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Fréchet Differentiability of Regular Locally Lipschitzian Functions

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This paper considers Fréchet differentiability almost everywhere in the sense of category of regular, locally Lipschitzian real-valued functions defined on open subsets of a Banach space. It is first shown that, for separable Banach spaces, Clarke's generalized gradient of such a function is a minimal, convex- and compact-valued, upper semicontinuous multifunction. Using a theorem of Christensen and Kenderov it is then shown that, for separable Asplund spaces, such a function is Fréchet differentiable on a dense G_δ subset of its domain. © 1991 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

In 1919 H. Rademacher [20] proved that any Lipschitzian mapping of an open set U in $\mathbb{R}^m$ into $\mathbb{R}^n$ is differentiable almost everywhere in U. Several people have extended this result to (real) Banach spaces; cf. [1, 7, 16]. There are two main difficulties to be overcome in these extensions: what kind of differentiability to use and what kind of exceptional sets to allow. For example, in 1976 Aronszajn [1] extended Rademacher's theorem to Lipschitzian mappings between separable Banach spaces. He proved that such a Lipschitzian mapping is Gâteaux differentiable almost everywhere with respect to a certain class of exceptional sets—see [1, Chap. II, Theorem 1]. Christensen, see [7], proved a similar theorem with the exceptional sets being the Haar-null sets. A discussion of the relationship between these two results and also some other results can be found in [16]. Examples of locally Lipschitzian functions that are Gâteaux differentiable everywhere, but that are nowhere Fréchet differentiable can be found in [1, 3, 21, 23]. On the other hand, if more is assumed about the spaces and the function, then it is sometimes possible to prove Fréchet differentiability almost everywhere in the sense of topology (category). For example, every continuous, convex, real-valued function on an open subset of certain Banach spaces, called Asplund spaces, is Fréchet differentiable

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except for the complement of a dense G_δ (see [13, 17] for references and for related work on theorems for Gâteaux differentiability). A natural question is whether *real-valued*, Lipschitzian functions on certain Banach spaces are Fréchet differentiable almost everywhere in some appropriate sense. Most of the examples of Lipschitzian nowhere Fréchet differentiable functions mentioned earlier are not real-valued, but there is an example in the literature (see [4, Example 3.3; 7, p. 124; 14, p. 125]) of a Lipschitzian function $f: L^2([0, \pi]) \to \mathbb{R}$ that is supposed to be nowhere Fréchet differentiable. Fortunately or unfortunately, that example *is* Fréchet differentiable everywhere (see [12, 17]). On the other hand, there is an example, due to Sova [21], of a Lipschitzian function $f: L^1([0, \pi]) \to \mathbb{R}$ that is nowhere Fréchet differentiable. However, the space $L^1([0, \pi])$ is not an Asplund space so this is not surprising. In this paper we give another class of Lipschitzian, real-valued functions that are Fréchet differentiable almost everywhere in the sense of category. In particular, we prove the following theorem.

THEOREM 3.3. *Let $f: U \to \mathbb{R}$ be a locally Lipschitzian, regular function on an open subset U of a real, separable Asplund space. Then f is Fréchet differentiable on a dense G_δ (equivalently, except for a set of the first category).*

Recently, Preiss [19] (see also [17, Theorem 4.12]) has proved a similar theorem, which we state here for purposes of comparison.

THEOREM (Preiss). *Any locally Lipschitzian real-valued function on an Asplund space is Fréchet differentiable at the points of a dense set.*

As is evident, our hypotheses are stronger. We require the space to be separable and that the function be regular (see below for the definition). However, our conclusion is stronger—we get Fréchet differentiability on a dense G_δ, not just on a dense subset. This raises the question of whether our conclusion is true without some hypothesis like regularity. The answer is no, as is shown in Example 4.3 (see also [17, p. 104]). Example 4.3, which uses the real line as the Asplund space, gives a Lipschitzian function with points of nondifferentiability of the second category.

