JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **179**, 396–402 (1993)

On an Inequality of H. Minc and L. Sathre

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Submitted by J. L. Brenner

Received June 2, 1992

We prove that if we let n be a positive integer, then we have for all positive real numbers r,

$$\frac{n}{n+1} \leqslant \left((n+1) \sum_{i=1}^{n} i^r \Big/ n \sum_{i=1}^{n+1} i^r \right)^{1/r} \leqslant \frac{(n!)^{1/n}}{((n+1)!)^{1/(n+1)}}.$$

Both bounds are best possible.

In 1964 H. Minc and L. Sathre [4] proved several interesting inequalities involving $(n!)^{1/n}$. One of their results states that if n is a positive integer, then

$$n/(n+1) < (n!)^{1/n}/((n+1)!)^{1/(n+1)}. \tag{1}$$

In 1988 J. S. Martins [3] published another lower bound for the ratio $(n!)^{1/n}/((n+1)!)^{1/(n+1)}$: Let r be a positive real number and let n be a natural number. Then

$$\left((n+1) \sum_{i=1}^{n} i^r \Big/ n \sum_{i=1}^{n+1} i^r \right)^{1/r} \leqslant (n!)^{1/n}/((n+1)!)^{1/(n+1)}. \tag{2}$$

It is natural to ask whether the expressions on the left-hand side of (1) and (2) can be compared. This is indeed possible! It is the aim of this paper to prove the following refinement of inequality (1).

THEOREM. *If r is a positive real number and if n is a positive integer, then*

$$\frac{n}{n+1} \leqslant \left((n+1) \sum_{i=1}^{n} i^r \Big/ n \sum_{i=1}^{n+1} i^r \right)^{1/r} \leqslant \frac{(n!)^{1/n}}{((n+1)!)^{1/(n+1)}}. \tag{3}$$

Proof. We define

$$a_{pn+1} = a_{pn+2} = \cdots = a_{(p+1)n} = (n+1-p)/(n+1) \qquad \text{for} \quad p = 0, 1, ..., n,$$

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and

$$b_{q(n+1)+1} = b_{q(n+1)+2} = \cdots = b_{(q+1)(n+1)} = (n-q)/n \quad \text{for } q = 0, 1, \ldots, n-1.$$

Because of

$$\sum_{i=1}^{n(n+1)} a_i^r = n(n+1)^{-r} \sum_{i=1}^{n+1} i^r \qquad \text{and} \qquad \sum_{i=1}^{n(n+1)} b_i^r = (n+1)\, n^{-r} \sum_{i=1}^{n} i^r$$

we conclude that the left-hand inequality of (3) is equivalent to

$$\sum_{i=1}^{n(n+1)} a_i^r \leqslant \sum_{i=1}^{n(n+1)} b_i^r. \tag{4}$$

It is known (see [2, p. 35]) that inequality (4) is valid if the following two assumptions are fulfilled:

(i) $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{n(n+1)} \geqslant 0$, $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{n(n+1)} \geqslant 0$,

(ii) $\prod_{i=1}^k a_i \leqslant \prod_{i=1}^k b_i$ for $k = 1, \ldots, n(n+1)$.

Obviously, assumption (i) holds. To prove (ii) we set

$$A_k = \prod_{i=1}^k a_i \qquad \text{and} \qquad B_k = \prod_{i=1}^k b_i;$$

then we obtain

$$A_k = \left(\frac{n+1-p}{n+1}\right)^{k-pn} \prod_{j=1}^{p-1} \left(\frac{n+1-j}{n+1}\right)^n$$
$$\text{for} \quad pn+1 \leqslant k \leqslant (p+1)n,\ p = 0, \ldots, n,$$

and

$$B_k = \left(\frac{n-q}{n}\right)^{k-q(n+1)} \prod_{j=1}^{q-1} \left(\frac{n-j}{n}\right)^{n+1}$$
$$\text{for} \quad q(n+1)+1 \leqslant k \leqslant (q+1)(n+1),\ q = 0, \ldots, n-1.$$

Let $k \in \{1, \ldots, n(n+1)\}$; then there exists a uniquely determined number $i \in \{0, \ldots, n\}$ such that $in+1 \leqslant k \leqslant (i+1)n$. We consider three cases.

