The Exterior Robin Problem for the Helmholtz Equation

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A new method is given for solving the exterior Robin problem for the Helmholtz equation. The problem is reformulated as a new integral equation which is continuous as the field point approaches the boundary. It is shown that its solution can be represented as a convergent Neumann series for convex surfaces, for small values of the wave number. Examples are included which illustrate the method.

1. Introduction

Exterior problems for the Helmholtz and Laplace equations have been extensively studied by mathematicians. In one such problem, known as the Robin problem, the quantity $\gamma u + \partial u / \partial n$ is given on the boundary. For the case when $\gamma = 0$, the problem reduces to the exterior Neumann problem. The important problem of constructing the solution, for arbitrarily shaped, compact, simply connected surfaces has received considerable attention (e.g. see [2], [3] and [28] and the references given there). In the limiting case $\gamma \to \infty$, the problem reduces to the exterior Dirichlet problem. Constructive methods for solving this problem for smooth, simply connected, compact scattering surfaces are also available (e.g. see [1], [12], [19] and [28]). For finite, but non-zero values of $\gamma$, the construction of the solution, however, becomes considerably more difficult. Günter (see [11, Chap. 5, Sec. 7]) represents the solution to the interior Robin problem for Laplace's equation as an infinite series, the terms of which involve integrals with the Neumann function as the kernel function. Unfortunately an explicit representation for this Neumann function is generally not available. Except for those geometries where the solution can be expanded in terms of known eigenfunctions, no other constructive methods for this problem are known.

This paper gives a method for solving the exterior Robin problem in three dimensions. The problem is reformulated as an integral representation which is continuous as the field point approaches the boundary. An iteration scheme is given for solving this integral equation for smooth, simply connected, compact,

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convex surfaces, for small values of $\gamma$, for sufficiently small values of the wave number. In the limiting case, $\gamma = 0$, the method here reduces to the generalization of Neumann's method obtained by Ahner and Kleinman [2] and by Kleinman and Wendland [20] for solving the exterior Neumann problem. Excellent numerical results have been obtained for the Neumann problem by Jones and Kleinman [15] using the method in [2] and [20].

Numerical computations for three-dimensional Dirichlet and Neumann scattering problems based on different integral equation formulations have been obtained by Brundrit [5], Chen and Schweikert [6], Chertok [7], Copley [8], Fenlon [10], Hess [13], Peterson and Ström [25] and for the limiting case $k = 0$ by Hess and Smith [14], Lynn and Timlake [22] and Wendland [29]. Unfortunately, a comparison of these methods along with the generalized Neumann method is not available. Numerical computations for the Robin problem have not been done.

In the next section, the notation used is defined, and precise statements are given of the potential and scattering problems considered. In Section 3, the potential and scattering problems are reformulated as new integral equations. In Section 4, it is shown that the solution to the potential problem can be expressed as a Neumann series provided the surface bounds a convex region and the constant appearing in the boundary condition satisfies a certain inequality. The method of shifting eigenvalues, used by Kantorovich and Krylov [17] for the interior Dirichlet potential problem, is employed there to demonstrate the convergence of the series. In Section 5, it is shown that the solution to the integral equation in Section 3 for the scattering problem can be expressed as a Neumann series for small, but non-zero values of the wave number. In Section 6, one potential and one scattering problem are given illustrating the method. In the last section a discussion is given on the results obtained here and the recent results of Kleinman and Wendland [20].

2. The Robin Problem

Here we first establish the notation that is to be used throughout this paper and then give a precise statement of the problems that will be considered. $S$ denotes a smooth, simply connected, compact surface in $\mathbb{R}^3$ on which a Hölder continuous normal exists everywhere. The regions inside and outside of $S$ are denoted by $V_j$ and $V_o$, respectively. Let $0 \in V_j$ be the origin of a rectangular coordinate system and the position vector of a typical point will be designated by $x$. $\hat{n}_y$ represents a unit normal vector on $S$ at $y$ and is directed into $V_o$. The distance between two points $x$ and $y$ will be denoted by $r$ and the distance $r(0, x)$ will be denoted by $\rho$. 
The first problem considered is the exterior Robin problem for Laplace’s equation, which we denote by \( P_0 \). Here we wish to find the total field \( u_0(x) \) such that

\[
\begin{align*}
\nabla^2 u_0(x) &= 0 & x &\in \Gamma, \\
\gamma u_0(x) + \frac{\partial}{\partial n} u_0(x) &= 0 & x &\in S,
\end{align*}
\]

\( u_0(x) \) is regular at infinity.

\( u_0^i(x) \) is a given incident field and satisfies Laplace’s equation in \( V_i \) and is continuous along with its first partial derivatives everywhere except possibly in some compact set in \( V_e \). A complex function is regular at infinity if both its real and imaginary parts are regular in the sense of Kellogg [18, p. 217]. For \( \gamma \) a complex constant, \( P_0 \) has a unique solution except for \( \gamma > 0 \) (see [24]). In this case a discrete set of eigenvalues \( \gamma_i \) exist. For example, in the case of a sphere of radius \( a, \gamma - 1/a \) and \( u^i = 0 \), the function

\[
\phi(x) = \frac{1}{2\pi} \int_S \frac{1}{r} \, dS
\]

(2.1)

satisfies Laplace’s equation in \( V_e \), is regular at infinity, and satisfies the Robin boundary condition.

