \) and \( 0 < d \in D \), (iii) \( p_\varphi(d_1) < p_\varphi(d_2) \) for \( 0 < d_1 < d_2 \), (iv) \( p_\varphi(d_1 + d_2) < p_\varphi(d_1) + p_\varphi(d_2) \) for \( d_1, d_2 > 0 \). For the proof of (iv), choose \( \varepsilon > 0 \) and \( d_1 < u_1 \in L \), \( d_2 < u_2 \in L \) such that \( \varphi(u_1) < p_\varphi(d_1) + \varepsilon \) and \( \varphi(u_2) < p_\varphi(d_2) + \varepsilon \). Then \( p_\varphi(d_1 + d_2) < \varphi(u_1) + \varphi(u_2) < p_\varphi(d_1) + p_\varphi(d_2) + 2\varepsilon \), and (iv) follows. Defining now \( p_\varphi(d) = p_\varphi(d^+) \) for any \( d \in D \), the functional \( p_\varphi \) is sublinear on \( D \) (compare Theorem 19.2 in Note VI), and \( \varphi \) is majorized by \( p_\varphi \) on \( L \) since \( \varphi(f) < \varphi(f^+) = p_\varphi(f) \) for any \( f \in L \). By the extension theorem there exists a linear functional \( \psi \) on \( D \) satisfying \( \psi = \varphi \) on \( L \) and \( \psi(d) < p_\varphi(d) \) on \( D \). If \( u \in D^+ \), then \( \psi(-u) < p_\varphi(-u) = p_\varphi(0) = 0 \), so \( \psi(u) > 0 \). Hence \( 0 < \psi \in D^+ \).

In order to prove that every \( 0 < \varphi \in \widetilde{L_n} \) has a unique positive normal extension, we observe that if we set \( M_d = \{ u : d < u \in L \} \) and \( m_d = \{ u : d > u \in L \} \) for any \( 0 < d \in D \), then \( \inf(M_d - m_d) = 0 \), and so

\[
p_\varphi(d) = \inf(\varphi(u) : d < u \in L) = \sup(\varphi(u) : d > u \in L)
\]

since \( \varphi \) is normal on \( L \). This implies that \( p_\varphi(d) \) is nonnegative and additive on \( D^+ \), and coincides with \( \varphi \) on \( L^+ \). The obvious linear extension \( \psi \) of \( p_\varphi \) to \( D \) is, therefore, an extension of \( \varphi \), and it follows easily that \( \psi \) is normal on \( D \). Also, any other positive normal extension \( \psi_1 \) of \( \varphi \) satisfies \( \psi_1(d) = \sup(\varphi(u) : d > u \in L) \) for every \( 0 < d \in D \), so \( \psi_1(d) = p_\varphi(d) = \psi(d) \).

The existence of the unique positive normal extension to \( D \) of any \( 0 < \varphi \in \widetilde{L_n} \) indicates why the results in Note IX about normal integrals on Archimedean Riesz spaces are similar to the results for Dedekind complete spaces in Note VIII.

In general it is not so simple to describe the Dedekind completion of an Archimedean Riesz space \( L \) in terms of \( L \). In the case, however, that \( \widetilde{L_n} \) separates the points of \( L \), i.e., in the case that \( \emptyset(L_n) = \{ 0 \} \), the Dedekind completion of \( L \) has an interesting representation as a subset of \( L_{nn} = (\widetilde{L_n})_n \).

We recall the results in section 28 of Note VIII, where it was shown that under the condition \( \emptyset(L_n) = \{ 0 \} \) the space \( L \) is imbedded as a Riesz subspace in \( L_{nn} \), and by Theorem 28.2 \( L \) is an ideal in \( L_{nn} \) if and only if \( L \) is Dedekind complete. Furthermore, if \( L \) is an ideal in \( L_{nn} \), then \( L \) is order dense in \( L_{nn} \). These results can be generalized as follows.

**Theorem 32.8.** If \( L \) is a Riesz space satisfying \( \emptyset(L_n) = \{ 0 \} \), then the ideal \( D \) generated by \( L \) in \( L_{nn} \) is a Dedekind completion of \( L \), and \( D \) is order dense in \( L_{nn} \).