We have introduced several ideas here that need to be more clearly defined. We now turn to that task. Let X be a real Banach space with norm $\|\cdot\|$ and dual space X^*, let U be an open subset of X, and let $f: U \to \mathbb{R}$. We say that f is *Gâteaux differentiable* at $x_0 \in U$ if, for every $v \in X$, the directional derivative $f'(x_0, v) = \lim_{t \downarrow 0}((f(x_0 + tv) - f(x_0))/t)$ exists and the mapping $D_G f(x_0; \cdot) = f'(x_0; \cdot)$ is linear and continuous on X. The mapping f is *strictly differentiable* at $x_0 \in U$ iff $\lim_{x \to x_0, t \downarrow 0}((f(x + tv) - f(x))/t) = D_S f(x_0; v)$ exists for each $v \in X$ and $D_S f(x_0; \cdot)$ is linear and continuous on X. f is Fréchet differentiable at $x_0 \in U$ if there exists a continuous linear functional $D_F f(x_0; \cdot)$ such that

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - D_F f(x_0; x - x_0)|}{\|x - x_0\|} = 0.$$

The strict differentiability concept here is a Gâteaux-type strict derivative. Clarke [5, p. 30] has a slightly different definition, but notes that the two are the same when f is locally Lipschitzian. Since we consider only locally Lipschitzian functions, we can use the above definition of strict differentiability and still use the results of Clarke. It is clear that either Fréchet or strict differentiability imply Gâteaux differentiability. There are known examples where the converses are not true; cf. [11] and Example 4.2 in Section 4.

A mapping $f: U \to \mathbb{R}$ is *locally Lipschitzian at a point* $x \in U$ if there exists a neighborhood V of x in U and a constant K such that, for every $x_1, x_2 \in V$, $|f(x_1) - f(x_2)| \leqslant K \|x_1 - x_2\|$. A mapping f is called *locally Lipschitzian* if it is locally Lipschitzian at each point $x \in U$. A valuable tool for dealing with locally Lipschitzian functions is the generalized gradient of Clarke; see [5]. It is defined as follows: The *generalized directional derivative* $f^\circ(x_0; v)$ of f at x_0 in the direction $v \in X$ is $f^\circ(x_0; v) = \overline{\lim}_{x \to x_0,\, t \downarrow 0}((f(x + tv) - f(x))/t)$. The *generalized gradient* ∂f of f at x_0 is $\partial f(x_0) = \{\xi \in X^* \mid \langle \xi, v \rangle \leqslant f^\circ(x_0; v)$ for all $v \in X\}$. From the above definitions it appears that the generalized gradient is related to the strict derivative. The following result, see [5, Proposition 2.2.4], shows this. A locally Lipschitzian mapping $f: U \to \mathbb{R}$ is strictly differentiable at x iff $\partial f: U \to X^*$ is single-valued, and in that case, $\{D_S f(x; \cdot)\} = \partial f(x)$. In the calculus of generalized gradients, formulas often involve inclusions. A class of mappings where these inclusions are equalities are the regular functions: A locally Lipschitzian function f is *regular* at x provided, for all $v \in X$, the usual one-sided directional derivative $f'(x; v)$ exists and equals $f^\circ(x; v)$. Properties, as well as classes, of regular functions are discussed in [5]. For instance, continuous convex functions are regular. Other functions important in optimization such as pointwise maxima and integral functions are also regular; see [5, Sects. 2.7, 2.8]. Further, a locally Lipschitzian function is Gâteaux differentiable and regular at x iff f is strictly differentiable at x [5, Propositions 2.3.6 and 2.2.4].

In Section 2, we prove that the generalized gradient is minimal in the class of upper semicontinuous compact- and convex-valued multifunctions from X to X^*. Then, in Section 3.3, we use this and a theorem of Christensen and Kenderov [8] on single-valuedness of multifunction to prove Theorem 3.3. Finally, in Section 4, we give some examples.

We generally follow the notation in [5].

2. The Generalized Gradient Is Minimal Convex USCO

A multifunction $F: X \to Y$ from a Hausdorff space X to a Hausdorff space Y is usco (upper semicontinuous and compact-valued) if

(i) For each $x \in X$, $F(x)$ is nonempty and compact;

(ii) For each open set $V \subset Y$, the set $\{x \in X \mid F(x) \subset V\}$ is open in X.

As in Section 1, let X be a Banach space, U an open subset of X, (X^*, ω^*) be the dual of X with the weak* topology, and $f: U \to \mathbb{R}$ be locally Lipschitzian. Then, as we indicate below, the generalized gradient $\partial f: U \to (X^*, \omega^*)$ has nonempty, convex, weak*-compact values. If X is separable, then ∂f is upper semicontinuous. This result is essentially contained in [5, Proposition 2.1.2 and Proposition 2.1.5] or in [2, Proposition 10, p. 422]. We sketch a proof of upper semicontinuity based on the treatment in [5], whose notation we are following.

