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On a Class of Variational Inequalities

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In this paper, we introduce and study a new unified and general class of variational inequalities. We derive the general error estimates for the finite element solutions of variational inequalities. It has been shown that a class of contact problems with friction terms arising in elastostatics can be studied in the framework of variational inequalities. Several special cases, which can be obtained from the general results, are also discussed. © 1987 Academic Press, Inc.

1. INTRODUCTION

The mathematical subject, we call variational inequalities, was introduced by Stampacchia and Fichera in potential theory and mechanics (problems in elasticity with unilateral constraints), developed by the French and Italian schools, has enjoyed a vigorous growth for the last twenty years. Variational inequalities not only have stimulated new and deep results dealing with nonlinear partial differential equations, but also have been used in a large variety of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences.

In 1970, using a transformation, Baiocchi found that the free boundary value problem associated with seepage through an earth dam is equivalent to a class of variational inequality. Since then variational inequalities have produced a tremendous impact in this field. So successful have been variational inequalities, that other methods are rarely used.

The development of the theory of variational inequalities can be viewed as the simultaneous pursuit of two different lines of research: On the one hand, it reveals the fundamental facts on the qualitative behavior of solutions (regarding existence, uniqueness, and regularity) to important classes of nonlinear boundary value problems; on the other hand, it also provides highly efficient new numerical methods to solve, for example, free and moving boundary value problems. Consequently, it is clear that the theory of variational inequalities provides a natural and elegant framework

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for the study of the many seemingly unrelated free boundary value problems arising in fluid flow through porous media, elasticity, transportation and economics equilibrium, operations research, etc.

Our main aim in this paper is to introduce and study a new class of variational inequalities, which is the most general and unifies all the previously known classes of variational inequalities. We focus our attention on an existence and uniqueness theorem for such abstract problems in Section 3 after introducing this class in Section 2 by using the fixed point theorem. We derive the general error estimates for the finite element approximation of the solution of variational inequalities in Section 4. In Section 5, we show that a class of contact problems with friction terms arising in elasticity can be studied in the framework of variational inequalities introduced in this paper. Furthermore, we prove that the error estimate for the finite element approximate solution is of order $h^{1/2}$ in the energy norm. Several special cases which can be obtained from our results are also discussed.

2. Preliminaries and Formulation

Let H be a real Hilbert space with its dual H', whose inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The pairing between elements of H' and H is denoted by $\langle \cdot,\cdot \rangle$. Let M be a closed nonempty convex subset of H.

Let $a(u, v)$ be a coercive and continuous bilinear form on H, that is, there exist constants $\alpha > 0$, and $\beta > 0$ such that

$$a(v, v) \geqslant \alpha \|v\|^2, \qquad \text{for all} \quad v \in H \tag{2.1}$$

and

$$a(u, v) \leqslant \beta \|u\| \|v\|, \qquad \text{for all} \quad u, v \in H. \tag{2.2}$$

It is clear that $\alpha \leqslant \beta$.

Consider the form $b(u, v)$: $H \times H \to \mathbb{R}$ satisfying the following properties:

(i) $b(\cdot,\cdot)$ is linear in the first variable.

(ii) $b(\cdot,\cdot)$ is bounded, that is, there exists a constant $\gamma > 0$ such that

$$|b(u, v)| \leqslant \gamma \|u\| \|v\|, \qquad \text{for all} \quad u, v \in H. \tag{2.3}$$

(iii) $b(\cdot,\cdot)$ is either convex or linear in the second argument.

(iv) For every u, v, $w \in H$,

$$|b(u, v) - b(u, w)| \leqslant b(u, v - w) \tag{2.4}$$

$$b(u, v \pm w) \leqslant b(u, v) + b(u, w). \tag{2.5}$$

If $a(u, v)$ is a coercive continuous bilinear form, $b(u, v)$ is a form satisfying the properties (i)–(iv) and f is a continuous functional, then we consider the functional $I[v]$, defined by

$$I[v] = a(v, v) + b(v, v) - 2f(v), \qquad \text{for all} \quad v \in H. \tag{2.6}$$

Many mathematical problems in physical and engineering sciences either arise or can be formulated in terms of a functional of this form. Here one seeks to minimize the functional $I[v]$, defined by (2.6) over a whole space or on a convex set in H, bearing in mind whether the real-valued functional f is linear or not. We point out that the whole theory of variational methods can be based on the minimum of the functional $I[v]$. In fact, the functional $I[v]$ represents the potential energy associated with the statistic friction problem for the Coulomb law, see Section 5, for applications.

For the linear continuous functional f, the minimum of $I[v]$ on a closed convex set M is equivalent to finding $u \in M$ such that

$$a(u, v-u) + b(u, v) - b(u, u) \geqslant \langle f, v-u \rangle \qquad \text{for all} \quad v \in M, \tag{2.7}$$

a case considered by Oden and Pires [1].

For a differentiable nonlinear continuous functional f, using the technique of Noor [2], we can show that the minimum of $I[v]$ on M can be characterized by a class of variational inequalities of the type

$$a(u, v-u) + b(u, v) - b(u, u) \geqslant \langle f'(u), v-u \rangle, \quad \text{for all } v \in M, \tag{2.8}$$

where $f'(u)$ is the Fréchet differential of f at $u \in M$.