**Proof.** Since \( L \) is a Riesz subspace of the Dedekind complete space \( L_{nn} \), Corollary 32.6 shows that all which remains to be proved for \( D \) being a Dedekind completion of \( L \) is that, for every \( 0 < u^* \in D \), there exists \( u_0 \in L \) satisfying \( 0 < u_0 < u^* \). We observe first that, given \( 0 < u^* \in D \), there exists \( u \in L \) such that \( u^* < u \), since \( D \) is the ideal generated by \( L \). Hence \( 0 < u^* < u \). The proof that \( 0 < u_0 < u^* < u \) for some \( u_0 \in L \) is now the same as in Theorem 28.2 (i), the only difference being that instead of to Lemma 27.8 an appeal is now made to Lemma 31.4 in Note IX.
The proof that $L_{nn}$ is the smallest normal subspace including $L$ is also almost the same as in Theorem 28.2 (ii), and this implies immediately that $D$ is order dense in $L_{nn}$. A second proof is obtained by observing that $D$ is Dedekind complete, so $D$ is order dense in $D_{nn}$ by Theorem 28.2 (ii). But $L^n$ and $D^n$ can be identified by the preceding theorem, and hence $L_{nn}$ and $D_{nn}$ can also be identified.

If $O(L^n) = \{0\}$, then (according to Theorem 28.4) $L = L_{nn}$ if and only if it follows from $0 < u_\tau \uparrow$ in $L$ and $\sup \varphi(u_\tau) < \infty$ for every $0 < \varphi \in L^n$ that $u_\tau < u$ exists in $L$. A generalization is as follows.

Theorem 32.9. If $O(L^n) = \{0\}$, then $L_{nn}$ is a Dedekind completion of $L$ if and only if it follows from $0 < u_\tau \uparrow$ in $L$ and $\sup \varphi(u_\tau) < \infty$ for every $0 < \varphi \in L^n$ that $u_\tau < u$ for some $u \in L$ and all $\tau$.

Proof. Assume first that $L_{nn}$ is a Dedekind completion of $L$, and let $0 < u_\tau \uparrow$ and $\sup \varphi(u_\tau) < \infty$ for every $0 < \varphi \in L^n$. Then $0 < u_\tau \uparrow$ in $L_{nn}$ and $\sup \varphi(u_\tau) < \infty$ for every $0 < \varphi \in L_{nn}$, since $L_{nnn} = L^n$ on account of $L^n$ being perfect (cf. Corollary 28.6 in Note VIII). But $L_{nn}$ is also perfect, so $u'' = \sup u_\tau$ exists in $L_{nn}$ by Theorem 28.4. Since, by hypothesis, $L_{nn}$ is a Dedekind completion of $L$, there exists $u \in L$ satisfying $u'' < u$, and so $u_\tau < u$ for all $\tau$.

Conversely, assume now that $0 < u_\tau \uparrow$ and $\sup \varphi(u_\tau) < \infty$ for every $0 < \varphi \in L^n$ implies the existence of $u \in L$ such that $u_\tau < u$ for all $\tau$. Since the ideal $D$ generated by $L$ in $L_{nn}$ is a Dedekind completion of $L$ and since $D$ is order dense in $L_{nn}$, it is easy to verify that $u'' = \sup (v : u'' > v \in L)$ for every $0 < u'' \in L_{nn}$, and so $\sup (\varphi(v) : u'' > v \in L) < u''(\varphi) < \infty$ for every $0 < \varphi \in L_n$. Hence, by hypothesis, there exists $u \in L$ such that $u > v$ for all $v$ satisfying $u'' > v \in L$. It follows that $u'' < u$, i.e., every $0 < u'' \in L_{nn}$ is in the ideal $D$ generated by $L$. This is the desired result that $D = L_{nn}$, i.e., $L_{nn}$ is a Dedekind completion of $L$.

The following converse of Theorem 32.8 holds.

Theorem 32.10. Let $L$ be a Riesz space and $B$ an ideal in $L^n$ such that $O_B = \{0\}$. Then $L$ can be identified with a Riesz subspace of $B^n$. If the ideal $D$ generated by $L$ in $B^n$ is a Dedekind completion of $L$, then $B$ is an order dense ideal in $L^n$.

