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Extension of a Theorem of Laguerre to Entire Functions of Exponential Type

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A *circular domain* is a domain whose boundary is a circle or a straight line. A well-known theorem of Laguerre ([6, pp. 56–63]; also see [7, p. 33]) can be stated as follows:

THEOREM A. *Let p be a polynomial of degree $n \geqslant 1$. If $p(z) \neq 0$ in a (closed or open) circular domain K, then*

$$np(z) - (z - \zeta)\, p'(z) \neq 0 \qquad \text{for} \quad z \in K, \zeta \in K$$

which in the case $\zeta = \infty$ means $p'(z) \neq 0$ for $z \in K$.

We prove the following resul which as we shall show constitutes an extension of Theorem A and has many applications.

THEOREM 1. *Let f be an entire function of exponential type $\tau > 0$ such that*

$$h_f(\pi/2) := \limsup_{y \to \infty} \frac{\log|f(iy)|}{y} = 0$$

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and denote by H the (closed or open) upper half-plane. If $f(z) \neq 0$ for $z \in H$, then

$$\tau f(z) + i(1-\zeta) f'(z) \neq 0 \qquad \textit{for} \quad z \in H \textit{ and } |\zeta| \leqslant 1. \tag{1}$$

Let us recall that the class of functions of exponential type τ consists of all functions of order less than 1 as well as those of order 1 type as most τ. Thus Theorem 1 says, in particular, that if f is of order less than 1 or of order 1 type 0 and $f(z) \neq 0$ for $z \in H$, then (1) holds for each $\tau > 0$.

Proof of Theorem 1. If $f(z) \neq 0$ for all $z \in \mathbb{C}$, then

$$f(z) = ae^{cz}$$

with $c \in \mathbb{R}$, $a \neq 0$ and $|c| \leqslant \tau$. A simple calculation shows that (1) holds in this case.

Next, suppose that f has zeros in the complement of H. Take an arbitrary point $z_0 = x_0 + iy_0 \in H$ where $x_0, y_0 \in \mathbb{R}$ and consider the function

$$g(z) := e^{i\tau z}\overline{f(\bar{z} + iy_0)}.$$

Obviously

$$\begin{aligned} &g(z) \neq 0 && \text{for} \quad \operatorname{Im} z \leqslant 0, \\ &h_g(-\pi/2) := \limsup_{y \to \infty} \frac{\log|g(-iy)|}{y} = \tau, && h_g(\pi/2) \leqslant 0 \end{aligned} \tag{2}$$

and $g(z)$ has zeros, $a_n + ib_n (a_n, b_n \in \mathbb{R};\ n = 1, 2,...)$ say, all with positive imaginary parts. The properties (2) ensure that g belongs to the class P introduced in [1, Definition 7.8.2]. Hence (see [1, Theorem 7.8.3 and formula (11.7.6)]),

$$\operatorname{Im} \frac{g'(x_0)}{g(x_0)} = \frac{1}{2}\left(h_g(-\pi/2) - h_g(\pi/2) + \sum_n \frac{b_n}{(x_0 - a_n)^2 + b_n^2}\right) > \frac{\tau}{2}.$$

It is geometrically evident that this inequality is equivalent to

$$\left|\frac{g'(x_0)}{g(x_0)} - i\tau\right| < \left|\frac{g'(x_0)}{g(x_0)}\right|,$$

i.e.,

$$|g'(x_0) - i\tau g(x_0)| < |g'(x_0)|. \tag{3}$$

But

$$g'(x_0) = e^{i\tau x_0}(i\tau\overline{f(z_0)} + \overline{f'(z_0)})$$

and so

$$|f'(z_0)| < |\tau f(z_0) + if'(z_0)|,$$

which implies that (1) holds for $z = z_0$ and $|\zeta| \leqslant 1$.