Special Cases

I. If the form $b(u, v) \equiv 0$, then it is clear that the minimum of $I[v]$ on M can be characterized by the following classes of variational inequalities depending upon whether f is a linear or nonlinear continuous functional

$$a(u, v-u) \geqslant \langle f, v-u \rangle, \qquad \text{for all} \quad v \in M, \tag{2.9}$$

fand

$$a(u, v-u) \geqslant \langle f'(u), v-u \rangle, \qquad \text{for all} \quad v \in M. \tag{2.10}$$

Variational inequality of the type (2.9) was originally considered and studied by Lions and Stampacchia [3]. Iterative and numerical methods for finding the approximate solutions of (2.9) have been studied by many authors including Noor [4], Falk [5], and Glowinski *et al.* [6], where one can also find various applications of this type of varitional inequalities.

Noor [7] introduced and considered the variational inequalities of the form (2.10) in the study of mildly nonlinear elliptic boundary value problems satisfying some extra constraint conditions. Recently, it has been shown [8] that unilateral problems with nonconvex potential can only be characterized by a class of variational inequality of the type (2.10).

II. If we restrict the dependenc of the form $b(u, v)$ to its second variable only, that is, if $b(u, v) = j(v)$, then it can be shown that the minimum of $I[v]$, defined by (2.6) can be characterized by a class of variational inequality of the type,

$$a(u, v-u) + j(v) - j(u) \geqslant \langle f'(u), v-u \rangle, \qquad \text{for all} \quad v \in M, \tag{2.11}$$

a problem considered and studied by Noor [9]. The variational inequality (2.11) characterizes a Signorini problem with friction, but with a law of friction different from Coulomb's law. Using the piecewise linear elements, it has been shown [9] that the error estimate in the energy norm is of order $h^{1/2}$. Furthermore, we note that if $f(u)$ is independent of u, that is, $f'(u), \equiv f$ (say), then (2.11) is equivalent to

$$a(u, v-u) + j(v) - j(u) \geqslant \langle f, v-u \rangle, \qquad \text{for all} \quad v \in M, \tag{2.12}$$

a case considered by Pires and Oden [10] and Glowniski *et al.* [6].

We also need the following standard results for the proof of our main results concerning the existence and uniqueness of the solution of variational inequality (2.8).

LEMMA 2.1. *Let M be a convex subset of H. Then, given $z \in H$, we have*

$$u = P_M z,$$

if and only if

$$u \in M: (u-z, v-u) \geqslant 0, \qquad \textit{for all} \quad v \in M,$$

where P_M is the projection of H into M.

LEMMA 2.2. *The project operator P_M is nonexpansive, that is,*

$$\| P_M u - P_M v \| \leqslant \| u - v \|, \qquad \textit{for all} \quad u, v \in H.$$

3. EXISTENCE RESULTS

In this section, we study those conditions under which there does exist a unique solution of a more general variational inequality of which (2.8) is a special case.

Let us consider the following problem:

PROBLEM 3.1. Find $u \in M$ such that

$$a(u, v-u)+b(u, v)-b(u, u) \geqslant \langle A(u), v-u\rangle, \qquad \text{for all} \quad v \in M, \tag{3.1}$$

where A is a nonlinear operator such that $A(u) \in H'$.

We also define the following concepts.

DEFINITION 3.1. The nonlinear operator $T: M \rightarrow H'$ is called

(i) *Antimonotone*, if for all $u, v \in M$,

$$\langle Tu - Tv, u-v\rangle \leqslant 0,$$

(ii) *Lipschitz continuous*, if there exists a constant $\xi > 0$ such that

$$\| Tu - Tv \| \leqslant \xi \, \| u - v \|, \qquad \text{for all} \quad u, v \in M.$$

Since $a(u, v)$ is a continuous bilinear form on H, then by the Riesz–Fréchet representation theorem, we have

$$a(u, v) \equiv \langle Tu, v\rangle, \qquad \text{for all} \quad v \in H. \tag{3.2}$$

It can be shown that $\| T \| \leqslant \beta$. Finally, we define Λ, a canonical isomorphism from H' onto H for all $f \in H'$,

$$\langle f, u\rangle = (\Lambda f, v), \qquad \text{for all} \quad v \in H, \tag{3.3}$$

Then $\| \Lambda \|_{H'} = 1 = \| \Lambda^{-1} \|_H$.

We make the following hypothesis.

Condition N. We assume that $\gamma + \xi < \alpha$, where α is the coercivity constant of $a(u, v)$, γ is the boundedness constant of the form $b(u, v)$, and ξ is the Lipschitz constant of the operator A.

We also need the following result, which is a generalization of a result of Noor [11] and Lions and Stampacchia [3].

LEMMA 3.1. *Let ρ be number such that $0 < \rho < 2(\alpha - \gamma - \xi)/(\beta^2 - (\gamma + \xi)^2)$ and $\rho < 1/(\gamma + \xi)$. Then there exists a θ with $0 < \theta < 1$ such that*

$$\| \phi(u_1) - \phi(u_2) \| \leqslant \theta \, \| u_1 - u_2 \|, \qquad \textit{for all} \quad u_1, u_2 \in H,$$

where for $u \in H$, $\phi(u) \in H'$ *is defined by*

$$\langle \phi(u), v \rangle = (u, v) - \rho a(u, v) - \rho b(u, v) + \rho \langle A(u), v \rangle, \qquad \textit{for all} \quad v \in H. \tag{3.4}$$

Here β *is the boundedness constant of the bilinear form* $a(u, v)$.