We also consider the exterior Robin problem for the Helmholtz equation which we denote by \( P \). Here we wish to find the function \( u(x) \) such that

\[
\begin{align*}
\nabla^2 u(x) + k^2 u(x) &= 0 & x &\in V, \\
\gamma u(x) + \frac{\partial}{\partial n} u(x) &= 0 & x &\in S
\end{align*}
\]

\( u(x) \) is a known incident field satisfying the Helmholtz equation in \( V_i \) and is continuous along with its first partial derivatives everywhere, except possibly in some compact set in \( V_e \). For \( \gamma \) a complex constant, \( P \) has a unique solution except for a discrete set of real values. We assume throughout that \( \gamma \) is a constant.

3. Integral Representations

Using Green’s identities (see [9, p. 256]) and the fact that \( u_0^i(x) \) satisfies Laplace’s in \( V_i \), the following integral representation for \( u_0(x) \) is obtained:
\[ u_0^i(x) + \frac{1}{4\pi} \int_S u_0(y) \left( \frac{\partial}{\partial n_\gamma} \frac{1}{r} + \gamma \frac{1}{r} \right) dS_y = u_0(x) \quad x \in \mathcal{V}_e \]
\[ = \frac{1}{2} u_0(x) \quad x \in S \quad (3.1) \]
\[ = 0 \quad x \in \mathcal{V}_i. \]

Letting
\[ K_0 u_0(x) = \frac{1}{2\pi} \int_S u_0(y) \frac{\partial}{\partial n_\gamma} \frac{1}{r} dS_y \quad (3.2) \]
and
\[ Q_0 u_0(x) = \frac{1}{2\pi} \int_S u_0(y) \frac{1}{r} dS_y \quad (3.3) \]

\( u_0(x) \) then satisfies the following integral equation for \( x \in S \)
\[ u_0(x) = 2u_0^i(x) + K_0 u_0(x) + \gamma Q_0 u_0(x). \quad (3.4) \]

We now employ the following form of Gauss' integral identity:
\[ \frac{1}{2\pi} \int_S \frac{\partial}{\partial n_\gamma} \frac{1}{r} dS_y = 0 \quad x \in \mathcal{V}_e \]
\[ = -1 \quad x \in S \quad (3.5) \]
\[ = -2 \quad x \in \mathcal{V}_i. \]

By multiplying both sides of Eq. (3.5) by \( \frac{1}{2} u_0(x) \) and subtracting from Eq. (3.1), the following integral representation is obtained for \( x \in S \cup \mathcal{V}_e \)
\[ u_0(x) = u_0^i(x) + Lu_0(x) + \frac{\gamma}{2} Q_0 u_0(x) \quad (3.6) \]
where
\[ Lu_0(x) = \frac{1}{4\pi} \int_S [u(y) - u(x)] \frac{\partial}{\partial n_\gamma} \frac{1}{r} dS_y. \quad (3.7) \]

Observe that (3.6) is valid for both points in \( \mathcal{V}_e \) and on \( S \) and does not exhibit the discontinuity that the standard integral representation (3.1) possesses as the field point approaches the boundary. For this reason (3.6) is said to be continuous as the field point approaches the boundary.

Now consider the scattering problem \( P \). Analogous to Eq. (3.6), the following integral equation may be obtained for \( x \in S \cup \mathcal{V}_e \):
\[ u(x) - u^i(x) + Lu(x) + \frac{\gamma}{2} Qu(x) + Mu(x) \quad (3.8) \]
where
\[ Qu(x) = \frac{1}{2\pi} \int_S u(y) \frac{e^{ikr}}{r} dS_y \quad (3.9) \]
and where
\[ M u(x) = \frac{1}{4\pi} \int_S u(y) \frac{\partial}{\partial n_y} \left( \frac{e^{ikr} - 1}{r} \right) dS_y. \] (3.10)

4. A Neumann Series for the Solution to the Potential Problem

Here it is shown, for suitably restricted values of \( \gamma \), that
\[ u_0(x) = \sum_{n=0}^{\infty} \left( L + \frac{\gamma}{2} Q_0 \right)^n u_0'(x) \] (4.1)
is the solution to \( P_0 \) on \( S \). Once the solution is known on \( S \), from the integral representation (3.6), it is also known in \( V_\epsilon \).

Let \( C_R[S] \) and \( C_C[S] \) denote the spaces of continuous functions defined on \( S \), real and complex respectively. If \( f \in C_R[S] \), define
\[ \| f \| = \sup_{x \in S} |f(x)| \] (4.2)
and if \( f \in C_C[S] \), define
\[ \| f \| = \sup_{x \in S} \left| \text{Re} f \cos \theta + \text{Im} f \sin \theta \right|. \] (4.3)
The spaces \( C_R[S] \) and \( C_C[S] \) are complete with respect to the norms (4.2) and (4.3) respectively (see [16]). For any real linear transformation \( R \), i.e., \( T: C_R[S] \to C_R[S] \), we have
\[ \| T \|_R = \sup_{\| f \| = 1} \| Tf \| \] (4.4)
with respect to the norm in (4.2). For any linear transformation \( T \) mapping \( C_C[S] \) into \( C_C[S] \), then
\[ \| T \|_C = \sup_{\| f \| = 1} \| Tf \| \] (4.5)
with respect to the norm in (4.3). If \( T \) is a real operator defined on \( C_C[S] \), then it can be shown (see [16, p. 500]) that
\[ \| T \|_R = \| T \|_C \] (4.6)
and consequently, the subscript of the norm will hereafter be omitted. The result in (4.6) will be useful later in keeping the same bound on the constant \( \gamma \), appearing of the Neumann series for both \( P_0 \) and \( P \).
Consider the following homogeneous equation corresponding to (3.4)