Remark 1. The conclusion of Theorem 1 does not hold with any ζ of modulus greater than 1. In fact, for every $\kappa > 1$ the function

$$f(z) := e^{i\tau z} + \kappa$$

satisfies the conditions of that theorem, whereas

$$\tau f(z) + i(1-\zeta) f'(z) = \tau(\kappa + \zeta e^{i\tau z}). \tag{4}$$

Given any ζ with $|\zeta| > 1$ we can find a point z_0 in H such that $-\zeta e^{i\tau z_0} > 1$. Hence there exists a $\kappa > 1$, namely $\kappa := -\zeta e^{i\tau z_0}$, for which (4) vanishes at z_0.

Remark 2. The conclusion of Theorem 1 does not hold if $h_f(\pi/2) < 0$ is admitted. In fact, for $0 < \varepsilon \leqslant \tau$ the function

$$f(z) := e^{i\varepsilon z} + e^{i\tau z}$$

which is of exponential type τ, does not vanish in the *open* upper half-plane H and $h_f(\pi/2) = -\varepsilon$. But for $\zeta := -(\tau - \varepsilon)/\tau$ the function

$$\tau f(z) + i(1-\zeta) f'(z)$$

vanishes identically if $\varepsilon = \tau$ and at the points

$$z := \frac{1}{\tau - \varepsilon}\left\{i \operatorname{Log}\left(\frac{\tau}{\tau - \varepsilon}\right) + 2k\pi\right\} \quad (k = 0, \pm 1, \pm 2, ...)$$

(lying in H) in case $\varepsilon < \tau$. If the half-plane is taken to be closed, the same function serves as a counterexample in the case $\varepsilon = \tau$; for $\varepsilon < \tau$ we may consider $f(z + (i/2(\tau - \varepsilon)) \operatorname{Log}(\tau/(\tau - \varepsilon)))$.

Remark 3. Theorem A can be deduced from Theorem 1. To see this let us first consider a polynomial p of degree n not vanishing in the closed unit disk. The entire function

$$f(z) := p(e^{iz})$$

is of exponential type n such that $h_f(\pi/2) = 0$ and $f(z) \neq 0$ in the closed upper half-plane H. Hence by Theorem 1

$$np(e^{iz}) - (1-\zeta)\, e^{iz} p'(e^{iz}) \neq 0 \quad \text{for } \operatorname{Im} z \geqslant 0 \text{ and } |\zeta| \leqslant 1$$

or equivalently (putting $w := e^{iz}$ and $\lambda = \zeta e^{iz}$)

$$np(w) - (w - \lambda)\, p'(w) \neq 0 \qquad \text{for} \quad |w| \leqslant 1 \text{ and } |\lambda| \leqslant |w|.$$

In particular, $|\lambda p'(w)| < |np(w) - wp'(w)|$ for $|w| = 1$ and all λ such that $|\lambda| \leqslant 1$; furthermore $np(w) - wp'(w) \neq 0$ for $|w| \leqslant 1$. Therefore by Rouché's theorem

$$np(w) - (w - \lambda)\, p'(w) \neq 0 \qquad \text{for} \quad |w| \leqslant 1 \text{ and } |\lambda| \leqslant 1. \tag{5}$$

This proves Theorem A in case K is the closed unit disk. As it is well-known, the result in its full generality can be deduced from this special case.

2. Applications

Theorem 1 can be used to obtain various old and new resuls for entire functions of exponential type. They are related to the famous inequality of S. Bernstein which states [1, Chap. 11] that if f is an entire function of exponential type τ such that $|f(x)| \leqslant M$ for $x \in \mathbb{R}$ then $|f'(x)| \leqslant M\tau$ for $x \in \mathbb{R}$. The first corollary is an analogue of [3, Theorem 4].