Case 1. $i = 0$. Then we have $1 \leqslant k \leqslant n$ which implies $A_k = B_k = 1$.

Case 2. $i = n$. Then we get $n^2+1 \leqslant k \leqslant n(n+1)$ which leads to

$$A_k = (n+1)^{n^2-k} \prod_{j=1}^{n-1} \left(\frac{n+1-j}{n+1}\right)^n \quad \text{and} \quad B_k = n^{n^2-k-1} \prod_{j=1}^{n-2} \left(\frac{n-j}{n}\right)^{n+1}$$

The inequality $A_k \leqslant B_k$ is equivalent to

$$(n/(n+1))^k \leqslant n!/(n+1)^n. \tag{5}$$

We have to show that (5) holds for $k = n^2+1$. A simple calculation yields that

$$(n/(n+1))^{n^2+1} \leqslant n!/(n+1)^n$$

is valid for $n = 1, 2, 3$. Let $n \geqslant 4$; from

$$n! > \sqrt{2\pi n}\, e^{-n} n^n \qquad \text{and} \qquad e^{-n} > (1+1/n)^{-n(n+1/2)}$$

(see [5, p. 181 ff. and p. 267]) we obtain

$$\frac{n!}{(n+1)^n}\left(\frac{n+1}{n}\right)^{n^2+1} > \sqrt{2\pi n}\, e^{-n}\left(\frac{n+1}{n}\right)^{n^2-n+1}$$

$$> \sqrt{2\pi n}\left(\frac{n+1}{n}\right)^{-3n/2} > \sqrt{2\pi n}\, e^{-3/2} > 1.$$

Case 3. $1 \leqslant i \leqslant n-1$. Then we have $in+1 \leqslant k \leqslant i(n+1)$ or $i(n+1)+1 \leqslant k \leqslant (i+1)n$. If $in+1 \leqslant k \leqslant i(n+1)$, then

$$A_k = \left(\frac{n+1-i}{n+1}\right)^{k-in} \prod_{j=1}^{i-1}\left(\frac{n+1-j}{n+1}\right)^n$$

and

$$B_k = \left(\frac{n+1-i}{n}\right)^{k-(i-1)(n+1)} \prod_{j=1}^{i-2}\left(\frac{n-j}{n}\right)^{n+1}.$$

Therefore, $A_k \leqslant B_k$ can be written as

$$\prod_{j=1}^{i-1}(n+1-j)^n \Big/ \prod_{j=1}^{i-2}(n-j)^{n+1} \leqslant (1+1/n)^k\,(n+1)^{-n}\, n^{n+1}(n+1-i)^{n+1-i}. \tag{6}$$

Inequality (6) is true for $i = 1$. Let $i \geqslant 2$; it suffices to prove that (6) is valid for $k = in+1$. This means we have to establish

$$\frac{(n+1-i)!\, n^{in}}{(n-1)!\,(n+1)^{(i-1)n}\,(n+1-i)^{n+1-i}} \leqslant n+1, \qquad 2 \leqslant i \leqslant n-1. \tag{7}$$

If $i(n+1) \leqslant k \leqslant (i+1)n$, then we obtain

$$A_k = \left(\frac{n+1-i}{n+1}\right)^{k-in} \prod_{j=1}^{i-1} \left(\frac{n+1-j}{n+1}\right)^n$$

and

$$B_k = \left(\frac{n-i}{n}\right)^{k-i(n+1)} \prod_{j=1}^{i-1} \left(\frac{n-j}{n}\right)^{n+1}$$

such that $A_k \leqslant B_k$ is equivalent to

$$\prod_{j=1}^{i-1} \frac{(n+1-j)^n}{(n-j)^{n+1}} \leqslant \left(\frac{(n+1)(n-i)}{n(n+1-i)}\right)^k \frac{n^{n+1}(n+1-i)^{in}}{(n+1)^n (n-i)^{i(n+1)}}. \tag{8}$$