Proof. Every $f \in L$ defines an element $f'' \in B^n$ by means of $f''(\varphi) = \varphi(f)$ holding for all $\varphi \in B$. The thus defined canonical mapping of $L$ into $B^n$ is linear, and also one-one on account of $O_B = \{0\}$. That the mapping preserves the partial order and also preserves finite suprema and infima is proved exactly as in section 28 of Note VIII. Hence, $L$ can be identified with a Riesz subspace of $B^n$. Assume now that the ideal $D$ generated by $L$ in $B^n$ is a Dedekind completion of $L$. Then $u_\tau \downarrow 0$ in $L$ implies that $u_\tau \downarrow 0$ in $D$ by the definition of a Dedekind completion, and so $u_\tau \downarrow 0$ in $B^n$. Since any $\varphi \in B$ acts as a normal integral on $B^n$, it follows that $\varphi(u_\tau) \downarrow 0$
for every $0 < \varphi \in B$. This shows that $B \subseteq L_\infty$. Observing now that
\[ \emptyset(L_\infty) \subseteq \emptyset B = \{0\}, \]
so that $L$ is Archimedean by Lemma 29.9 (v) in Note IX, we may apply all the results in section 31 (Note IX). In particular, since $B$ is an ideal in $L_\infty$ and $\emptyset B = \emptyset(L_\infty) = \{0\}$, we have $\{B\} = L_\infty$ by Corollary 31.6, i.e., $B$ is order dense in $L_\infty$.

Assuming again that $B$ is an ideal in $L_\infty$ such that $\emptyset B = \{0\}$, we indicate
now a pair of conditions, necessary and sufficient in order that the ideal $D$ generated by $L$ in $B_\infty$ is a Dedekind completion of $L$. In particular, it will follow that we may take for $B$ any arbitrary order dense ideal in $L_\infty$.

**Theorem 32.11.** Let $L$ be a Riesz space and $B$ an ideal in $L_\infty$ such that $\emptyset B = \{0\}$. Then, in order that the ideal $D$ generated by $L$ in $B_\infty$ is a Dedekind completion of $L$ it is necessary and sufficient that

(i) for every $\varphi \in B$ the null ideal $N_\varphi$ is a normal subspace of $L$,

(ii) for $\varphi, \psi \in B$ and $\varphi \perp \psi$ we have $C_\varphi \perp C_\psi$ for the carriers of $\varphi$ and $\psi$ (or, equivalently, $C_\varphi \subseteq N_\psi$ on account of $N_\varphi$ being the disjoint complement of $C_\psi$ since $N_\varphi$ is a normal subspace).

**Proof.** Let the ideal $D$ generated by $L$ in $B_\infty$ be a Dedekind completion of $L$. Then $B \subseteq L_\infty$ by the preceding theorem, and so the conditions (i) and (ii) are satisfied (cf. Theorem 31.2 (ii) in Note IX).

Assume now, conversely, that (i) and (ii) are satisfied. Then the following property, similar to the property in Lemma 31.4 of Note IX, holds.

(*) If $0 < \varphi \in B$ and $0 < u \in L$ are such that $\varphi(u) > 0$, then there exists $0 < v < u$ such that $\varphi(v) > 0$ and $\psi(v) = 0$ for all $\psi \in B$ satisfying $\psi \perp \varphi$.

Indeed, since $C_\varphi \oplus N_\varphi$ is order dense, we have

\[ u = \sup U = \sup \{v + w : 0 < v \in C_\varphi, \ 0 < w \in N_\varphi, \ v + w < u\} \]

It is impossible that $v = 0$ in all elements $v + w \in U$, for this would imply that $u \in N_\varphi$ since $N_\varphi$ is a normal subspace, contradicting the fact that $\varphi(u) > 0$. Hence, there exists an element $0 < v \in C_\varphi$ such that $v < u$. This element satisfies $\varphi(v) > 0$ and in addition, given $\psi \in B$ such that $\psi \perp \varphi$, it follows from (ii) that $C_\varphi \subseteq N_\psi$, so $\psi(v) = 0$.

Now, let $D$ be the ideal generated by $L$ in $B_\infty$. In order to show that $D$ is a Dedekind completion of $L$, it is sufficient to prove that for any given $0 < u'' \in D$ there exists $u_0 \in L$ such that $0 < u_0 < u''$. This can be done exactly as in the proof of Theorem 28.2 (i) in Note VIII (the roles of $L_\infty$ and $L_\infty$ are taken over by $B$ and $B_\infty$ respectively; note that property (*) is used instead of Lemma 27.8).