Corollary 1. *If f is an entire function of exponential type $\tau > 0$ satisfying $h_f(\pi/2) \leqslant 0$ and H is the (closed or open) upper half-plane, then*

$$f(z) + \frac{1}{\tau}\, i(1 - \zeta)\, f'(z) \in \overline{f(H)} \qquad \textit{for} \quad z \in H \textit{ and } |\zeta| \leqslant 1. \tag{6}$$

Proof. First let $h_f(\pi/2) = 0$. Then for every $w \notin \overline{f(H)}$ the function

$$F(z) := f(z) - w$$

satisfies the assumptions of Theorem 1 and so

$$\tau F(z) + i(1 - \zeta)\, F'(z) \neq 0 \qquad \text{for} \quad z \in H \text{ and } |\zeta| \leqslant 1.$$

This means that

$$f(z) + \frac{1}{\tau}\, i(1 - \zeta)\, f'(z) \neq w \qquad \text{for} \quad z \in H \text{ and } |\zeta| \leqslant 1$$

which is equivalent to (6). In case $h_f(\pi/2) < 0$, we may consider the function $g(z) := f(z) + 1$ for which $h_g(\pi/2) = 0$.

COROLLARY 2. *Under the assumptions of Corollary 1 let R be the supremum of the radii of all disks that can be placed inside $\overline{f(H)}$. Then*

$$|f'(z)| \leqslant \tau R \qquad \text{for} \quad z \in H. \tag{7}$$

Proof. From (6) it follows that $\overline{f(H)}$ contains the disk with centre at $f(z)+(1/\tau)\, if'(z)$ and radius $(1/\tau)|\zeta f'(z)|$ as long as $|\zeta| \leqslant 1$. This clearly implies (7).

Remark 4. An entire function of exponential type 0 is also of exponential type τ for every $\tau > 0$. Therefore we can conclude from (7) that for a nonconstant function f of exponential type 0 the closure of $f(H)$ must contain disks of arbitrarily large radius.

If f is an entire function of exponential type such that $h_f(\pi/2) \leqslant 0$ and

$$|f(x)| \leqslant 1 \qquad \text{for} \quad x \in \mathbb{R}, \tag{8}$$

then $|f(x)| \leqslant 1$ throughout the upper half-plane (see [1, Theorem 6.2.4] or (10)). Thus the following result is a consequence of Corollary 2.

COROLLARY 3 [2, Theorem 2]. *Let f be an entire function of exponential type τ satisfying* (8). *If $h_f(\pi/2) \leqslant 0$ and $f(z) \neq 0$ for* $\operatorname{Im} z > 0$, *then $|f'(x)| \leqslant \tau/2$ for $x \in \mathbb{R}$.*

Corollary 2 also leads us to

COROLLARY 4 [4, Theorem 1]. *Let f be an entire function of exponential type τ such that $|f(x)|$ is bounded on the real axis and $h_f(\pi/2) \leqslant 0$. If $|\operatorname{Re} f(x)| \leqslant 1$ for $x \in \mathbb{R}$, then*

$$|f'(x)| \leqslant \tau \qquad \text{for} \quad x \in \mathbb{R}. \tag{9}$$

Proof. According to a formula in [5, Lemma 9],

$$f(x+iy) = \tau y \sum_{k=-\infty}^{\infty} \frac{1-(-1)^k e^{-\tau y}}{(\tau y)^2 + (k\pi)^2} f\left(x + \frac{k\pi}{\tau}\right), \tag{10}$$

where (taking $f(z) \equiv 1$ in (10))

$$\begin{aligned} 1 &= \tau y \sum_{k=-\infty}^{\infty} \frac{1-(-1)^k e^{-\tau y}}{(\tau y)^2 + (k\pi)^2} \\ &= |\tau y| \sum_{k=-\infty}^{\infty} \left| \frac{1-(-1)^k e^{-\tau y}}{(\tau y)^2 + (k\pi)^2} \right| \qquad \text{for} \quad y \geqslant 0. \end{aligned}$$

Hence $|\mathrm{Re}\, f(x+iy)| \leqslant 1$ for $y \geqslant 0$, i.e., f maps the closed upper half-plane into the strip $\{w: |\mathrm{Re}\, w| \leqslant 1\}$. Now we may apply Corollary 2 to obtain the desired estimate.

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