Proof. For all $u_1, u_2 \in H$, consider

$$\begin{aligned} \langle \phi(u_1) - \phi(u_2), v \rangle &= (u_1 - u_2, v) - \rho a(u_1 - u_2, v) \\ &\quad - \rho b(u_1 - u_2, v) + \rho \langle A(u_1) - A(u_2), v \rangle \\ &= (u_1 - u_2, -\rho(\Lambda T u_1 - \Lambda T u_2), v) \\ &\quad - \rho b(u_1 - u_2, v) + \rho(\Lambda A(u_1) - \Lambda A(u_2), v) \end{aligned}$$

Hence

$$\begin{aligned} |\langle \phi(u_1) - \phi(u_2), v \rangle| &\leqslant \| u_1 - u_2 - \rho \Lambda (T u_1 - T u_2) \| \, \| v \| \\ &\quad + \rho\gamma \, \| u_1 - u_2 \| \, \| v \| + \rho\xi \, \| u_1 - u_2 \| \, \| v \|. \end{aligned}$$

Now using (2.1), (2.2), (3.2), and (3.3), we obtain

$$\| u_1 - u_2 - \rho \Lambda (T u_1 - T u_2) \|^2 \leqslant (1 + \rho^2 \beta^2 - 2\alpha\rho) \, \| u_1 - u_2 \|^2.$$

Thus

$$\begin{aligned} |\langle \phi(u_1) - \phi(u_2), v \rangle| &\leqslant \{ \sqrt{(1 + \rho^2\beta^2 - 2\alpha\rho)} + \rho(\gamma + \xi) \} \, \| u_1 - u_2 \| \, \| v \| \\ &= \theta \, \| u_1 - u_2 \| \, \| v \|, \end{aligned}$$

where $\theta = [\sqrt{(1 + \rho^2\beta^2 - 2\alpha\rho)} + \rho(\gamma + \xi)] < 1$ for $0 < \rho < 2(\alpha - \gamma - \xi)/(\beta^2 - (\gamma + \xi)^2)$ and $\rho < 1/(\gamma + \xi)$, by Condition N, it follows that

$$\| \phi(u_1) - \phi(u_2) \| = \sup_{v \in H} \frac{|\langle \phi(u_1) - \phi(u_2), v \rangle|}{\| v \|} \leqslant \theta \, \| u_1 - u_2 \|,$$

the required result.

Remark 3.1. Note that for $b(u, v) \equiv 0$, Lemma 3.1 is exactly the same as proved in Noor [2]. Furthermore, if $A(u)$ is independent of u, that is, $A(u) = f$, then Lemma 3.1 reduces to a result of Noor [11]. For $A(u) \equiv f$ (say) and $b(u, v) \equiv 0$, we get a result of Lions and Stampacchia [3].

Now using the technique of Lions and Stampacchia [3] and Noor [11, 7], we prove the main existence result of this section.

THEOREM 3.1. *Let $a(u, v)$ be a coercive continuous bilinear form and $b(u, v)$ be a form satisfying the properties* (i)–(iv). *If the operator A is antimonotone and Lipschitz continuous and Condition N holds, then there exists a unique solution $u \in M$ such that*

$$a(u, v-u) + b(u, v) - b(u, u) \geqslant \langle A(u), v-u \rangle, \qquad \textit{for all } v \in M. \quad (3.5)$$

Proof. (a) *Uniqueness.* See Oden and Pires [1] and Noor [11].

(b) *Existence.* For a fixed ρ as in Lemma 3.1, and $u \in H$ define $\phi(u) \in H'$ by (3.4). By Lemma 2.1, there exists a unique $w \in M$ such that

$$(w, v-w) \geqslant \langle \phi(u), v-w \rangle, \qquad \text{for all} \quad v \in M,$$

and w is given by

$$w = P_M \Lambda \phi(u) = Tu,$$

which defines a map from H into M.

Now for all $u_1, u_2 \in H$,

$$\begin{aligned} \| Tu_1 - Tu_2 \| &= \| P_M \Lambda \phi(u_1) - P_M \Lambda \phi(u_2) \| \\ &\leqslant \| \Lambda \phi(u_1) - \Lambda \phi(u_2) \|, \qquad \text{by Lemma 2.2} \\ &\leqslant \| \phi(u_1) - \phi(u_2) \| \\ &\leqslant \| u_1 - u_2 \|, \qquad \text{by Lemma 3.1.} \end{aligned}$$

Since $\theta < 1$, Tu is a construction mapping and has a fixed point $Tu = u$, which belongs to M, a closed convex set of H and satisfies

$$\begin{aligned} &a(u, v-u) + b(u, v) - b(u, u) \\ &\qquad \geqslant \langle A(u), v-u \rangle, \qquad \text{for all } v \in M, \end{aligned}$$

showing that $u \in M$ is a unique solution of Problem 3.1.