Proposition 2.1. *Let X be a separable Banach space. Then the generalized gradient $\partial f: U \to (X^*, \omega^*)$ is a convex-valued usco.*

Proof. For $x \in U$, $\partial f(x)$ is nonempty, convex, and weak*-compact [5, Proposition 2.1.2]. To complete the proof we must show ∂f is upper semicontinuous. To show this, let $x_0 \in U$ and choose an open neighbourhood V of x_0 in U such that f is Lipschitzian with constant K on V. By [5, Proposition 2.1.2], the set $\partial f(V)$ is norm bounded by K; that is, $\partial f(V) \subset B_K(X^*)$, the closed ball of radius K in X^*. It is known that $B_K(X^*)$ is ω^*-compact, and since X is separable, the ω^*-topology on $B_K(X^*)$ is metrizable. But this means that $\partial f: U \to B_K(X^*)$, where $B_K(X^*)$ is a compact metric space. Now it is a standard result that, for closed-valued multifunctions with compact metrizable range, closed graph is equivalent to upper semicontinuity. But in [5, Proposition 2.1.5] it is shown that ∂f has closed graph (in the ω^*-topology). Thus ∂f is usco.

By Zorn's lemma, every usco multifunction $F: X \to Y$ contains a minimal (with respect to inclusion of graphs) usco mapping. The analogous result is also true for convex-valued usco mappings. We next show that ∂f is a minimal convex usco when f is locally Lipschitzian and regular.

Theorem 2.2. *Let X be a separable Banach space, U an open subset of X, $f: U \to \mathbb{R}$ a locally Lipschitzian, regular function. Then the multifunction $\partial f: U \to (X^*, \omega^*)$ is a minimal convex-valued usco.*

In order to prove Theorem 2.2 we will use some lemmas.

Lemma 2.3 (Thibault [22, Proposition 2.2]). *Let X, U be as in Theorem 2.2 and let $f: U \to \mathbb{R}$ be locally Lipschitzian. Let f be Gâteaux differentiable* *on a set M, where $U \backslash M$ is a Haar-null set (see* [7] *for the definition of Haar-null and the existence of such an M). Then*

(i) $f^{\circ}(x; v) = \max\{\langle \xi, v\rangle \mid \xi = \lim_{n\to\infty} D_G f(x_n; \cdot),\ x_n \to x,\ x_n \in M\}$ *for all $v \in X$. (The convergence $\lim_{n\to\infty} D_G f(x_n; \cdot) = \xi$ is in the ω^*-topology while $x_n \to x$ in the norm topology of X.)*

(ii) $\partial f(x) = \overline{\text{co}}\{\lim_{n\to\infty} D_G f(x_n; \cdot) \mid x_n \to x,\ x_n \in M\}$. *(The closure is in the ω^*-topology on X^*.)*

LEMMA 2.4 (Drewnowski and Labuda [9, Proposition 4.1 and Theorem 4.3]). *Let $F: U \to (X^*, \omega^*)$ be a convex-valued usco, where U is an open subset of a Banach space X. The following are equivalent:*

(i) *F is a minimal convex-valued usco;*

(ii) *for each open set $V \subset U$ and each closed half-space P in (X^*, ω^*), $F(x) \cap P \neq \varnothing$ for each $x \in V$ implies $F(V) \subset P$;*

(iii) *$vF: U \to \mathbb{R}$ is a minimal convex-valued usco for all $v \in X$.*

A proof of Lemma 2.4 can be found in [9], where somewhat more general and other related results are given.

LEMMA 2.5. *Let U be an open subset of a Banach space X. If $f: U \to \mathbb{R}$ is locally Lipschitzian, then*

$$v\partial f(x) = \{y \in \mathbb{R} \mid -f^{\circ}(x; -v) \leqslant y \leqslant f^{\circ}(x; v)\}$$

for each $v \in X$.

Proof. Let $y = \langle \xi, v\rangle$ for some $\xi \in \partial f(x)$. Then, by the definition of the generalized gradient,

$$f^{\circ}(x; v) \geqslant \langle \xi, v\rangle = y \qquad \text{and} \qquad f^{\circ}(x; -v) \geqslant \langle \xi, -v\rangle = -y.$$

Therefore $v\partial f(x) \subset \{y \in \mathbb{R} \mid -f^{\circ}(x; -v) \leqslant y \leqslant f^{\circ}(x; v)\}$. To prove the opposite inclusion, recall that for all $v \in X$, $f^{\circ}(x; v) = \max\{\langle \xi, v\rangle \mid \xi \in \partial f(x)\}$ (see [5, Proposition 2.1.2(b)]). Since $\partial f(x)$ is ω^*-compact, there exist $\hat{\xi}, \hat{\hat{\xi}} \in \partial f(x)$ such that $\langle \hat{\xi}, v\rangle = f^{\circ}(x; v)$ and $\langle \hat{\hat{\xi}}, -v\rangle = f^{\circ}(x; -v)$. Thus both $f^{\circ}(x; v)$ and $-f^{\circ}(x; -v)$ belong to the set $v\partial f(x)$. But $v\partial f(x)$ is convex and so the inclusion follows.

