On the focal defect group of a block, characters of height zero, and lower defect group multiplicities

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Abstract

We discuss the focal subgroup of the defect group $D$ of a $p$-block $B$, which we refer to as the focal defect group, and denote by $D_0$. We note that (the character group) of $D/D_0$ acts (in a defect (or height) preserving fashion) on irreducible characters in $B$, and prove that the action on irreducible characters of height zero is semi-regular. We also prove that all orbits under this action have length divisible by $[Z(D) : D_0 \cap Z(D)]$. As applications, we prove that all Cartan invariants for $B$ are divisible by $[Z(D) : D_0 \cap Z(D)]$, that if $\text{Out}(D)$ is a $p$-group (and $D \neq 1$), then the number of irreducible characters of height zero in $B$ is divisible by $p$ and that if $Z(D) \nsubseteq D_0$, then the block $B$ is of Lefschetz type (see [R. Knörr, G.R. Robinson, Some remarks on a conjecture of Alperin, J. London Math. Soc. (2) 39 (1) (1989) 48–60]).

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Notation, terminology and general background

Let $G$ be a finite group, $p$ be a prime, $(K, R, F)$ be a $p$-modular system satisfying the “usual properties,” and let $B$ be a block of $RG$ with defect group $D$, of order $p^d$. Let $(D, b_D)$ denote a fixed maximal $B$-subpair, and for each subgroup $Q$ of $D$, let $(Q, b_Q)$ denote the unique $B$-subpair contained in $(D, b_D)$ with subgroup component $Q$. Let $p^d$ be the order of a Sylow $p$-subgroup of $G$. Overlooking slight abuses of notation, we let $\mathcal{F}$ denote the fusion system associated to the $B$-subpairs in $G$ and $N_{\mathcal{F}}(Q)$ denote $N_{G}(Q, b_Q)$. As usual, we let $\text{Irr}(B)$

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denote the set of (ordinary) irreducible characters in $B$, and $\text{Irr}_e(B)$ denote the set of (ordinary) irreducible characters of defect $e$ in $B$. We denote $|\text{Irr}(B)|$ by $k(B)$, and $|\text{Irr}_e(B)|$ by $k_e(B)$. Notice that, in this notation, the number of irreducible characters of height zero in $B$ is $k_d(B)$.

We define the subgroup $D_0$ of $D$ as follows:

$$D_0 = \left\{ [D, N_\mathcal{F}(D)], [U, N_\mathcal{F}(U)] : U \in \mathcal{E}(B) \right\},$$

where $\mathcal{E}(B)$ denotes the set of (subgroup components of) the essential $B$-subpairs contained in $(D, b_D)$. Henceforth, we will refer to $D_0$ as the *focal defect group* of $B$. Then $D_0 \vartriangleleft D$, $D/D_0$ is Abelian, and by the Alperin–Broué fusion theorem for blocks (from [1]), we see that whenever $\lambda$ is a linear character of $D$ which contains $D_0$ in its kernel and $(x, b_x)$ and $(y, b_y)$ are $G$-conjugate Brauer elements contained in $(D, b_D)$, then $\lambda(x) = \lambda(y)$. In fact, it follows from [8] that $D_0 = D \cap G'_{\mathcal{F}}$, where $G'_{\mathcal{F}}$ is the iterated amalgam realising the saturated fusion system $\mathcal{F}$ associated to $B$ (see [7]).

1. Character group action on $\text{Irr}(B)$

We will show that if $\chi$ is an irreducible character in $B$ and $\lambda$ is an irreducible character of $D/D_0$, then $\lambda \ast \chi$ is also an irreducible character in $B$, where (as in the Broué–Puig “star” construction from [2]), $\lambda \ast \chi$ is defined as follows: set $\chi = \sum_{(x, b_x)} \text{Ind}^G_{C_G(x)}(d_{x, j}^x \tilde{\phi}_j)$, where $(x, b_x)$ runs through a full set of $G$-conjugacy classes of the Brauer elements contained in $(D, b_D)$ and $\phi_j$ runs through the irreducible Brauer characters in $b_x$, and $d_{x, j}^x$ is the usual generalised decomposition number. Here, $\tilde{\phi}_j(xy) = \phi_j(y)$ for $p$-regular $y \in C_G(x)$ and $\tilde{\phi}_j$ vanishes on all elements of $C_G(x)$ which do not have $p$-part $x$.

Then $\lambda \ast \chi = \sum_{(x, b_x)} \text{Ind}^G_{C_G(x)}(\lambda(x)d_{x, j}^x \tilde{\phi}_j)$ (retaining the same notation). It is clear that $\lambda \ast \chi$ is a linear combination of irreducible characters in $B$, by Brauer’s second main theorem. Clearly, $\lambda \ast \chi$ is an irreducible character if and only if it is a generalised character, and $\lambda \ast \chi(xy) = \lambda(x)\chi(xy)$ whenever $x \in D$ and $y$ is a $p$-regular element of $C_G(x)$.