Remarks 2.2. (i) It is obvious that for $A(u) = f'(u)$, the existence of a unique solution of the nonlinear variational inequality (2.8) follows under the assumptions of Theorem 3.1.

(ii) If A is independent of u, that is, $A(u) = f$ (say), then the Lipschitz constant ξ is zero and consequently Theorem 3.1 is exactly the same as proved by Noor [11] for a class of variational inequalities arising in elasticity.

(iii) If $b(u, v) \equiv 0$, and $A(u) = f$ (say), then we get a result of Lions and Stampacchia [3] for a class of variational inequalities of type (2.9).

4. Abstract Error Estimates

We shall now establish a general error estimate for finite element approximation of the type (3.5). Our estimate is quite general. It holds for any finite element subspace S_h and approximate constraint set M_h and represents a significant improvement of all estimates for variational inequalities that can be found in the literature. For definiteness, we shall assume that there exists a Hilbert space U which is densely and continuously embedded in the dual space H'. It is then possible to identify H with a subspace of U', i.e., dense in U' by a continuous injection.

In order to derive the error estimate for the approximate solutions for variational inequalities of types (3.5) and (2.7), we consider an approximate form of the variational inequalities (3.5) and (2.7). Thus, let $S_h \subset H$ be a finite dimensional subspace and $M_h \subset H$ be a finite dimensional convex set; for the construction of S_h and M_h, see Section 5.

An approximation of (3.5) is that of finding $u_h \in M_h$ such that

$$a(u_h, v_h - u_h) + b(u_h, v_h) - b(u_h, u_h) \geqslant \langle A(u_h), v_h - u_h \rangle, \qquad \text{for all} \quad v_h \in M_h. \tag{4.1}$$

Similarly, we can also construct an approximation of (2.7) of the form: Find $u_h \in M_h$ such that

$$a(u_h, v_h - u_h) + b(u_h, v_h) - b(u_h, u_h) \geqslant \langle f, v_h - u_h \rangle, \qquad \text{for all} \quad v_h \in M_h. \tag{4.2}$$

and of (2.11) of the form: Find $u_h \in M_h$ such that

$$a(u_h, v_h - u_h) + j(v_h) - j(u_h) \geqslant \langle A(u_h), v_h - u_h \rangle, \qquad \text{for all} \quad v_h \in M_h. \tag{4.3}$$

With these hypotheses and preliminaries established, we can now derive the following abstract error estimate.

THEOREM 4.1. *Let $u \in M$ and $u_h \in M_h$ be the solutions of* (3.5) *and* (4.1), *respectively. Let $a(u, v)$ be a coercive continuous bilinear form and $b(u, v)$ satisfy the properties* (i)–(iv) *and denote by $a(u, v) = \langle Tu, v \rangle$, for all $v \in H$.*

If A is antimonotone Lipschitz continuous and $Tu - A(u) \in U$, then there exists a constant $c > 0$, such that

(i) *For $M_h \not\subset M$,*

$$\begin{aligned}\|u-u_h\|_H \leqslant c[\|u-v_h\|_H + \|v-u_h\|_H^2 + \|v_h - v\|_H \\ + \{(\|A(u)-Tu\|_U \|u-v_h\|_{U'})^{1/2} \\ + (\|A(u_h)-Tu_h\|_U \|u_h - v\|_{U'})^{1/2} \\ + (\|u\|_H [\|u_h - v\|_H \\ + \|u-v_h\|_H]]^{1/2}\}], \qquad \textit{for all } v \in M \textit{ and } v_h \in M_h. \quad (4.4)\end{aligned}$$

(ii) *For $M_h \subset M$,*

$$\begin{aligned}\|u-u_h\|_H \leqslant c[\|u-v_h\|_H + (\|A(u_h)\|_U \|u-v_h\|_{U'})^{1/2} \\ + (\|u\|_H \|u-v_h\|_H^{1/2})], \qquad \textit{for all} \quad v_h \in M_h. \qquad (4.5)\end{aligned}$$

Proof. (i) $M_h \not\subset M$. Since $u \in M$ and $u_h \in M_h$ are solutions of (3.5) and (4.1), respectively, then adding (3.5) and (4.1), we have for all $v \in M$ and $v_h \in M_h$,

$$\begin{aligned}a(u,u) + a(u_h,u_h) \leqslant \langle A(u), v-u\rangle + \langle A(u_h), v_h - u_h\rangle \\ - b(u,v) + b(u,u) + b(u_h,u_h) - b(u_h,v_h).\end{aligned}$$

Subtracting $a(u, u_h) + a(u_h, u)$ from both sides, using (i)–(iv), and rearranging terms, we get

$$\begin{aligned}a(u-u_h, u-u_h) \leqslant a(u-u_h, v-u_h) + a(u-u_h, u-v_h) \\ + \langle A(u) - A(u_h), v_h - v\rangle \\ + \langle A(u) - Tu, u - v_h\rangle + \langle A(u_h) - Tu_h, u_h - v\rangle \\ + b(u-u_h, u-u_h) + b(u, u_h - v) + b(u_h, u - v_h). \qquad (4.6)\end{aligned}$$