We leave it as an exercise to the reader to prove that property (*) implies, conversely, the conditions (i) and (ii) of the last theorem. In connection with condition (ii) we finally remark that $0 < \varphi, \psi \in L_\infty$ and $\varphi \perp \psi$ does not always imply $C_\varphi \perp C_\psi$. Indeed, if $L$ consists of all real continuous functions on $\{x : 0 < x < 1\}$, $\varphi(f) = \int_0^1 f dx$ and $\psi(f) = \sum_{n=1}^\infty n^{-2} f(r_n)$,
where \(\{r_n\}\) is the set of all rational numbers in \([0, 1]\), then \(\varphi \perp \psi\), but \(C_\varphi = C_\psi = L\) since \(\varphi\) and \(\psi\) are strictly positive.

33. Discussion around a result due to H. Nakano

In this section we present several theorems related to a result due to H. Nakano. This result states that if \(L_q\) is a normed Riesz space (as introduced in Note VII) such that \(L_q\) is \(\sigma\)-Dedekind complete and satisfies \(L_q = L^q_\theta\), then \(L_q\) is super Dedekind complete. Since \(L_q = L^q_\theta\) is equivalent to \(L^*_q = L^*_\theta\), by Corollary 24.3 in Note VII, an alternative way of expressing Nakano’s result is that if \(L_q\) is \(\sigma\)-Dedekind complete and every bounded linear functional on \(L_q\) is an integral, then \(L_q\) is super Dedekind complete. We will prove Nakano’s result in a somewhat more complete form, since one of our theorems will be that \(L_q\) is \(\sigma\)-Dedekind complete and \(L^*_q\) consists only of integrals if and only if \(L_q\) is super Dedekind complete and \(L^*_q\) consists only of normal integrals (cf. Theorem 33.4). Another important theorem will be that \(L^*_q\) consists only of normal integrals if and only if \(L_q\) consists only of integrals and, in addition, every increasing order-bounded sequence is a \(\varrho\)-Cauchy sequence (cf. Theorem 33.8).

We point out already that Nakano’s theorem, in the above-mentioned general form, has a remarkable parallel, which is obtained if we replace orderbounded increasing sequences by normbounded increasing sequences. The parallel theorem states that every normbounded increasing sequence in \(L_q\) has a supremum and \(L^*_q\) consists only of integrals if and only if every normbounded set which is directed upwards has a supremum and \(L^*_q\) consists only of normal integrals. This parallel theorem, to be proved in the next section, will be of importance in obtaining for T. Ogasawara’s reflexivity criterion for normed Riesz spaces a proof avoiding any use of weak and weak* topologies.

We consider the following set of conditions, labelled \((A, i)-(A, iv)\); each of these conditions may or may not hold in the normed Riesz space \(L_q\).

\((A, i)\) \(u_n \downarrow 0\) implies \(\varrho(u_n) \downarrow 0\),

\((A, ii)\) \(u_t \downarrow 0\) implies \(\varrho(u_t) \downarrow 0\),

\((A, iii)\) \(0 < u_n \uparrow < u_0\) implies that \(\{u_n\}\) is a \(\varrho\)-Cauchy sequence, i.e., every increasing orderbounded sequence is a \(\varrho\)-Cauchy sequence,

\((A, iv)\) \(0 < u_t \uparrow < u_0\) implies that \(\{u_t\}\) is a \(\varrho\)-Cauchy net, i.e., every orderbounded set which is directed upwards is a \(\varrho\)-Cauchy net.

The condition \((A, i)\) is equivalent to \(L_q = L^q_\theta\) (cf. Theorem 24.2 (ii) in Note VII) and also to \(L^*_q = L^*_\theta\) (cf. Corollary 24.3 in Note VII), i.e., \((A, i)\) holds if and only if every bounded linear functional on \(L_q\) is an integral. Similarly, \((A, ii)\) holds if and only if \(L^*_q = L^*_\theta\), where \(L^*_\theta\) is defined by \(L^*_\theta = L^*_\theta \cap L^*_\theta\). In other words, \((A, ii)\) holds if and only if every bounded linear functional on \(L_q\) is a normal integral. Indeed, \((A, ii)\) implies immediately that \(L^*_q = L^*_\theta\), and conversely, if \(L^*_q = L^*_\theta\) is given,
an application of Mazur's theorem (exactly as in Lemma 22.6 of Note VII) shows that (A, ii) holds.