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. By Proposition 2.1, the generalized gradient ∂f is a convex-valued usco. By Lemma 2.4 ((i) and (iii)), it suffices to show that $v\partial f(\cdot)$ is a minimal convex-valued usco for each $v \in X$. But $v\partial f(\cdot)$ is clearly a convex-valued usco, so we need only show minimality.

In view of Lemma 2.4 ((i) and (ii)), we need to show that $v\partial f(V) \subset P$ for every closed half-space $P \subset \mathbb{R}$ and every open set $V \subset U$ such that $v\partial f(x) \cap P \neq \varnothing$ for each $x \in V$. So let P and V be such that $v\partial f(x) \cap P \neq \varnothing$ for $x \in V$. We may asume $P = \{y \in \mathbb{R} \mid y \leqslant a\}$ since the case $P = \{y \in \mathbb{R} \mid y \geqslant b\}$ is not essentially different. Since $v\partial f(x) \cap P \neq \varnothing$ for each $x \in V$, we have $-f^\circ(x; -v) \leqslant a$ by Lemma 2.5. Suppose we could have $f^\circ(x; v) > a$ for some $x \in V$. Then, by Lemma 2.3(i), we would have $\overline{\lim}_{y \to x,\, y \in M} D_G f(y; v) = \max\{\lim D_G f(x_n; v) \mid x_n \to x \text{ in norm, } x_n \in M\} = f^\circ(x; v) > a$, where M is the set of points on which f is Gâteaux differentiable. The assumption that f is regular gives, by [5, Proposition 2.3.6(d)], $\partial f(y) = \{D_G f(y; \cdot)\}$ and hence $v\partial f(y) = D_G f(y; v)$ for $y \in M$. Thus there exists $y \in M \cap V$ such that $v\partial f(y) = D_G f(y; v) > a$. But this means that $v\partial f(y) \cap P = \varnothing$, a contradiction. Thus $f^\circ(x; v) \leqslant a$ and $v\partial f(y) \subset P$ for all $y \in V$, and we have shown that ∂f is a minimal convex-valued usco.

Note that $\partial f(x)$ is single-valued on M, which certainly makes $\partial f(x)$ minimal for $x \in M$.

The next proposition, which is somewhat of a tangent to the rest of this paper, explains the relationship between strict differentiability and Gâteaux differentiability for locally Lipschitzian functions. It is a slight extension of [6, Proposition 1.13].

PROPOSITION 2.6. *Let $f\colon U \to \mathbb{R}$ be a locally Lipschitzian mapping on an open subset U of a separable Banach space X. The following are equivalent*:

(i) *f is strictly differentiable at x*;

(ii) *f is Gâteaux differentiable at x and the mapping $D_G f\colon U \to (X^*, \omega^*)$ is continuous at x relative to the set upon which the Gâteaux derivative exits.*

Proof. (i) $\Rightarrow$ (ii). It is obvious that strict differentiability implies Gâteaux differentiability. Since strict differentiability of f at x implies $\partial f(x) = \{D_S f(x; \cdot)\}$ [5, Proposition 2.2.4], Lemma 2.3(ii) yields the continuity of the mapping $D_G f$ at $x \in U$ with the norm topology on X and the ω^*-topology on X^*.

(ii) $\Rightarrow$ (i). Again Lemma 2.3(ii) shows that $\partial f(x)$ is a singleton and by [5, Proposition 2.2A (converse)] we get $D_S f(x; \cdot)$ exists and is that singleton.