Since, by assumptions, $Tu - A(u) \in U$, $a(u, v)$ is coercive continuous and f' is Lipschitz continuous, so we obtain

$$\begin{aligned}\alpha \|u-u_h\|_H^2 \leqslant \beta \|u-u_h\|_H \|v-u_h\|_H + \beta \|u-u_h\|_H \\ \times \|u-v_h\|_H + \xi \|u-u_h\|_U \|v_h - v\|_{U'} \\ + \|A(u) - Tu\|_U \|u-v_h\|_{U'} \\ + \|A(u_h) - Tu_h\|_U \|u_h - v\|_{U'} \\ + \gamma[\|u-u_h\|_H^2 + \|u\|_H \|u_h - v\|_H \\ + \|u_h\|_H \|u-v_h\|_H]. \qquad (4.7)\end{aligned}$$

Since the inequality

$$ab \leqslant \varepsilon a^2 + (1/4\varepsilon) b^2$$

holds for positive a, b, and any $\varepsilon > 0$, we have

$$\xi \, \| u - u_h \|_H \, \| v - v_h \|_H \leqslant \frac{\alpha}{6} \, \| u - u_h \|_H^2 + \frac{3\xi^2}{2\alpha} \, \| v_h - v \|_H^2$$

$$\beta \, \| u - u_h \|_H \, \| v - u_h \|_H \leqslant \frac{\alpha}{6} \, \| u - u_h \|_H^2 + \frac{3\beta^2}{2\alpha} \, \| v - u_h \|_H^2$$

$$\beta \, \| u - u_h \|_H \, \| u - v_h \|_H \leqslant \frac{\alpha}{6} \, \| u - u_h \|_H^2 + \frac{3\beta^2}{2\alpha} \, \| u - v_h \|_H^2.$$

Thus, from (4.7) and the above inequalities, we have, for $\alpha > 3\gamma$,

$$\begin{aligned} \| u - u_h \|_H^2 \leqslant & \frac{9(\beta^2 + \gamma^2)}{2\alpha(\alpha - 3\gamma)} \, \| u - v_h \|_H^2 + \frac{9\beta^2}{\alpha(\alpha - 3\gamma)} \, \| v - u_h \|_H^2 \\ & + \frac{9\xi^2}{\alpha(\alpha - 3\gamma)} \, \| v_h - v \|_H^2 \\ & + \frac{3}{\alpha - 3\gamma} \, \| A(u) - Tu \|_U \, \| u - v_h \|_{U'} \\ & + \frac{3}{\alpha - 3\gamma} \, \| A(u_h) - Tu_h \|_U \, \| u_h - v \|_{H'} \\ & + \frac{3}{\alpha - 3\gamma} \, \{ \| u \|_H (\| u_h - v \|_H + \| u - v_h \|_H) \}. \end{aligned}$$

Hence, the estimate (4.3) follows from the fact that for positive a, b, c, d, and e, $a \leqslant b + c + d + e$ implies that $\sqrt{a} \leqslant \sqrt{b} + \sqrt{c} + \sqrt{d} + \sqrt{e}$.

(ii) $M_h \subset M$. Setting $v = u_h$ in (4.5), we get

$$\begin{aligned} a(u - u_h, u - u_h) \leqslant & \, a(u - u_h, u - v_h) + \langle A(u) - A(u_h), v_h - u_h \rangle \\ & + \langle A(u) - Tu, u - v_h \rangle \\ & + b(u - u_h, u - u_h \rangle + b(u_h, u - v_h), \end{aligned}$$

which can be written as

$$\begin{aligned} a(u-u_h, u-u_h) \leqslant a(u-u_h, u-v_h) &\\ + \langle A(u), u-v_h\rangle + a(u, v_h-u) &\\ + \langle A(u)-A(u_h), u-u_h\rangle &\\ + \langle A(u)-A(u_h), v_h-u\rangle &\\ + b(u-u_h, u-u_h) + b(u_h, u-v_h). \end{aligned}$$

Now using the antimononicity of $A(u)$, we obtain

$$\begin{aligned} a(u-u_h, u-u_h) \leqslant a(u-u_h, u-v_h) + a(u, v_h-u) + \langle A(u_h), u-v_h\rangle &\\ + b(u-u_h, u-u_h) + b(y_h, u-v_h). \end{aligned} \tag{4.8}$$

From the coercivity, continuity of $a(u, v)$ and boundedness of $b(u, v)$, we obtain

$$\begin{aligned} \alpha \|u-u_h\|_H^2 \leqslant \beta \|u-u_h\|_H \|u-v_h\|_H + \beta \|u\|_H \|v_h-u\|_H &\\ + \|A(u_h)\|_U \|u-v_h\|_{U'} &\\ + \gamma \|u-u_h\|_H^2 + \gamma \|u_h\|_H \|u-v_h\|_H \end{aligned}$$

or

$$\begin{aligned} (\alpha-\gamma) \|u-u_h\|_H^2 \leqslant (\alpha+\beta) \|u-u_h\|_H \|u-v_h\|_H &\\ + \|A(u_h\|_U \|u-v_h\|_{U'} &\\ + (\gamma+\beta) \|u\|_H \|u-v_h\|_H. \end{aligned}$$

Hence, for $\alpha > \gamma$, we have

$$\begin{aligned} \|u-u_h\|_H \leqslant \frac{(\alpha+\beta)}{(\alpha-\gamma)} \|u-v_h\|_H + \left(\frac{2}{\alpha-\gamma} \|A(u_h)\|_U \|u-v_h\|_{U'}\right)^{1/2} &\\ + \left(\frac{2(\gamma+\beta)}{\alpha-\gamma} \|u\|_H \|u-v_h\|_H\right)^{1/2}, \end{aligned}$$

the required estimate (4.3).