Next, we observe that (A, iii) and (A, iv) are equivalent. Obviously (A, iv) implies (A, iii), and assuming the inverse to be false, there exists 0 < u_\tau \uparrow < u_0 such that \{u_\tau\} is not a \rho-Cauchy net. Then there exists \epsilon > 0 and a sequence u_{n_\tau} \uparrow in \{u_\tau\} such that \rho(u_{n_\tau+1} - u_{n_\tau}) > \epsilon for n = 1, 2, .... This contradicts (A, iii).

Example 33.1. (i) It is evident that (A, ii) implies (A, i), but the converse is not true. To show this, we consider the space presented earlier in Example 29.11 of Note IX, where X is an uncountable point set and L_\rho the Riesz space of all real f(x) on X for which there exists a finite number f(\infty) such that, given any \epsilon > 0, we have |f(x) - f(\infty)| > \epsilon for at most finitely many x. The norm \rho(f) = \rho(f) = \sup |f(x)|. The space L_\rho has the following properties, which are either easily verified or proved in Example 29.11.

(a) L_\rho is norm complete,

(b) L_\rho is not \sigma-Dedekind complete; in fact, L_\rho does not even have the weaker property that for every 0 < u, v \in L_\rho the element sup_u \{inf(v, nu)\} exists,

(c) \frac{1}{2}(L_\rho^*) = \{0\},

(d) \rho(f) = f(\infty) is an integral, but not a normal integral,

(e) (A, i) holds, but (A, ii) and (A, iii) do not hold.

(ii) Let L_\rho be the Riesz space of all real continuous functions on \{x : 0 < x < 1\} with \rho(f) = \int_0^1 |f| dx. Then (A, iii) holds but not (A, i).

It was observed already that (A, ii) holds if and only if every bounded linear functional on L_\rho is a normal integral. If (A, ii) holds, it is also possible to say something about some unbounded linear functionals.

Theorem 33.2. If (A, ii) holds and 0 < u_\tau \uparrow u, then there exists a sequence \{u_{n_\tau}\} \subset \{u_\tau\} such that 0 < u_{n_\tau} \uparrow u. Hence, every integral on L_\rho is a normal integral, i.e., L_\rho^* = L_\rho^*.

Proof. If 0 < u_\tau \uparrow u, then u - u_\tau \downarrow 0, so \rho(u - u_\tau) \downarrow 0 by (A, ii). It follows that \{u_\tau\} contains a sequence u_{n_\tau} \uparrow such that \rho(u - u_{n_\tau}) \downarrow 0, and by Lemma 26.1 in Note VIII we have 0 < u_{n_\tau} \uparrow u.

In order to obtain a smooth presentation of the main theorems, we first prove a lemma.

Lemma 33.3. (i) If 0 < u_\tau \uparrow is a \rho-Cauchy net and \epsilon_n \downarrow 0 is a sequence of positive numbers, then there exists a sequence \{u_{n_\tau}\} \subset \{u_\tau\} such that u_{n_\tau} \uparrow and

\sup_{\tau} \rho (\sup (u_{n_\tau}, u_\tau) - u_{n_\tau}) < \epsilon_n

for all n. Furthermore, any upper bound of the sequence \{u_{n_\tau}\} is an upper bound of the net \{u_\tau\}.
(ii) If every orderbounded increasing ϕ-Cauchy sequence has a norm limit, and \(0 < u_\tau \uparrow < u_0\) is a ϕ-Cauchy net, then \(u = \sup u_\tau\) exists, and the sequence \(\{u_\tau\}\), existing by (i) of the present lemma, satisfies \(\sup u_\tau = u = \sup u_\tau\). Furthermore, \(\varphi(u - u_\tau) \to 0\).

(iii) If \(L_0\) is \(\sigma\)-Dedekind complete, and \(0 < u_\tau \uparrow < u_0\) is a ϕ-Cauchy net, then \(u = \sup u_\tau\) exists, and the sequence \(\{u_\tau\}\), existing by (i) of the present lemma, satisfies \(\sup u_\tau = u = \sup u_\tau\).