Because of $(n+1)(n-i) < n(n+1-i)$ it is enough to prove inequality (8) for $k = (i+1)n$. This means we have to show

$$\frac{(n+1-i)!\, n^{in}}{(n-1)!\, (n+1)^{(i-1)n} (n+1-i)^{n+1-i}} \leqslant n, \qquad 2 \leqslant i \leqslant n. \tag{9}$$

Since inequality (9) implies (7) it suffices to prove (9). Let

$$c_n = (n-1)!\, (n+1)^{(n-1)n} n^{1-n^2}.$$

If we replace in Bernoulli's inequality

$$(1+x)^n \geqslant 1 + nx, \qquad x > -1,$$

the value x by $-1/n^2$, then we obtain

$$(c_n/c_{n-1})^{1/(n-1)} = ((n^2-1)/n^2)^n\, n/(n-1) \geqslant 1$$

which implies

$$c_n \geqslant c_{n-1} \geqslant \cdots \geqslant c_1 = 1.$$

This proves (9) for $i = n \geqslant 2$.

The double-inequality

$$\sqrt{2\pi k}\, k^k e^{-k} \exp(1/(12k+1)) < k! < \sqrt{2\pi k}\, k^k e^{-k} \exp(1/(12k)) \tag{10}$$

holds for $k \geqslant 2$ (see [5, p. 181 ff.]). If $2 \leqslant i \leqslant n-1$, then we conclude from (10)

$$\frac{(n+1-i)!\, n^{in}}{(n-1)!\,(n+1)^{(i-1)n}\,(n+1-i)^{n+1-i}}$$
$$< \frac{(n+1-i)^{1/2}\, n^{in}}{(n+1)^{(i-1)n}\,(n-1)^{n-1/2}} \exp\left[i-2+\frac{1}{12(n+1-i)}-\frac{1}{12(n-1)+1}\right]. \tag{11}$$

We prove that the right-hand side of (11) is not greater than n. This is equivalent to

$$(i-1)n\log(n+1)-(in-1)\log(n)-\frac{1}{2}\log(n+1-i)-(i-2)$$
$$-\frac{1}{12(n+1-i)}+\left(n-\frac{1}{2}\right)\log(n-1)+\frac{1}{12(n-1)+1}\geqslant 0. \tag{12}$$

First we show that the sum on the left-hand side of (12) attains its minimum at $i=2$. We define for $x\in[2, n-1]$

$$f_n(x)=(x-1)n\log(n+1)-(xn-1)\log(n)-\frac{1}{2}\log(n+1-x)$$
$$-(x-2)-\frac{1}{12(n+1-x)}.$$

Differentiation yields

$$f_n'(x)=\tfrac{1}{2}(n+1-x)^{-1}-\tfrac{1}{12}(n+1-x)^{-2}+n\log(n+1)-n\log(n)-1.$$

Since $f_n''(x)>0$ and because of $((n+1)/n)^{n+1/2}>e$ we obtain

$$f_n'(x)>\tfrac{1}{2}(n-1)^{-1}-\tfrac{1}{12}(n-1)^{-2}-\tfrac{1}{2}\log((n+1)/n). \tag{13}$$

The sequence on the right-hand side of (13) is decreasing on $[2,\infty)$ and tends to 0 if n tends to ∞. Hence we have $f_n'(x)>0$ which implies $f_n(i)\geqslant f_n(2)$ for $2\leqslant i\leqslant n-1$.