THEOREM 4.2. *Let $u \in M$ and $u_h \in M_h$ are solutions of* (2.11) *and* (4.3), *respectively, then under the assumptions of Theorem* 4.1, *there exists a constant $c > 0$, independent of h, such that*

(i) *for* $M_h \not\subset M$,

$$\begin{aligned}\|u-u_h\|_H \leqslant c[\|u-u_h\|_H &+ \|v-u_h\|_H \\ &+ \|v_h - v\|_H + \{\|A(u)-Tu\|_U \|u-v_h\|_{U'} \\ &+ \|A(u_h)-Tu_h\|_U \|u_h - v\|_{U'}\}^{1/2} \\ &+ \{\|v-u_h\|_H + \|v_h - u\|_H\}^{1/2}]\end{aligned} \tag{4.9}$$

for all $v \in M$ *and* $v_h \in M_h$.

(ii) *for* $M_h \subset M$.

$$\begin{aligned}\|u-u_h\|_H \leqslant c[\|u-v_h\|_H &+ \{\|A(u_h)-Tu\|_U \|u-v_h\|_{U'}\}^{1/2} \\ &+ \{\|u-v_h\|_H\}^{1/2}], \qquad \textit{for all} \quad v_h \in M_h.\end{aligned} \tag{4.10}$$

Special Cases

If A is independent of u, that $A(u)=f$ (say), then our results are exactly the same as obtained by Pires and Oden [10] recently for variational inequalities characterizing the Signorini problem with nonlocal friction arising in plane elasticity.

THEOREM 4.3. [10]. *Let* $u \in M$ *and* $u_h \in M_h$ *be the solutions of* (2.7) *and* (4.2) *respectively. If* $a(u,v)=\langle Tu, v\rangle$, *and* $b(u,v)$ *satisfies* (i)–(iv), *and* $Tu - f \in U$, *then we have the following error estimates*:

(1) $M_h \not\subset M$,

$$\begin{aligned}\|u-u_h\|_H \leqslant c\{\|u-v_h\|_H &+ [\|Tu-f\|_U(\|u-v_h\|_{U'} + \|u_h - v\|_{U'})]^{1/2} \\ &+ [\gamma\ \|u\|_H(\|u-v_h\|_H + \|u_h - v\|_H)]^{1/2}\},\end{aligned}$$

for all $v \in M$ *and* $v_h \in M_h$.

(2) $M_h \subset M$,

$$\begin{aligned}\|u-u_h\|_H \leqslant c\{\|u-v_h\|_H &+ (\|Tu-f\|_U \|u-v_h\|_{U'})^{1/2} \\ &+ [\gamma\ \|u\|_H \|u-v_h\|_H]^{1/2}\}, \qquad \textit{for all} \quad v_h \in M_h.\end{aligned} \tag{4.12}$$

(b) *For* $b(u,v)=j(v)$, *then*

(3) $M_h \not\subset M$,

$$\begin{aligned}\|u-u_h\|_H \leqslant c\{\|u-v_h\|_H &+ [\|Tu-f\|_U(\|u-v_h\|_{U'} + \|u_h - v\|_{U'})]^{1/2} \\ &+ [\gamma(\|u-v_h\|_H + \|u_h - v\|_H)]^{1/2}\},\end{aligned}$$

for all $v \in M$ *and* $v_h \in M_h$. (4.13)

(4) $M_h \subset M$,

$$\| u - u_h \|_H \leqslant c[\| u - v_h \|_H + (\| Tu - f \|_U \, \| u - v_h \|_{U'})^{1/2} + (\gamma \, \| u - v_h \|_H)^{1/2}], \quad \textit{for all} \quad v_h \in M_h. \tag{4.14}$$

(c) *For* $b(u, v) \equiv 0$, *then*

(5) $M_h \not\subset M$,

$$\| u - u_h \|_H \leqslant c\{ u - v_h \|_H + [\| Au - f \|_U (\| u - v_h \|_{U'} + \| v_h - v \|_{U'})^{1/2}]\}, \quad \textit{for all } v \in M \textit{ and } v_h \in M_h \tag{4.15}$$

(6) $M_h \subset M$,

$$\| u - u_h \|_H \leqslant c\{ \| u - v_h \|_H + (\| Au - f \|_U \, \| u - v_h \|_{U'})^{1/2} \}, \quad \textit{for all} \quad v_h \in M_h. \tag{4.16}$$