$$\phi(x) = \lambda(K_0 + \gamma Q_0) \phi(x)$$  \hspace{1cm} (4.7)

and let $\lambda_j$ and $\phi_j$ denote its eigenvalues and eigenfunctions respectively. Consider also the adjoint equation of (4.7), obtained by interchanging the variables in the kernel functions contained in the integral operators:

$$\psi(x) = r(K_0^* - \gamma Q_0) \psi(x)$$  \hspace{1cm} (4.8)

where

$$K_0^* \psi(x) = \frac{1}{2\pi} \int_S \psi(y) \frac{\partial}{\partial n_x} \frac{1}{r} dS_y$$  \hspace{1cm} (4.9)

and where the asterisk of $Q_0$ is omitted, since the kernel of $Q_0$ is symmetric. Let $\tau_j$ and $\psi_j$ denote the eigenvalues and eigenfunctions of (4.8) respectively. In Appendix A the following results are derived for $\gamma \leqslant 0$:

The eigenvalues $\tau_j$ are real; thus $\lambda_j = \tau_j$  \hspace{1cm} (4.10)

$$\lambda_j = \frac{D_i + D_e}{D_i - D_e + 2\gamma H}$$  \hspace{1cm} (4.11)

where $D_i$ and $D_e$ denote the Dirichlet integrals of $Q_0 \psi_j$ over $V_\gamma$ and $V_\epsilon$ respectively and

$$H = \int_S (Q_0 \psi_j)^2 dS.$$  \hspace{1cm} (4.12)

For the case $\gamma = 0$, it is seen from (4.11) that

$$|\lambda_j| \geqslant 1$$

which corresponds to the result found in Kellogg (p. 310). This result, however, is not true in general when $\gamma \neq 0$.

We now prove the following lemma

**Lemma 4.1.** If $\gamma < 0$, then the integral equation (4.8) has a unique solution for $0 \leqslant \lambda \leqslant 1$, where $\tau = \lambda$.

**Proof.** Suppose the integral equation has an eigenvalue $\lambda_j \in [0, 1]$. Then from (4.11) it follows that

$$(-1 + \lambda_j) D_i + 2\gamma \lambda_j H = (1 + \lambda_j) D_e.$$  \hspace{1cm} (4.13)

Now

$$D_i \geqslant 0, \quad D_e \geqslant 0, \quad H \geqslant 0$$
and thus it is seen that the right hand side of (4.13) is nonnegative and that the left hand side is nonpositive. If \(0 \leq \lambda_j < 1\), then \(D_i = D_e = H = 0\) and thus \(Q_0\phi_j \equiv 0\) everywhere. Taking the normal derivative of \(Q_0\phi_j\) from \(V_i\) and from \(V_e\) and subtracting it follows that (see \([27]\))

\[
2\phi_j(x) = \frac{\partial}{\partial n_-} Q_0\phi_j - \frac{\partial}{\partial n_+} Q_0\phi_j = 0 \quad x \in S
\]

which contradicts our assumption that \(\lambda_j\) is an eigenvalue.

If \(\lambda_j = 1\), then it follows that \(D_e = H = 0\) and thus \(Q_0\phi_j \equiv 0\) in \(V_e\) and on \(S\). This function, however, satisfies Laplace’s equation inside \(V_i\) and vanishes on \(S\). By the uniqueness of the interior Dirichlet problem it follows that \(Q_0\phi_j\) vanishes inside \(V_i\) and hence in this case also \(\phi_j = 0\) on \(S\).

Next we establish

**Lemma 4.2.** If \(\gamma < 0\), then the integral equation (4.7) has a unique solution when

\[
\frac{-1}{\|K_0\| - \gamma \|Q_0\|} < \lambda < \frac{1}{\|K_0\| - \gamma \|Q_0\|}
\]

*Proof.* Let \(R_\gamma\) denote the spectral radius of \(K_0 + \gamma Q_0\). Then

\[
R_\gamma = \sup_{i=1,2,3,...} \frac{1}{|\lambda_i|} = \frac{1}{|\lambda_1|} \quad (4.14)
\]

where \(\lambda_1\) denotes the eigenvalue of (4.7) with smallest magnitude. The integral equation (4.7) has a unique solution for

\[
- |\lambda_1| < \lambda < |\lambda_1| \quad (4.15)
\]

It can be shown that

\[
\|K_0\| - \gamma \|Q_0\| \geq \|K_0 + \gamma Q_0\| \geq R_\gamma = \frac{1}{|\lambda_1|} \quad (4.16)
\]

From this inequalities (4.15) and (4.16), the conclusion of the lemma follows.