Proof. (i) First determine a sequence \(\{u_{r_n}\} \subset \{u_\tau\}\) such that \(0 < u_{r_n} \uparrow\) and

\[
\sup \{\varphi(u_\tau - u_{r_n}) : u_\tau > u_{r_n}\} < \varepsilon_n
\]

for all \(n\). If \(u_{r_n}\) is fixed, then \(\sup (u_{r_n}, u_{r_{n+1}}) - u_{r_n} < u_{r_{n+1}} - u_{r_n}\) for a suitable \(u_\tau > u_{r_n}\), so \(\varphi\{\sup (u_{r_n}, u_{r_{n+1}}) - u_{r_n}\} < \varepsilon_n\), and it follows that

\[
\sup \varphi\{\sup (u_{r_n}, u_{r_{n+1}}) - u_{r_n}\} < \varepsilon_n.
\]

Now, assume that \(u_{r_n} < v\) for all \(n\). Then

\[
\sup (u_\tau, v) - v = \sup (u_\tau, v) - \sup (u_{r_n}, v) = \sup (u_\tau, u_{r_n}, v) - \sup (u_{r_n}, v) < \sup (u_\tau, u_{r_n}) - u_{r_n}
\]

for all \(n\) and \(\tau\), so \(\varphi\{\sup (u_\tau, v) - v\} < \varepsilon_n\) for all \(n\) and \(\tau\), and this implies that \(\sup (u_\tau, v) = v\) for all \(\tau\), i.e., \(u_\tau < v\) for all \(\tau\).

(ii) Let \(0 < u_\tau \uparrow < u_0\) be a ϕ-Cauchy net. Determining the sequence \(\{u_{r_n}\} \subset \{u_\tau\}\) as in part (i), we have

\[
\varphi(u_{r_n+m} - u_{r_n}) = \varphi\{\sup (u_{r_n+m}, u_{r_n}) - u_{r_n}\} < \varepsilon_n,
\]

so \(\{u_{r_n}\}\) is an orderbounded increasing ϕ-Cauchy sequence. Hence, by hypothesis, there exists \(f \in L_0\) such that \(\varphi(f - u_{r_n}) \to 0\) as \(n \to \infty\). But then, by Lemma 26.1 in Note VIII, \(f = \sup u_{r_n}\). Hence, writing now \(u\) instead of \(f\), and observing that \(u\) is also an upper bound of \(\{u_\tau\}\) by part (i), we have obviously \(\sup u_{r_n} = u = \sup u_\tau\). It follows immediately from \(\varphi(u - u_{r_n}) \to 0\) that \(\varphi(u - u_\tau) \to 0\).

(iii) Let \(0 < u_\tau \uparrow < u_0\) be a ϕ-Cauchy net. Determining the sequence \(\{u_{r_n}\}\) as in part (i), the sequence is increasing and orderbounded. Hence, since \(L_0\) is \(\sigma\)-Dedekind complete, \(u = \sup u_{r_n}\) exists, and exactly as before we have \(u_{r_n} = u = \sup u_\tau\).

Theorem 33.4. The following conditions (a), (b), (γ) on the space \(L_0\) are mutually equivalent.

(a) \(L_0\) is \(\sigma\)-Dedekind complete, and (A, i) holds, i.e., \(u_n \downarrow 0\) implies \(\varphi(u_n) \downarrow 0\).

(b) Every orderbounded increasing sequence in \(L_0\) has a norm limit.

(γ) \(L_0\) is super Dedekind complete, and (A, ii) holds, i.e., \(u_\uparrow 0\) implies \(\varphi(u_\uparrow) \downarrow 0\).
Observe that (β) can also be expressed by saying that (A, iii) holds and that every orderbounded increasing q-Cauchy sequence has a norm limit.

Proof. (α)⇒(β). Let 0<u_n↑<u_0. The space L_q is σ-Dedekind complete, so u=sup u_n exists, and since u−u_n↓0, we have ϕ(u−u_n)↓0 by (A, i). This is the desired result.

(β)⇒(γ). We will prove first that (A, ii) holds, so let u_τ↓0. In order to show that ϕ(u_τ)↓0, we may replace u_τ by inf (u_τ, u_n) for any fixed u_n, i.e., we may assume immediately that u_0>u_τ↓0. Then 0<v_τ=u_0−u_τ↑u_0, so {v_τ} is a q-Cauchy net by (A, iv)⇔(A, iii). The hypotheses of part (ii) in the preceding Lemma 33.3 are now satisfied for {v_τ}, so ϕ(u_0−v_τ)→0, i.e., ϕ(u_τ)→0.