5. APPLICATIONS

We consider the following Signorini problem with nonlocal friction:

$$\sigma_{ij}(u)_{,j} = f_i(u), \qquad \sigma_{ij}(u) = E_{ijkl} u_{k,l} \qquad \text{in} \quad \Omega,$$

$$u_i = 0 \qquad \text{on} \quad \Gamma_D$$

$$\sigma_{ijkl} u_{k,l} n_j = t_i \qquad \text{on} \quad \Gamma_F;$$

and on Γ_C, (5.1)

$$\left.\begin{array}{c} u \cdot N - s \leqslant 0, \qquad \sigma_N(u) \leqslant 0, \qquad \sigma_N(u)(u \cdot N - s) \leqslant 0 \\ |\sigma_T(u)| < \mu S(\sigma_N(u)) \Rightarrow u_T = 0 \\ |\sigma_T(u)| = \mu S(\sigma_N(u)) \Rightarrow \text{there exists a number} \\ \lambda \geqslant 0 \text{ such that } u_T = -\lambda \sigma_T(u). \end{array}\right\}$$

In this section we consider a varitional formulation of problem (5.1) and show that how this problem can be studied in the framework of the abstract theory of variational inequalities developed in the previous sections. We would like to point out that problem (5.1) is a generalization of a problem considered by Oden and Pires [1], when the nonlinear function $f_i(u)$ is a function of space variable only, i.e., $f_i(u) \equiv f_i$. It is well known that contact is precisely the physical event through which loads are delivered to a structure and by which a structure transmits forces to its supports. The underlying difficulty is that contact problems in solid mechanics are inherently nonlinear due to the facts that the area of contact is not known a priori to the applicaton of loads and complex physical phenomena are experienced on the contact surfaces which often require special mechanical and mathematical considerations. It is known [1], that problems of type (5.1) can be studied in the framework of variational inequalities.

Here, we use the following notations and conventions, see Kikuchi and Oden [12]:

— Ω is the elastic body in a bounded open domain $\mathbb{R}^N$ with Lipschitz boundary $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_F \cup \bar{\Gamma}_C$.

— $\Gamma_D(\Gamma_F)$ are portions of Γ on which the displacements (tractions) are prescribed.

— Γ_C is the (candidate) contact surface on which the body may come in contact with the foundation upon the application of loads; it is assumed throughout that $\bar{\Gamma}_C \cap \bar{\Gamma}_D = \phi$, $u = (u_1, u_2, ..., u_N)$ is the displacement vector $u = u(x)$, where $x = (x_1, x_2, ..., x_N)$ is a point in Ω.

— $\sigma_{ij}(u)$ are components of the stress tensor; its value at a displacement u is defined by the relation $\sigma_{ij}(u) = E_{ijkl} u_{k,l}$, where E_{ijkl} are the elasticities of material of which the body is composed. These are given functions of x satisfying the usual ellipticity (coercivity) and symmetry conditions.

— H is the space of admissible displacement vector $\{v \in H^1(\Omega),\ v = 0$ a.e. on $\Gamma_D\}$, where $H^1(\Omega)$ is the usual Sobolev space of classes of functions with L_2-partial derivatives of order 1. The space H is a Hilbert space and is equipped with the energy norm

$$\| v \|^2 = \int_\Omega v_{i,j} v_{i,j}\, dx,$$

where the commas denote partial differentiation $v_{i,j} = \partial v_i / \partial x_j$. We shall also use the notation

$$\| v \|_0^2 = \int_\Omega v \cdot v\, dx.$$

— $W = H^{1/2}(\Gamma_C)$ is the space of normal traces of the admissible displacements v on the contact surface Γ_C. Here n denotes the unit outward normal to Γ_C. If $u \in H$, and $\Lambda(u)$ is the trace of u on Γ_C, then $\Lambda(u) \cdot n \in W$.

— M is the constraint set corresponding to the unilateral condition $\Lambda(u) \cdot n \leqslant s = \{v \in H,\ \Lambda(v) \cdot n \leqslant s \text{ in } W\}$, where s is given in W; s represents the initial gap between the body Ω and the rigid foundation. M is a closed convex subset of H.

— $$a(u, v) = \int_\Omega E_{ijkl} u_{k,l} v_{i,j}\, dx$$

$$= \text{the virtual work produced by the stresses.} \tag{5.2}$$

— f is a continuous (nonlinear) functional on H representing the virtual work of the external forces with values given by

$$f(v) = \int_\Omega \int_0^v f_i(\eta)\, d\eta\, dx + \int_{\Gamma_F} t_i \gamma(v)\, ds, \tag{5.3}$$

where f_i is the body force depending upon the displacement vector, t_i is the prescribed surface tractions on Γ_F, and ds, the elemental surface area measure. Here we assume that $f_i \in L_2(\Omega)$ and $t_i \in L_2(\Gamma_F)$.

— S: $W' \to L_2(\Gamma_C)$ is a smoothing operator from the dual space W' into $L_2(\Gamma_C)$ representing a transformation of the normal stress $\sigma_N(u)$ in W' into a mollified stress in $L_2(\Gamma_C)$. We assume that S is endowed with the property that $S(\sigma_N(u)) \geqslant 0$ a.e. on Γ_C.

— μ is the coefficient of friction of the contact surface. We assume that $\mu \in L_\infty(\Gamma_C)$, $\mu > \mu_0 \geqslant 0$ a.e. on Γ_C.