From this lemma, we may derive the following important result:

**Lemma 4.3.** If \(\gamma < 0\), then the integral equation (4.7) has a unique solution for \(-\frac{1}{\alpha} \leq \lambda < 0\), provided \((\alpha - \|K_0\|)/\|Q_0\| > -\gamma > 0\).

*Proof.* Suppose

\[
\frac{\alpha - \|K_0\|}{\|Q_0\|} > -\gamma > 0 \quad (4.17)
\]

Then

\[
\alpha > \|K_0\| - \gamma \|Q_0\| > 0 \quad (4.18)
\]
and it follows that

\[ \frac{-1}{\|K_0\| - \gamma \|Q_0\|} < \frac{1}{\alpha}. \]  

(4.19)

From Lemma 4.2 it follows that the integral equation (4.7) has a unique solution for \(-1(1/\alpha) \leq \lambda < 0\).

Later we shall set \(\alpha = 3\) in Lemma 4.3, which presupposes that \(3 - \|K_0\| > 0\). The next lemma gives a sufficient condition, when this inequality is valid.

**Lemma 4.4.** If \(S\) bounds a convex region, then \(\|K_0\| = 1\).

**Proof.** For \(f \in C_b(S)\) and \(\|f\| = 1\), we have

\[ \|K_0f\| \leq \sup_{x \in S} \frac{1}{2\pi} \int_S \left| \frac{\partial}{\partial n_y} \frac{1}{|x-y|} \right| dS_y. \]

It follows that

\[ \left| \frac{\partial}{\partial n_y} \frac{1}{|x-y|} \right| = \frac{\cos \beta}{|x-y|^2} = \frac{\cos \beta}{|x-y|^2} = \frac{\partial}{\partial n_y} \frac{1}{|x-y|} \]

where \(\beta\) is the angle between the outward unit normal at \(y\) and the vector from \(x\) to \(y\) and is less than or equal to \(\pi/2\) since \(S\) is convex. From Eq. (3.5) we obtain

\[ \|K_0f\| \leq 1. \]

Setting \(f = 1\), it follows from Eq. (3.5) that \(K_01 = -1\) and thus \(\|K_0\| = 1\).

We now may prove the following theorem

**Theorem 4.1.** For convex surfaces \(S\), the integral equation

\[ \phi = \lambda(K_0 + \gamma Q_0) \phi \]

has only the trivial solution for \(-1(1/\alpha) \leq \lambda \leq 1\), provided \((\alpha - 1)/\|Q_0\| > -\gamma > 0\).

**Proof.** First suppose \(\gamma < 0\). Then the conclusion follows immediately from Lemmas 4.1–4.4. If \(\gamma = 0\), then the theorem follows from Kellogg [18, pp. 309–312], where in fact the result is established for \(-1 < \lambda \leq 1\) and no convexity requirement is assumed.

Using Theorem 4.1, we now show that the integral equation (3.6) can be solved iteratively. This can be done by the method of shifting eigenvalues (see [17]). In essence we transform the parameter \(\lambda\) in

\[ u_0(x) = 2u_0'(x) + \lambda(K_0 + \gamma Q_0) u_0(x) \]

(4.21)
so that the solution to the integral equation (3.6), corresponding to $\lambda = 1$, can be represented as a convergent Neumann series. Let $\lambda = \eta/(1 - \eta)$. Using Gauss' identity we obtain after some simplification

$$ u_0(x) = 2(1 - \eta) u_0^i(x) + \eta(2L + \gamma Q_0) u_0(x) $$

(4.22)

which corresponds to (3.6) for $\eta = \frac{1}{2}$. Solving for $\eta$ in terms of $\lambda$ we have

$$ \eta = \frac{\lambda}{1 + \lambda} $$

(4.23)

and it follows that Eq. (4.22) has a unique solution for

$$ |\eta| < |\eta_1| = \inf_{j=1,2,3...} \frac{|\lambda_j|}{1 + \lambda} $$

(4.24)

where $\lambda_j$ denotes the eigenvalues of (4.21) and $\eta_1$ is the smallest eigenvalue of Eq. (4.22). The transformation for $\eta$ maps the interval $-\frac{1}{2} \leq \lambda \leq 1$ continuously onto the interval $-\frac{1}{2} \leq \eta \leq \frac{1}{2}$. Setting $\alpha = 3$ in Theorem 4.1, it follows that $\frac{1}{2} < |\eta_1|$ and thus (4.22) has a unique solution for $|\eta| \leq \frac{1}{2}$. Putting $\eta = \frac{1}{2}$ in (4.22), it follows that the solution can be represented as the Neumann series in (4.1) which converges for $x \in S$, when $S$ is convex and when

$$ \frac{2}{\|Q_0\|} > -\gamma \geq 0. $$

(4.25)

5. A Neumann Series for the Solution to the Scattering Problem

In this section it is shown that the solution to (3.8) can be represented as the Neumann series

$$ u(x) = \sum_{n=0}^{\infty} \left( L + \frac{\gamma}{2} Q + M \right)^n u^i(x) \quad x \in S $$

(5.1)

for small values of the wave number $k$.