To prove inequality (12) it remains to show

$$g(n)=n\log(n+1)-(2n-1)\log(n)+(n-1)\log(n-1)$$
$$+\frac{1}{12(n-1)+1}-\frac{1}{12(n-1)}\geqslant 0 \quad \text{for} \quad n\geqslant 3.$$

Differentiation reveals

$$g'(n)=\frac{24n-23}{12(n-1)^2\,(12n-11)^2}+\frac{1}{n(n+1)}-\log(n^2/(n^2-1)).$$

Because of $((n+1)/n)^{n+1/2} > e$ we get $\log(n^2/(n^2-1)) > (n^2-1/2)^{-1}$. This leads (after some elementary computations) to

$$g'(n) < 0 \qquad \text{for} \quad n \geqslant 3.$$

Hence we have

$$g(n) > \lim_{n \to \infty} g(n) = 0$$

which completes the proof of the Theorem.

Remarks 1. Double-inequality (3) provides upper and lower bounds for the ratio $Q_n(r) = ((n+1)\sum_{i=1}^{n} i^r/n \sum_{i=1}^{n+1} i^r)^{1/r}$ which do not depend on the parameter r. This leads to the question: What are the greatest number α_n and the smallest number β_n such that

$$\alpha_n \leqslant Q_n(r) \leqslant \beta_n$$

holds for all $r > 0$? Because of $\lim_{r \to \infty} Q_n(r) = n/(n+1)$ and $\lim_{r \to 0} Q_n(r) = (n!)^{1/n}/((n+1)!)^{1/(n+1)}$ we conclude $\alpha_n = n/(n+1)$ and $\beta_n = (n!)^{1/n}/((n+1)!)^{1/(n+1)}$. Thus, the bounds given in (3) are best possible.

2. Let $a_n(\mathbf{x})$ and $g_n(\mathbf{x})$ be the unweighted arithmetic and geometric means of $\mathbf{x} = (x_1, ..., x_n)$ with $x_i > 0$ $(i = 1, ..., n)$, i.e.,

$$a_n = a_n(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad \text{and} \qquad g_n = g_n(\mathbf{x}) = \prod_{i=1}^{n} x_i^{1/n}.$$

The well-known Popoviciu inequality states

$$(a_{n+1}/g_{n+1})^{n+1} \geqslant (a_n/g_n)^n. \tag{14}$$

Martins [3] pointed out that inequality (2) presents a refinement of (14) for the special case $x_i = i^r$ $(i = 1, ..., n+1)$. Indeed, (2) is equivalent to

$$a_{n+1}/g_{n+1} \geqslant a_n/g_n \tag{15}$$

with $x_i = i^r$ $(i = 1, ..., n+1)$ and $r > 0$. It is worth mentioning that there exists an additive analogue of (15),

$$a_{n+1} - g_{n+1} \geqslant a_n - g_n \tag{16}$$

with $x_i = i^r$ $(i = 1, ..., n+1)$ and $r > 0$. A proof of (16) runs as follows.

From (15) we get

$$a_{n+1} - g_{n+1} \geqslant g_{n+1}(a_n/g_n - 1). \tag{17}$$

Because of $g_{n+1} \geqslant g_n$ (which is equivalent to the elementary inequality $((n+1)!)^{1/(n+1)} \geqslant (n!)^{1/n}$) we conclude from (17) and the arithmetic mean-geometric mean inequality

$$a_{n+1} - g_{n+1} \geqslant g_n(a_n/g_n - 1) = a_n - g_n.$$

We note that inequality (16) is a sharpening of the classical Rado inequality

$$(n+1)(a_{n+1} - g_{n+1}) \geqslant n(a_n - g_n)$$

which is valid for all $\mathbf{x} = (x_1, ..., x_{n+1})$ with $x_i > 0$ $(i = 1, ..., n+1)$. Many remarkable extensions and refinements of the inequalities of Rado and Popoviciu can be found in monograph [1].

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