By the Broué–Puig induction theorem (again from [2]), it suffices to prove that

$$\langle \text{Res}^G_{C_G(Q)}(\lambda \ast \chi), \mu \ast \Phi \rangle \in \mathbb{Z}$$

whenever $Q$ is a subgroup of $D$, and $\Phi$ is the character of a Projective indecomposable $b_Q$-module (where $\mu \ast \Phi(tu) = \mu(t)\Phi(u)$ for $t \in Q$ $p$-regular $u \in C_G(Q)$ and $\mu \ast \Phi(tu) = 0$ if the $p$-part of $u$ lies outside $Z(Q)$ (but $u \in C_G(Q)$ and $t \in Q$). But it is clear that

$$\langle \text{Res}^G_{Q C_G(Q)}(\lambda \ast \chi), \mu \ast \Phi \rangle = \langle \text{Res}^G_{Q C_G(Q)}(\chi), \text{Res}^D_{Q(\tilde{\lambda})}(\mu \ast \Phi) \rangle,$$

which is certainly an integer.

We next observe that if $\chi$ has height zero in $B$ and $\lambda$ is a non-trivial linear character of $D/D_0$, then $\lambda \ast \chi \neq \chi$. For choose an element $x \in D \setminus (\ker \lambda)$. There is a $p$-regular element $y \in G$ such that $D \in \text{Syl}_p(C_G(y))$ and

$$[G : C_G] \frac{\chi(y)}{\chi(1)} \notin J(R).$$
Then $\chi(y) \notin J(R)$ so that $\chi(xy) \notin J(R)$. In particular $\chi(xy) \neq 0$, so that $\lambda \ast \chi(xy) \neq \chi(xy)$ as $\lambda(x) \neq 1$. (This is an argument which really dates back to R. Brauer.)

When $|D| > 1$, it has long been known that the number of irreducible characters of height zero in $B$ at least 2, is divisible by $p$ if $p < 5$, and is divisible by 4 if $p = 2$ and $|D| > 2$. In the case that $D_0 < D$, this can be strengthened somewhat.

We proved above that (the character group of) $D/D_0$ acts semi-regularly on $\text{Irr}_d(B)$ (which is also the set of irreducible characters of height zero in $B$). If $[D : D_0] = p$ and $|D| > p$, then we have $p \sum_{i \in I} \chi_1^2 \equiv 0$ (mod $p^2$), where $I$ is a set of representatives for the orbits of $D/D_0$ on the irreducible characters of height 0 in $B$. By a standard argument, this implies that in $\mathbb{Z}/p\mathbb{Z}$, we may express 0 as a sum of $|I|$ non-zero squares. If $p < 5$, this implies that $|I|$ is divisible by $p$ and if $p \equiv 3$ (mod 4), it implies that $|I| \geq 3$. Hence we have:

**Theorem 1.** If $D \neq D_0$, then the number of irreducible characters of height zero in $B$ is a multiple of $[D : D_0]$ so is, in particular, divisible by $p$. If $p = 2$ or 3 and we also have $|D| > p$ (as well as the previous hypothesis), then $k_d(B) \equiv 0$ (mod $p^2$).

For irreducible characters of positive height, the action of $D/D_0$ obtained via the $\ast$-construction need not be semi-regular. However, we do have the following result (which proves, in particular, that the above action is semi-regular on all irreducible characters in $B$ in the case that the defect group is Abelian, even though (at present, at least), we do not know that all irreducible characters in $B$ have height 0).

**Theorem 2.** For every non-negative integer $e$, $k_e(B)$ is divisible by $[Z(D) : D_0 \cap Z(D)]$.

**Proof.** We know that (the character group of) $D/D_0$ acts on $\text{Irr}_e(B)$ via $\chi \rightarrow \lambda \ast \chi$. We first claim that if the linear character $\lambda$ of $D$ (with $D_0$ in its kernel) fixes an irreducible character $\chi$ in this action, then $Z(D)D_0 \subseteq \ker \lambda$. For if $z \in Z(D) \setminus \ker \lambda$, and $\lambda \ast \chi = \chi$, then $\chi$ vanishes identically on the $p$-section of $z$, contrary to a theorem of Brauer, as $z$ is central in $D$. Hence the number of elements of $D/D_0$ which fix any irreducible character at all in $B$ in this action is at most $[D : Z(D)D_0]$, so, in particular, the length of any orbit is at least

$$\frac{[D : D_0]}{[D : Z(D)D_0]} \geq [Z(D) : D_0 \cap Z(D)]$$

(so is divisible by this integer, as both are powers of $p$). \qed

2. Further consequences and applications

The first application may have some independent interest, and is easily seen to be in accordance with the predictions of the Alperin–McKay conjecture (although in somewhat restrictive circumstances).

**Corollary 3.** Suppose that $|D| > 1$, and that $N_F(D) = DCG(D)$. Then the number of irreducible characters of height zero in $B$ is divisible by $p$. 