Consider the integral equation

$$ u(x) = 2(1 - \eta) u^i(x) + \eta(2L + \gamma Q + 2M) u(x) \quad x \in S. $$

(5.2)

We assume that $S$ is convex and that $\gamma$ satisfies the inequality (4.25). Rewrite Eq. (5.2) as

$$ T_{\eta} u(x) = 2(1 - \eta) u^i(x) + \eta(\gamma N + 2M) u(x) $$

(5.3)

where

$$ T_{\eta} u(x) = (I - \eta(2L + \gamma Q_0)) u(x), $$

(5.4)
where $I$ denotes the identity operator, and where

$$Nu(x) = (Q - Q_0) u(x).$$  \hfill (5.5)

In the last section it was shown that $T_n$ is invertible for $|\eta| \leq \frac{1}{2}$. From Eq. (5.3) we have

$$\{I - \eta T_n^{-1}(\gamma N + 2M)\} u(x) = 2T_n^{-1}u(x)$$  \hfill (5.6)

which has a unique solution whenever

$$\|\eta T_n^{-1}(\gamma N + 2M)\| < 1.$$  \hfill (5.7)

Although Eq. (5.6) is soluble as a Neumann series, it involves the operator $T_n^{-1}$ which is not known explicitly. Thus the usefulness of such a representation is limited.

From the Bounded Inverse Theorem (see [4, p. 271]), $T_n^{-1}$ is a bounded linear operator on $C_c[S]$. The operators $M$ and $N$ have continuous kernel functions for all values of $x$ and $y$ on $S$. Therefore these operators are bounded and it can be shown that

$$\|2M\| \leq \frac{2}{\pi} Ak^2$$  \hfill (5.8)

and

$$\|\gamma N\| \leq \frac{\gamma}{\pi} Ak$$  \hfill (5.9)

where $A$ is the surface area of $S$. From Eqs. (5.7), (5.8), and (5.9) it follows that the operator equation (5.6) has a unique solution for $|\eta| \leq \frac{1}{2}$ when

$$|k| < \left[ \frac{\pi}{2A} \left(\|\eta\| \|T_n^{-1}\|^{-1} + \frac{\gamma^2}{16}\right)^{1/2} - \frac{\gamma}{4} \right].$$  \hfill (5.10)

Therefore Eq. (5.2) also has a unique solution for sufficiently small values of $k$, for $|\eta| \leq \frac{1}{2}$. Setting $\eta = \frac{1}{2}$ we get (3.8) whose solution may thus be represented as the Neumann series (5.1).

While the convergence of the series (4.1) and (5.1) has been proven only for values of $\gamma$ satisfying Eq. (4.25) for values of $k$ satisfying Eq. (5.10), it is believed that these series will also converge for complex values of $\gamma$, whose magnitudes satisfy the inequality (4.25), provided in the case of (5.1), the wave number is sufficiently small. To give support of this conjecture, two examples are solved in the next section for $\gamma$ a complex constant.

6. Examples

In this section one potential problem and one scattering problem will be solved by calculating the solution on $S$ from the Neumann series (4.1) and (5.1)
respectively. \( \gamma \) is assumed to be a complex constant. Consider first the problem \( P_0 \) for a sphere of radius \( a \) and

\[
u_0^i(x) = \frac{1}{|x - x_e|} \quad x_e \in V_e.\]  

(6.1)

To calculate the solution from the series (4.1), we shall first find the general term and then sum the Neumann series. From [26, p. 851] we have that

\[
\frac{1}{|x - x_e|} = \sum_{m=0}^{\infty} \frac{a^m}{\rho_e^{m+1}} P_m(\cos \beta) \]  

(6.2)

where \( \rho_e = |x_e| > a = |x| \) and \( \cos \beta = \hat{r}(0, x_e) \cdot \hat{r}(0, x) \). To evaluate the term

\[
(L + \frac{\gamma}{2} Q_0) u_0^i(x) = \frac{1}{2} \frac{1}{|x - x_e|} + \frac{1}{4\pi} \int_S \frac{1}{|y - x_e|} \frac{\partial}{\partial n_x} \frac{1}{|x - y|} dS_y 
+ \frac{\gamma}{4\pi} \int_S \frac{1}{|y - x_e|} \cdot \frac{1}{|x - y|} dS_y \]  

(6.3)

we employ an average of two expansions for \( 1/|x - y| \) similar to (6.2) with \( a_+ \) in one and \( a_- \) in the other. On the sphere, \( \hat{\theta}/\partial n_x = \hat{\theta}/(\hat{\theta} |y|) \); using the orthogonality of the Legendre polynomial and then letting \( a_+ = a_- = a \), we obtain after some simplification

\[
(L + \frac{\gamma}{2} Q_0) u_0^i(x) = \sum_{m=0}^{\infty} \frac{m + \gamma a}{2m + 1} \frac{a^m}{\rho_e^{m+1}} P_m(\cos \beta). \]  

(6.4)

By an induction argument it can be shown that the general term of the series (4.1) is

\[
(L + \frac{\gamma}{2} Q_0)^n u_0^i(x) = \sum_{m=0}^{\infty} \left( \frac{m + \gamma a}{2m + 1} \right)^n \frac{a^m}{\rho_e^{m+1}} P_m(\cos \beta). \]  

(6.5)

The total field may now be found from (6.5) by summing the geometric series with terms \( (m + \gamma a)/(2m + 1) \), provided this term is less than unity in absolute value for \( m = 0, 1, 2, \ldots, \)

\[
u_0(x) = \sum_{n=0}^{\infty} \left( L + \frac{\gamma}{2} Q_0 \right)^n u_0^i(x) = \sum_{m=0}^{\infty} \frac{2m + 1}{m + 1 - \gamma a} \frac{a^m}{\rho_e^m} P_m(\cos \beta) \]  

(6.6)

where \( x \in S \). This agrees with the result using separation of variables. If

\[
a |\gamma| < 1 \]  

(6.7)

it is seen for \( m \geq 0 \) that

\[
\left| \frac{m + \gamma a}{2m + 1} \right| \leq \frac{m |a| \gamma}{2m + 1} < 1. \]  

(6.8)
Thus from the example in Section 2, the series (4.1) converges for \(|\gamma|\) less than the smallest eigenvalue of \(P_{0}\).