3. FRÉCHET DIFFERENTIABILITY ALMOST EVERYWHERE

The next proposition relates strict differentiability and "upper semicontinuity" of the generalized gradient of a locally Lipschitzian function to

Fréchet differentiability. The type of upper semicontinuity we will use here is sometimes called upper semicontinuity with respect to inclusion (usci) and can be defined as follows: Let $F: X \to (Y, d)$ be a multifunction where (Y, d) is a metric space. F is usci at $x_0 \in X$ if, for each $\varepsilon > 0$, there is a neighborhood $U(x_0)$ of x_0 such that $F(x) \subset \{y \in Y \mid d(y, F(x_0)) < \varepsilon\}$ for each $x \in U(x_0)$. The set $\{y \in Y \mid d(y, F(x_0)) < \varepsilon\}$ is called an ε-neighborhood of the set $F(x_0)$. Note that usci is implied by upper semicontinuity since upper semicontinuity requires the above property to hold for every neighborhood of $F(x_0)$ and not just ε-neighborhoods.

PROPOSITION 3.1. *Let $f: U \to \mathbb{R}$ be a locally Lipschitzian function on an open subset U of a Banach space X with norm $\|\cdot\|$. If the generalized gradiant $\partial f: U \to X^*$ is single-valued and usci at x_0 from the norm topology on X into the norm topology on X^*, then f is Fréchet differentiable at x_0.*

Proof. Since the mapping f is usci at x_0, for each $\varepsilon > 0$ there is $\delta > 0$ such that if $\|x - x_0\| < \delta$, then $\partial f(x) \subset \partial f(x_0) + \varepsilon B$, where B is the unit ball in X^*. By the mean value theorem for generalized gradients [5, Theorem 2.3.7], there exists $u = x_0 + t(x - x_0)$, $0 < t < 1$, such that $f(x) - f(x_0) \in \langle \partial f(u), x - x_0 \rangle$. Hence, for some $u', u'' \in \partial f(u)$, we have $\langle u', x - x_0 \rangle \leqslant f(x) - f(x_0) \leqslant \langle u'', x - x_0 \rangle$. Therefore

$$\begin{aligned}\langle u', x - x_0 \rangle - \langle \partial f(x_0), x - x_0 \rangle &\leqslant f(x) - f(x_0) - \langle \partial f(x_0), x - x_0 \rangle \\ &\leqslant \langle u'', x - x_0 \rangle - \langle \partial f(x_0), x - x_0 \rangle.\end{aligned}$$

By the usci of ∂f at x_0 we have

$$|\langle u'', x - x_0 \rangle - \langle \partial f(x_0), x - x_0 \rangle| \leqslant \varepsilon \|x - x_0\|$$

and

$$|\langle u', x - x_0 \rangle - \langle \partial f(x_0), x - x_0 \rangle| \leqslant \varepsilon \|x - x_0\|$$

if $\|x - x_0\| < \delta$. Thus, for $\|x - x_0\| < \delta$,

$$|f(x) - f(x_0) - \langle \partial f(x_0), x - x_0 \rangle| \leqslant \varepsilon \|x - x_0\|$$

and hence f is Fréchet differentiable at x_0.

To show that locally Lipschitzian, regular real-valued mappings are Fréchet differentiable almost everywhere, we will apply the following theorem of Christensen and Kenderov.

THEOREM 3.2 [8, Lemma 1.6]. *Let $F: U \to (X^*, \omega^*)$ be a minimal usco multifunction from an open (Baire) subset U of an Asplund space X into the* *dual space* (X^*, ω^*). *Then there exists a dense* G_δ *subset* C *of* U *such that, for each* $x \in C$, *the set* $F(x)$ *is a singleton and* $F: U \to X^*$ *is usci at* x *with respect to the norm topologies.*

Remark. A Banach space X is an Asplund space iff every continuous real-valued convex function on an open convex subset is Fréchet differentiable at every point of some dense G_δ subset of its domain. In [8, Lemma 1.6] it is assumed that X^* has the Radon–Nikodym property; it is known (cf. [10, p. 213]) that this is equivalent to assuming that X is an Asplund space.

THEOREM 3.3. *Let* $f: U \to \mathbb{R}$ *be a locally Lipschitzian, regular function on an open subset* U *of a separable Asplund space* X. *Then* f *is Fréchet differentiable on a dense* G_δ *subset of* U.

Proof. We apply Theorem 3.2 to the generalized gradient of the mapping f. According to Theorem 2.2, $\partial f: U \to (X^*, \omega^*)$ is a minimal convex-valued usco. By Zorn's lemma, there exists a minimal usco G contained in ∂f. Theorem 3.2 says that there is a dense G_δ set $C \subset U$ such that, for each $x \in C$, $G(x)$ is a singleton and G is usci at x. Clearly $\partial f(x)$ is the closed convex hull of $G(x)$ in the ω^*-topology, and so, ∂f is single-valued and usci with respect to the norm topology on X^* at each point of C. By Proposition 3.1, f is Fréchet differentiable at each $x \in C$.