For the proof that L_q is super Dedekind complete, let 0<u_τ↑<u_0. By (A, iv)⇔(A, iii) the set {u_τ} is a q-Cauchy net, so the hypotheses of part (ii) in the preceding Lemma 33.3 are satisfied for {u_τ}. It follows that u=sup u_τ exists, and sup u_τ=u=sup u_τ for an appropriate sequence {u_τ} ⊂ {u_τ}. This shows that L_q is super Dedekind complete.

(γ)⇒(α). Evident.

The theorem of H. Nakano ([3], p. 321–322; reprinted from Proc. Imp. Acad. Tokyo 19, 1943), stating that if L_q=L_q (i.e., (A, i) holds) and L_q is σ-Dedekind complete, then L_q is super Dedekind complete, is included in the present Theorem 33.4.

Corollary 33.5. If L_q is σ-Dedekind complete, then L_q as a Riesz space on its own, is super Dedekind complete and has the property that u_τ↓0 (with u_τ∈L_q for all τ) implies ϕ(u_τ)↓0.

Proof. L_q is an ideal in L_q. Hence, L_q as a Riesz space on its own is σ-Dedekind complete and has the property (A, i). It follows from the preceding theorem that L_q is super Dedekind complete and has the property (A, ii).

We observe, incidentally, that the two hypotheses in condition (α) of Theorem 33.4 (namely, L_q=L_q and σ-Dedekind completeness) are independent, as shown by the following example.

Example 33.6. If (A, i) holds but not (A, ii), then L_q is not σ-Dedekind complete, and so L_q is not super Dedekind complete (cf. Example 33.1 (i)).

If (A, i) holds but L_q is not super Dedekind complete, then L_q is not σ-Dedekind complete. It may happen now that (A, ii) does not hold (cf. again Example 33.1 (i)), and it may also happen that (A, ii) does hold. By way of example, let L_q be the Riesz space of all real step functions on Lebesgue measurable sets in {x : 0<x<1}, with only a finite number of steps and with the usual identification of functions differing only on a set of measure zero, and let ϕ(f) be the L_1 norm of |f|.
If \( L_0 \) is \( \sigma \)-Dedekind complete but not super Dedekind complete, then (A, i) does not hold, and so (A, ii) does not hold either. By way of example, let \( L_0 \) be the Riesz space of all real bounded \( f(x) \) on an uncountable point set \( X \) such that \( f(x) \neq 0 \) for at most countably many \( x \in X \), and with \( \phi(f) = \sup |f(x)| \).

If \( L_0 \) is \( \sigma \)-Dedekind complete but (A, ii) does not hold, then (A, i) does not hold either. It may happen now that \( L_0 \) is super Dedekind complete (the sequence space \( l_\infty \)), and it may also happen that \( L_0 \) is not super Dedekind complete (cf. the example in the preceding paragraph).

As shown in Theorem 33.4, the condition (A, i), when taken together with \( \sigma \)-Dedekind completeness, implies the condition (A, ii). On the other hand, as shown in the last example, it may very well happen that (A, ii) holds without \( L_0 \) being \( \sigma \)-Dedekind complete. Hence, it is of interest to have a condition different from \( \sigma \)-Dedekind completeness which, when taken together with (A, i), is necessary and sufficient in order that (A, ii) holds. We will prove now that (A, iii) is such a condition. In the proof we will use the following simple characterization of Archimedean Riesz spaces.

**Lemma 33.7.** The Riesz space \( L \) is Archimedean if and only if 
\[ 0 < u_\tau \uparrow < u_0 \quad \text{and} \quad V = \{ v : v > u_\tau \text{ for all } \tau \} \]
implies that the directed set \( \{ v - u_\tau : v \in V, \tau \in \{ \tau \} \} \) satisfies \( v - u_\tau \downarrow 0 \).

**Proof.** Let \( L \) be Archimedean, and let \( \{ u_\tau \} \) and \( V \) be as defined above. If \( v \) runs through \( V \) and \( u_\tau \) through \( \{ u_\tau \} \), then the set of all \( v - u_\tau \) is directed downwards. Assume that \( 0 < w < v - u_\tau \) for all \( v \in V \) and all \( \tau \). Then \( v - w \in V \) for all \( v \in V \), so \( v - kw \in V \) for \( k = 1, 2, \ldots \), and hence \( v - kw > 0 \), i.e., \( kw < v \) for all \( v \in V \). In particular, \( kw < u_0 \) for \( k = 1, 2, \ldots \), and since \( L \) is Archimedean, this implies \( w = 0 \). Hence \( v - u_\tau \downarrow 0 \).