We now compare the condition (6.7) with (4.25), replacing \(-\gamma\) by \(|\gamma|\) there. For a sphere, it can be shown

\[
\frac{\partial}{\partial n_x} \frac{1}{r(x, y)} = -\frac{1}{2ar(x, y)} \quad x, y \in S. \quad (6.9)
\]

Thus

\[
(Q_0) (x) = \frac{1}{2\pi} \int_S \frac{1}{r(x, y)} dS_y = -\frac{a}{\pi} \int_S \frac{\partial}{\partial n_x} \frac{1}{r(x, y)} dS_y = 2a \quad (6.10)
\]

where the last equation follows from the Gauss integral identity (3.5). For \(f \in C_r[S]\) and \(\|f\| = 1\), we have

\[
\|Q_0 f\| \leq \frac{1}{2\pi} \int_S \frac{1}{r(x, y)} dS_y = 2a. \quad (6.11)
\]

Consequently, from (6.10) and (6.11)

\[
\|Q_0\| = 2a. \quad (6.12)
\]

Thus for a sphere it is seen that the inequalities (6.7) and (4.25) are equivalent.

Next consider the exterior Robin scattering problem for a plane wave incident on a sphere of radius \(a\). Let the coordinate system be oriented so that the origin coincides with the center and the \(z\) axis is aligned with the direction \(\hat{k}\) of the plane wave so that \(\hat{z} = -\hat{k}\). From [23, pp. 107–108] we have

\[
e^{ikr_0 \cos \theta} = \sum_{m=0}^{\infty} i^m (2m + 1) j_m(kr_0) P_m(\cos \theta) \quad (6.13)
\]

\[
e^{ik|x-y|} = ik \sum_{m=0}^{\infty} (2m + 1) j_m(kr) h_m^{(1)}(kr) P_m(\cos \nu) \quad (6.14)
\]

where \(\cos \nu = \hat{r}(0, y) \cdot \hat{r}(0, x)\) and \(j_m(kr)\) and \(h_m^{(1)}(kr)\) are spherical Bessel and Hankel functions. To evaluate the series (5.1) we first calculate

\[
\left( L + \frac{1}{2} Q + M \right) w^i(x)
\]

\[
= \frac{1}{2} \sum_{m=0}^{\infty} i^m (2m + 1) j_m(ka) P_m(\cos \theta_x)
\]

\[
+ \frac{1}{4\pi} \int_S \left( \sum_{m=0}^{\infty} i^m (2m + 1) j_m(ka) P_m(\cos \theta) \right) \]
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\[ \times \left\{ \frac{ik^2}{2} \sum_{n=0}^{\infty} (2n + 1) \left[ j_n(ka) h_n^{(1)}(ka) + j_n(ka)^* h_n^{(1)}(ka^*) \right] P_n(\cos \nu) \right\} dS_r \]

\[ + \frac{\gamma}{4\pi} \int_S \left\{ \sum_{m=0}^{\infty} i^m(2m + 1) j_m(ka) P_m(\cos \theta) \right\} \]

\[ \times \left\{ ik \sum_{n=0}^{\infty} (2n + 1) j_n(ka) h_n^{(1)}(ka) P_n(\cos \nu) \right\} dS_r . \]  

(6.15)

Using the Wronskian identity (see [23, p. 68])

\[ ik^2 a^2 \left[ j_n(ka) h_n^{(1)}(ka)^* - j_n(ka)^* h_n^{(1)}(ka) \right] = -1 \]  

(6.16)

and the orthogonality of the Legendre polynomials, it follows that

\[ \left( L + \frac{\gamma}{2} Q + M \right) u^i(x) = \sum_{m=0}^{\infty} i^m(2m + 1) j_m(ka) A_m(ka) P_m(\cos \theta_x) \]  

(6.17)

where

\[ A_m(ka) = ika^2 h_m^{(1)}(ka) [b_j_m(ka)^* + \gamma j_m(ka)]. \]  

(6.18)

From an induction argument, the following is established

\[ \left( L + \frac{\gamma}{2} Q + M \right)^n u^i(x) = \sum_{m=0}^{\infty} i^m(2m + 1) j_m(ka) [A_m(ka)]^n P_m(\cos \theta_x). \]  

(6.19)

In Appendix B it is shown that

\[ |A_m(ka)| < 1 \]  

(6.20)

for \( m \geq 0 \) and \( a \frac{\gamma}{2} \sigma < 1 \) provided \( k \) is sufficiently small. With these restrictions and using the Wronskian identity we obtain

\[ u(x) = \sum_{n=0}^{\infty} \left( L + \frac{\gamma}{2} Q + M \right)^n u^i(x) \]  

(6.21)

\[ = \sum_{m=0}^{\infty} i^m(2m + 1) P_m(\cos \theta_x) \]  

(6.22)

where the geometric series in powers of \( A_m(ka) \) has been summed.