\textbf{Proof.} If $p \in \{2, 3\}$, then (as Brauer knew), we have that

$$\sum_{\mu \in \text{Irr}(B)} \frac{\mu(1)^2}{p^{2a-2d}}$$

is divisible by $p^d$, but not by $p^{d+1}$. Hence the number of irreducible characters of height zero in $B$ is divisible by $p$ in each case. Thus we may suppose that $p \geq 5$.

Set $K = K_\infty(D)$ (the characteristic subgroup defined by Glauberman in [4]). From the recent article (Diaz, Glesser, Mazza, Park [3]), we have $[U, N_{\mathcal{F}}(U)] \leq D \cap N_{\mathcal{F}}(K)'$ for each subgroup component $U$ of a centric $B$-subpair contained in $(D, b_D)$.

Now we claim (still in the case $p \geq 5$) that if $N_{\mathcal{F}}(D) = DCG(D)$, then $D_0 < D$. Using [3] again, it suffices to deal with the case that $G = N_{\mathcal{F}}(K)$. If $K = D$, then $D_0 \leq [D, G] \leq D_0$, so that $D_0 = D'$, and the claim clearly holds. Otherwise, we may work with the Kühlhammer–Puig extension $L(K)$, which is an extension of $G/KCG(K)$ by $K$. The group $L(K)$ has Sylow $p$-subgroup $D$ and has the same fusion on $p$-subgroups as $G$ does on $B$-subpairs (see [6]). From (Glauberman [4]), $L(K)$ has a normal subgroup $T$ of index $p$. Let $D_1$ be a Sylow $p$-subgroup of $T$. Then (using the focal subgroup theorem (for groups) in $L(K)$, and translating the fusion information back to $G$) means that

$$D_0 = \left[\left[D, N_{\mathcal{F}}(D)\right], \left[U, N_{\mathcal{F}}(U)\right] : U \in \mathcal{E}(B)\right] \leq D_1 < D,$$

as required. □

\textbf{Remarks.} It follows from these arguments that if $D_0 < D$ and $|D| \geq p^2$, then $k_d(B) \geq 2p$. For we obtain

$$\sum_{\chi \in \text{Irr}_d(B)} \chi(1)^2_p \equiv 0 \pmod{p^2}.$$ 

If there were only $p$ irreducible characters of height zero, then they would all have the same degree, say $p^{a-d}t$. Then we would have $pt^2 \equiv 0 \pmod{p^2}$, a contradiction. Notice that if $p$ is odd and $\text{Out}(D)$ is a $p$-group (but $D \neq 1$), then must have $|D| \geq p^3$. Hence:

\textbf{Corollary 4.} If $D \neq 1$ and $\text{Out}(D)$ is a $p$-group, then $k_d(B) \equiv 0 \pmod{p}$. If also $p > 3$, then $k_d(B) \geq 2p$, and if $p \equiv 3 \pmod{4}$ (still $p > 3$), then we have $k_d(B) \geq 3p$.

In general, it need not be the case that the Cartan invariants of the block $B$ with defect group $D$ are divisible by $[D : D_0]$. For example, when $p = 2$ and $B$ is the principal 2-block of $S_4$, we have $D_0 < D$, but there are odd Cartan invariants, since there is a unique 2-block, but 1 occurs as an elementary divisor of the Cartan matrix, as there is a 2-regular class of defect 0.

In general, it seems to be relatively difficult to provide $p$-locally determined conditions which guarantee that 1 does not occur as an elementary divisor for the defect group of the block $B$, so we consider the following result in that direction worth remarking on. We also take the opportunity to include an application related to Alperin’s conjecture. We recall that in [5], the block $B$ was said to be of \textit{Lefschetz type} if the alternating sum occurring in the reformulation of Alperin’s conjecture vanishes (see [5] for a more precise statement).
Corollary 5.

(i) All Cartan invariants for $B$ are divisible by $[Z(D) : D_0 \cap Z(D)]$. In particular, if $Z(D) \nsubseteq D_0$, then all Cartan invariants of $B$ are divisible by $p$.

(ii) If $Z(D) \nsubseteq D_0$, then $B$ is a block of Lefschetz type.

Proof. (i) If $\theta$ and $\psi$ are characters of projective indecomposable $B$-modules, then all irreducible constituents of $\theta$ in a given orbit under the above action occur with equal multiplicity in $\theta$, and likewise for $\psi$. Since each such orbit has length divisible by $[Z(D) : D_0 \cap Z(D)]$, we see that $\langle \theta, \psi \rangle$ is divisible by $[Z(D) : D_0 \cap Z(D)]$.

(ii) Using (i), we see that whenever $\sigma$ is a chain of $B$-subpairs as in Knörr and Robinson [5] (and arguing as in Proposition 5.1 of [5]), we have (in the notation of [5]) $\ell_0(B_\sigma) = 0$, so the result follows by Corollary 4.3 of [5].

References