Conversely, assume that \( L \) satisfies the mentioned condition, and let 
\[ 0 < nu < u_0 \quad \text{for} \quad n = 1, 2, \ldots \]
Then, if \( V = \{ v : v > nu \text{ for } n = 1, 2, \ldots \} \), we have \( v - nu \downarrow 0 \) by hypothesis. Since \( v - nu = v - (n + 1)u + u > u \) for any \( v \in V \) and any \( n \), this implies that \( u = 0 \), so \( L \) is Archimedean.

**Theorem 33.8.** The following conditions (a), (b) on the space \( L_0 \) are equivalent.

(a) The space \( L_0 \) satisfies (A, i) and (A, iii), i.e., \( u_n \downarrow 0 \) implies \( \phi(u_n) \downarrow 0 \), and \( 0 < u_n \uparrow < u_0 \) implies that \( \{ u_n \} \) is a \( \sigma \)-Cauchy sequence.

(b) The space \( L_0 \) satisfies (A, ii), i.e., \( u_\tau \downarrow 0 \) implies \( \phi(u_\tau) \downarrow 0 \).

**Proof.** (a) \( \Rightarrow \) (b). Let \( u_\tau \downarrow 0 \). In order to prove that \( \phi(u_\tau) \downarrow 0 \), we may replace \( u_\tau \) by \( \inf (\{ u_n \}) \) for any fixed \( u_n \), i.e., we may assume immediately that \( u_0 \uparrow u_\tau \downarrow 0 \). Then \( 0 < u_\tau - u_0 \uparrow u_0 \), so \( \{ u_\tau \} \) is a \( \sigma \)-Cauchy net by (A, iv) \( \Leftrightarrow \) (A, iii). But then \( \{ u_\tau \} \) is also a \( \sigma \)-Cauchy net, and by Lemma 33.3 (i) there is a decreasing sequence \( \{ u_{n_0} \} \subseteq \{ u_\tau \} \) such that any
lower bound of \( \{u_n\} \) is also a lower bound of \( \{u_r\} \). Since \( u_r \downarrow 0 \), it follows that \( u_n \downarrow 0 \), and so \( \varrho(u_n) \downarrow 0 \) by (A, i). But then \( \inf \varrho(u_r) = 0 \), i.e., (A, ii) holds.

\( (\beta) \Rightarrow (\alpha) \). We need only prove that \( L_\varrho \) satisfies (A, iii), so let \( 0 < u_n \uparrow < u_0 \). Observing now that \( L_\varrho \) is Archimedean and introducing the set \( V = \{v : \varrho v > u_n \text{ for } n = 1, 2, \ldots\} \), we have \( v - u_n \downarrow v, n \downarrow 0 \) by the preceding lemma, so \( \varrho(v - u_n) \downarrow v, n \downarrow 0 \) by (A, ii). If \( m > n \), then \( 0 < u_m - u_n < v - u_n \) for any \( v \in V \), so \( \varrho(u_m - u_n) \) is arbitrarily small if \( n \) is sufficiently large, and this shows that \( \{u_n\} \) is a \( \varrho \)-Cauchy sequence. Hence, (A, iii) holds.

Corollary 33.9. If every orderbounded increasing \( \varrho \)-Cauchy sequence in \( L_\varrho \) has a norm limit (in particular, if \( L_\varrho \) is norm complete), then (A, ii) and (A, iii) are equivalent. Furthermore, if in this case the equivalent conditions (A, ii) and (A, iii) hold, then \( L_\varrho \) is super Dedekind complete.

Proof. If (A, ii) holds, then (A, iii) holds by the preceding theorem. Assume now that \( L_\varrho \) has the stated property, and that (A, iii) holds. This means exactly that condition (\( \beta \)) in Theorem 33.4 is satisfied, so it follows from that theorem that (A, ii) holds and that \( L_\varrho \) is super Dedekind complete.

We are indebted to Professor T. Andô for some stimulating discussions on the contents of this note.

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