Furthermore, we assumed that the nonlinear given function $f(u)$ satisfies the following:

— $f(u)$ is antimonotone, that is,

$$\int_\Omega (f(u) - f(v))(u - v)\, dx \leqslant 0 \tag{5.4}$$

and

$$\| f(u) \|_{L_2(\Omega)} \leqslant r\{ \| u \|_{H^1} \}, \tag{5.5}$$

where $r \equiv r(t)$ is a nondecreasing function for $t \in \mathbb{R}$, $t > 0$. We note that, if the function $f(u)$, defined by (5.3) is a Frechet differentiable, then

$$\langle f'(u), v \rangle = \int_\Omega f(u) v\, dx, \qquad \text{see Noor and Whiteman [13].} \tag{5.6}$$

from which it follows that $f'(u)$ is antimonotone and

$$\| f'(u) \| \leqslant \| f(u) \|_{L_2(\Omega)} \leqslant r\{ \| u \|_{H^1} \}. \tag{5.7}$$

With these preliminaries now established and using the technique of Oden and Pires [1], we can show that the Signorini problem with nonlocal friction can be characterized by a class of highly nonlinear variational inequalities, which is the motivation of our next result.

THEOREM 5.1. *Let $u \in (H^2(\Omega))^N \cap M$ be a solution of problem* (5.1), *then u, is also a solution of the variational inequality* (2.8) *and conversely in the distributional sense, where*

$$b(u, v) = \int_{\Gamma_C} \mu S(\sigma_n(u)) \, |v_T| \, ds \tag{5.8}$$

satisfies the properties (i)–(iv) *and* $\langle f'(u), v \rangle = \int_\Omega f(u) v \, dx$, *see Noor and Whiteman* [13].

Remark 5.1. If $F_n = \sigma_n$ is assumed to be given on all of Γ_C; that is, $\mu \, |\sigma_n(u)| = g$, g given in $L_\infty(\Gamma_C)$, then the contact surface Γ_C is known in advance and u is not prescribed on Γ_C. Then the solution of problem (5.1) can be characterized by a class of variational inequalities of the type:

$$a(u, v-u) + j(v) - j(u) \geqslant \langle f'(u), v-u \rangle, \qquad \text{for all} \quad v \in M, \tag{5.9}$$

where $j(v)$ is given by the relation

$$j(v) = \int_{\Gamma_C} g \, |v_T| \, ds$$

is obviously convex, proper, and lower semi-continuous.

In order to apply the result of Sections 3, we must show that all the hypotheses of Theorem 2.2 are satisfied. Now, in view of the symmetry and ellipticity of E_{ijkl}, the bilinear form $a(u, v)$ defined by (5.2) is coercive and continuous. The form $b(u, v) = \int_{\Gamma_C} \mu S(\sigma_n(u)) \, |v_T| \, ds$ also satisfies the properties, (i)–(iv) as shown by Oden and Pires [1]. As the nonlinear function $f(u)$ defined by (5.3) is antimonotone and Lipschitz continuous by the assumptions, thus showing that all the hypotheses of Theorem 2.2 are satisfied. Hence, it follows from Theorem 2.2, that there does exist a solution of problem 5.1.

Finite Element Approximation

We now consider finite element approximation of the variational inequality (2.8). Following standard finite element techniques, we construct a partition of $\bar{\Omega}$ into a mesh of finite elements over which the displacements are approximated by piecewise polynomials. This defines a finite dimensional subspace S_h of H. By constructing a sequence of regular refinemens of the mesh, we generate a family $\{S_h\}$, $h>0$, of subspaces of H. It is well known [14] that the subspaces S_h exhibit the following asymptotic interpolation properties.

If $u \in (H^r(\Omega))^N$, $r \geqslant 0$ then there exists a constant $c>0$ such that

$$\inf_{v_h \in S_h} \{\|u-v_h\|_0 + h\,\|u-v_h\|_H\} \leqslant Ch^s\,\|u\|_{r,\Omega}, \qquad s=\min(3, r-1). \tag{5.10}$$

Let Γ_C^h denote the boundary of the mesh that approximates Γ_C and Σ_e denote the set of all odal points e on Γ_C^h. We assume that $\Sigma_e = \Gamma_C^h \cap \Gamma_C$. As an approximation of the constraint set M, we introduce

$$M_h = \{\phi_h \in C^0(\Gamma_C^h)\colon \phi_h = \gamma(v_h)\cdot N,\, v_h \in S_h,\, \phi_h(e) \leqslant s_h(e), \quad \text{for all} \quad e \in \Sigma_e\}, \tag{5.11}$$

where s_h is the $L_2(\Gamma_C)$-projection of s on the space W_h of normal traces of functions in S_h. Thus, in our discrete model of the friction problem (2.8), we impose the contact condition only at the modal points on Γ_C^h. Clearly, in general, $M_h \not\subset M$.

We also need the following result, which can be easily proved by using the methods of Noor [15] and Janovsky and Whiteman [16].

LEMMA 5.1. *There exists a constant C_1 such that*

$$\|f(u_h)\|_{L_2(\Omega)} \leqslant C_1, \qquad \textit{for all} \quad h>0. \tag{5.12}$$

For simplicity, we only consider the special case of Theorem 4.1(ii), where $M_h \subset M$. Taking $U = U' = (L_2(\Omega))^2$ and using Lemma 5.1, we obtain

$$\|u-u_h\|_1 = O(h^{1/2}). \tag{5.13}$$

Note that in the absence of the friction term $b(u, v)$, it has been shown in [15], that the error estimate in the energy norm is of order h.

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