Remark. In both Theorems 2.2 and 3.3 we could replace the assumption that f is regular with the assumption that f is strictly differentiable on a subset M of U such that $U \setminus M$ is Haar-null. The conclusions would remain as before and the proofs would differ little.

4. EXAMPLES

We give three examples that partially indicate the reasons for the various hypotheses in Theorem 3.3.

EXAMPLE 4.1. The mapping $f: L^1([0, \pi]) \to \mathbb{R}$ defined by $f(x) = \int_0^\pi \sin x(s)\, ds$, $x \in L^1([0, \pi])$, is strictly differentiable at each $x \in L^1([0, \pi])$, but is nowhere Fréchet differentiable. This example is essentially due to Sova [21] who proved that f is Gâteaux differentiable everywhere and nowhere Fréchet differentiable. We refer the reader to [21] or to [12] for the proof of the nowhere Fréchet differentiability. To prove that f is strictly differentiable at $x \in L^1([0, \pi])$, notice that there exists a sequence (x_n) in $L^1([0, \pi])$ with $x_n \to x$ in the norm topology and a sequence of real numbers (t_n) decreasing to zero such that

$$f^\circ(x; v) = \overline{\lim_{\substack{x' \to x \\ t \downarrow 0}}} \frac{\int_0^\pi [\sin(x'(s) + tv(s)) - \sin x'(s)]\, ds}{t}$$

$$= \lim_{n \to \infty} \frac{\int_0^\pi [\sin(x_n(s) + t_n v(s)) - \sin x_n(s)]\, ds}{t_n}$$

for a given $v \in L^1([0, \pi])$. Since (x_n) converges to x in $L^1([0, \pi])$, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k}(s) \to x(s)$ pointwise a.e. Clearly

$$f^\circ(x; v) = \lim_{k \to \infty} \int_0^\pi \frac{[\sin(x_{n_k}(s) + t_{n_k} v(s)) - \sin x_{n_k}(s)]}{t_{n_k}}\, ds.$$

But

$$\left| \frac{\sin(x_{n_k}(s) + t_{n_k} v(s)) - \sin x_{n_k}(s)}{t_{n_k}} \right| \leqslant |v(s)|,$$

so by the dominated convergence theorem,

$$f^\circ(x; v) = \int_0^\pi \lim_{k \to \infty} \frac{[\sin(x_{n_k}(s) + t_{n_k} v(s)) - \sin x_{n_k}(s)]}{t_{n_k}}\, ds$$

$$= \int_0^\pi v(s) \cos x(s)\, ds.$$

From this we get that

$$\partial f(x) = \{\xi \in L^\infty([0, \pi]) \mid f^\circ(x; v) \geqslant \langle \xi, v \rangle \text{ for } v \in L^1([0, \pi])\}$$

$$= \left\{ \xi \in L^\infty([0, \pi]) \,\middle|\, \int_0^\pi v(s) \cos x(s)\, ds \geqslant \int_0^\pi v(s) \xi(s)\, ds, \right.$$

$$\left. v \in L^1([0, \pi]) \right\}$$

$$= \{\cos x\}.$$

Since f is clearly Lipschitzian, this shows $D_S f(x; \cdot) = \cos x$ by [5, Proposition 2.2.4].

The next example is classical and shows that Fréchet differentiability does not imply strict differentiability.

EXAMPLE 4.2. $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is Fréchet differentiable at $x=0$, but is not strictly differentiable since it is easy to prove that $\partial f(0)=\{t\in\mathbb{R} \mid -1\leqslant t\leqslant 1\}$.

The next example shows that Theorem 3.3 without the assumption that f is regular is false.

EXAMPLE 4.3. The real line can be decomposed into two complementary sets A and B such that A is of the first category and B is a G_δ of (Lebesgue) measure zero; see [15, Theorem 1.6]. Zahorski [24, Lemma III] (see also [18, Lemma 2]) proved that, for any G_δ set M of measure zero on the real line, there exists a Lipschitzian function f differentiable everywhere except on M. If we take $M=B$, then we get a Lipschitzian function with points of nondifferentiability of the second category.

Note added in proof. Since the paper was first written (Summer 1988), it has come to our attention that other people were working on this and related problems; in particular, at the 1990 AMS meeting in Louisville, J. M. Borewein described a more general framework that apparently gives Theorem 3.3 as a corollary.

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