7. COMPARISON OF THE RESULTS HERE WITH THOSE OF KLEINMAN AND WENDLAND

Kleinman and Wendland [20] have recently shown the generalized Neumann method given by Ahner and Kleinman [2] for solving \( P \) for \( \gamma = 0 \) to be valid.
for the general class of boundary surfaces characterized by Kral [21] which include piecewise Lyapunoff surfaces excluding spines. Thus the series representation they obtain for the solution on the boundary is identical to the series (5.1) for the case $\gamma = 0$. Their work is particularly important in that first, no condition of convexity is imposed on the scattering surface, and second, they do not assume an everywhere continuously turning normal. Both of these conditions were imposed here. Thus the results of Kleinman and Wendland removes an old objection (see [9, p. 303]) for using integral equations for solving potential problems: “in spite of its elegance the method of integral equations is inferior to the previously developed procedure (the Schwartz alternating procedure), since the occurrence of even an ordinary corner leads to singularities of the kernel, so that the immediate application of Fredholm’s theory is ruled out”. It should be pointed out that in [9] the discussion was directed to the two-dimensional problem for Laplace’s equation. But the difficulties with corners has also been a serious problem for solving three-dimensional scattering problems based on an approach by integral equations. Thus their work is extremely significant. Unfortunately their results were not available when the research for this paper was completed.

On the other hand, the “generalized Neumann method” has been shown here to be valid also for finding the solution to $P$ for small but non-zero values of $\gamma$. This is particularly important since practical methods for solving this problem, unlike the corresponding Neumann and Dirichlet problems, have not previously been available.

There are certain points of similarity between the method of Kleinman and Wendland [20] and the one here. In both [20] and here the results of Plemelj (see [18, pp. 309–310]) are extended. In [20] to include non-smooth surfaces and here for $\gamma$ different from 0 (see Appendix A). Also, both use the method of shifting eigenvalues which is based on an analytic transformation of integral operators (see [17], p. 118).

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**APPENDIX A**

Consider the integral equation

$$\psi(x) = \tau(K_0^* + \gamma Q_0) \psi(x) \quad x \in S$$

(A.1)
where $K^*_\alpha$ and $Q_\alpha$ are defined in (4.9) and (3.3) respectively and where $\gamma \leq 0$. Let $\tau_j$ and $\psi_j(x)$ denote the eigenfunctions of (A.1) respectively. We prove here the following results

(1) The $\tau_j$'s are real

(2) 
\[ \tau_j = \frac{D_i + D_e}{D_i - D_e + 2\gamma H} \]

where $D_i$ and $D_e$ are the Dirichlet integrals of $Q_\alpha \psi_j$ over $V_i$ and $V_e$ respectively and $H$ is defined in (4.12). These results have been established for the case $\gamma = 0$ (see [18], pp. 309–310) and by generalizing the proof given in [18], we prove their validity for the case $\gamma < 0$.

Let

\[ V(x) = Q_\alpha \psi(x). \]  

(A.2)

Taking the normal derivatives of $V(x)$ as $x$ approaches $S$ from $V_e$ and $V_i$ respectively we have that

\[ \frac{\partial}{\partial n_+} V(x) = -\psi(x) + K^*_\alpha \psi(x), \]

\[ \frac{\partial}{\partial n_-} V(x) = \psi(x) + K^*_\alpha \psi(x). \]

Thus

\[ \frac{1 - \tau}{2} \left\{ \frac{\partial}{\partial n_-} + \gamma \right\} V - \frac{1 + \tau}{2} \left\{ \frac{\partial}{\partial n_+} + \gamma \right\} V = \psi - \tau(K^* \alpha + \gamma Q_\alpha) \psi = 0. \]  

(A.3)

Let $\tau = \alpha + i\beta$ be an eigenvalue of (A.1) with corresponding eigenfunction $\psi(x) = \psi_1(x) + i\psi_2(x)$ and let us suppose that $\beta \neq 0$. Let $V = V_1 + iV_2$. The real and imaginary parts of the left hand side of (A.3) yield

\[ (1 - \alpha) \frac{\partial V_1}{\partial n_-} - (1 + \alpha) \frac{\partial V_1}{\partial n_+} + \beta \left( \frac{\partial V_2}{\partial n_-} + \frac{\partial V_2}{\partial n_+} \right) - 2\alpha \gamma V_1 + 2\beta \gamma V_2 = 0 \]  

(A.4)

\[ (1 - \alpha) \frac{\partial V_2}{\partial n_-} - (1 + \alpha) \frac{\partial V_2}{\partial n_+} - \beta \left( \frac{\partial V_1}{\partial n_-} + \frac{\partial V_1}{\partial n_+} \right) - 2\alpha \gamma V_2 - 2\beta \gamma V_1 = 0. \]  

(A.5)

Multiplying these equations by $V_2$ and $V_1$ respectively, subtracting and integrating over $S$ we obtain

\[ \beta[D_{i,1} + D_{i,2}] - \beta[D_{e,1} + D_{e,2}] + 2\beta \gamma[H_1 + H_2] = 0. \]  

(A.6)

Multiplying (A.4) by $V_1$ and (A.5) by $V_2$, adding and integrating over $S$ we get

\[ (1 - \alpha) [D_{i,1} + D_{i,2}] + (1 + \alpha) [D_{e,1} + D_{e,2}] - 2\alpha \gamma[H_1 + H_2] = 0. \]  

(A.7)
Solving for $D_{e,1} + D_{e,2}$ from (A.6) and (A.7) we find

$$D_{e,1} + D_{e,2} = \gamma[H_1 + H_2].$$  \hspace{1cm} (A.8)

Since $\gamma < 0$, (A.8) implies that $D_{e,1} = D_{e,2} = H_1 = H_2 = 0$. Solving (A.6) and (A.7) for $D_{i,1} + D_{i,2}$ we have

$$D_{i,1} + D_{i,2} = -\gamma[H_1 + H_2] = 0. \hspace{1cm} (A.9)$$

Thus $D_{i,1} = D_{i,2} = 0$. It follows that $\psi = 0$ on $S$, contradicting the fact that $\tau$ is an eigenvalue. Thus our assumption that $\beta \neq 0$ is invalid.

The second result we wished to establish follows immediately from (A.7), where $D_{e,2} = D_{e,2} = H_2 = 0$.

**APPENDIX B**

Here the inequality (6.20) is established for $m \geq 0$ and $|\gamma a| < \sigma < 1$ provided $ka$ is small. For $m = 0$ we have from [23, p. 73]

$$j_0(ka) = \frac{\sin ka}{ka} \hspace{1cm} (B.1)$$

and

$$h_0^{(1)}(ka) = -i \frac{e^{ika}}{ka} \hspace{1cm} (B.2)$$

After some simplification, it can be shown that

$$|\Lambda_m(ka)| = \left| (1 - \gamma a) \frac{\sin ka}{ka} - \cos ka \right|. \hspace{1cm} (B.3)$$

From the power series representations for $\sin ka$ and $\cos ka$, we have

$$(1 - \gamma a) \frac{\sin ka}{ka} - \cos ka = -\gamma a \sum_{n=0}^{\infty} (-1)^n w_{1,n} - 2 \sum_{n=1}^{\infty} (-1)^n w_{2,n} \hspace{1cm} (B.4)$$

where

$$w_{1,n} = \frac{(ka)^{2n}}{(2n + 1)!} \quad \text{and} \quad w_{2,n} = \frac{n(ka)^{2n}}{(2n + 1)!}. \hspace{1cm} (B.5)$$

It can be shown that $\{w_{1,n}\}$ and $\{w_{2,n}\}$ are nonincreasing for $6 \geq (ka)^2$ and that $\lim_{n \to \infty} w_{1,n} = 0$ and $\lim_{n \to \infty} w_{2,n} = 0$. Thus from Leibnitz's alternating series theorem we have

$$|\Lambda_0(ka)| < |\gamma a| + \frac{1}{3}(ka)^2 \hspace{1cm} (B.6)$$

and the inequality (6.20) is verified for $|\gamma a| < \sigma < 1$ and $ka$ sufficiently small.
Next suppose that \( m \geq 1 \). From [23] we have

\[
j_m(ka) = \frac{\pi^{1/2}}{2} \left( \frac{ka}{2} \right)^m \sum_{n=0}^{\infty} \frac{(-1)^n \left( \frac{ka}{2} \right)^{2n}}{n! T(m + n + \frac{3}{2})}\tag{B.7}
\]

and

\[
h_m^{(1)}(ka) = \frac{\exp \left[ -i \frac{\pi}{2} (m + 1) + ika \right]}{ka} \sum_{n=0}^{\left\lfloor m/2 \right\rfloor} (-1)^n \psi_n - i \sum_{n=0}^{\left\lfloor (m-1)/2 \right\rfloor} (-1)^n \omega_n\tag{B.8}
\]

where

\[
\psi_n = \frac{(2m-2n)! (2ka)^{2n}}{(2n)! (m-2n)!} \quad \text{and} \quad \omega_n = \frac{(2m-2n-1)! (2ka)^{2(n+1)}}{(2n+1)! (m-2n-1)!}\tag{B.9}
\]

where \( \lfloor \cdot \rfloor \) denotes the greatest integer function. From Leibnitz’s theorem we have for \( ka \leq 2^{1/2} \)

\[
| h_m^{(1)}(ka) | \leq \frac{1}{ka (2ka)^m} \left( \frac{2m}{m+1} \right) (1 + ka)\tag{B.10}
\]

Using (B.7) and again applying Leibnitz’s theorem we find

\[
| kj_m(ka)' + \gamma j_m(ka) | \leq \frac{\pi^{1/2}}{2a} \left( \frac{m+ \gamma \pi a}{T(m + \frac{3}{2})} \right) ka^m\tag{B.11}
\]

for \( ka \leq 2^{1/2} \). From Eqs. (B.10) and (B.11) we have

\[
| A_m(ka) | < \frac{\sqrt{5}}{3} (1 + ka)\tag{B.12}
\]

for small values of the wave number. From the first part it follows that the inequality (6.20) is valid for \( m \geq 0 \).

**References**