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# Continuity and Uniqueness of Regularized Output Least Squares Optimal Estimators

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The continuity and uniqueness properties of optimal estimators with respect to data are considered for different regularizations. It is found that there is a weak stability for optimal estimators as set-valued mappings under a weak regularization. For stronger regularization results are obtained that give stability in stronger topologies and the finiteness of the set of optimal estimators. Finally, we give conditions that imply uniqueness of optimal estimators and Lipschitz continuity with respect to data. 

## 1. INTRODUCTION AND PRELIMINARIES

The dependence of solutions of identification problems with respect to perturbations of the data is studied in this paper. Of particular interest is the behavior of solutions to so-called regularized output least squares estimation problems. To fix ideas, we direct our attention to the following well-studied model. Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$ with a Lipschitz boundary $\Gamma$ over which is posed the elliptic boundary value problem

$$\begin{aligned} -\nabla \cdot (a \nabla u) &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \Gamma. \end{aligned} \tag{1.1}$$

The so-called weak formulation of (1.1) seeks a function $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega a \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle \tag{1.1$'$}$$

for all $\varphi \in W_0^{1,2}(\Omega)$. We assume unless specified otherwise that



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$$f \in W^{-1,2}(\Omega) \tag{1.2}$$

and that

$$a \in L^{\infty}(\Omega) \tag{1.3}$$

with

$$0 < \mu_0 \leqq a \leqq \mu_1 \qquad a.e. \text{ in } \Omega, \tag{1.4}$$

where $\mu_0$ and $\mu_1$ are constants. In general, we suppress the dependence of the function spaces on $\Omega$ and use $\|\cdot\|$ to denote the $L^2(\Omega)$-norm, $\|\cdot\|_k$ to denote the $W^{k,2}(\Omega)$-norm for $k = -1, 2, \nu$, etc. We note that

$$\|\nabla \varphi\| = \left( \int_\Omega |\nabla \varphi(x)|^2 \, dx \right)^{1/2}$$

is a norm on $W_0^{1,2}$ because of Poincaré's inequality

$$\|\varphi\| \leqq \kappa_0 \|\nabla \varphi\|$$

which holds for all $\varphi \in W_0^{1,2}$ where $\kappa_0$ is a positive constant independent of $\varphi$. For $\varphi \in W^{2,2}$ we also will use the inequality

$$\|\varphi\|_{L^\infty} \leqq \kappa_1 \|\varphi\|_{W^{2,2}}$$

which holds for $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$, and with $\kappa_1$ a positive constant that is independent of $\varphi$ [2, 17].

In applications [21] it often occurs that the coefficient $a$ is unknown. The data available to deduce $a$ are frequently only available as measurements $z$ of $u$. The problem then is to construct an estimate of the coefficient $a$ from the data $z$. This problem has been studied extensively in the literature [5–7, 9–13, 18, 20, 21] using many different approaches. For uniqueness and continuity we refer the reader to [5] but mention the following. In [18], Eq. (1.1) is treated as a first-order hyperbolic equation in the unknown coefficient $a$. Continuous dependence and uniqueness results of $a$ are proved. In this and much of the work it is assumed that there are sufficient data to allow reliable evaluation of functions and their derivatives. Alternatively, it is often assumed in least square approaches that the data is available only as a function in $L^2$. In fact, in many applications the first assumption is too strong while the second is weaker than is actually the case. Nevertheless, it is known [6] that even with an abundance of data, in general, the estimation of the coefficient $a$ is an ill-posed problem. However, we mention [12] in which sufficiently strong conditions are imposed that imply there is a unique solution.

In most applications data are not available in great quantity. For situations in which data are sparse the regularized output least squares method provides a versatile approach to gain information about $a$. The regularized output least squares method is formulated as a minimization problem to find the coefficient $a$ minimizing a fit-to-data functional given by

$$J(a) = \|Cu(a) - z\|_Z^2 + \beta N(a) \tag{1.5}$$

for $\beta > 0$ over an admissible set $Q_{ad}$. Here $Z$ is an observation space satisfying

$$Z \text{ is a separable Hilbert space.} \tag{1.6}$$

The data $z$ belong to $Z$. The state $u(a)$ associated with a coefficient $a$ is observed by means of a linear mapping $C$ from the state space into $Z$. The regularized output least squares estimation problem is thus formulated as

(E.$\beta$) Find $a_0 \in Q_{ad} \subseteq Q$ such that $J(a_0) = \inf\{J(a)\colon a \in Q_{ad}\}$.

It is well known that, by suitably selecting the regularization term $\beta N(a)$, the behavior of the problem may be improved at the expense of smoothing [13, 21]. The purpose of this study is to investigate the continuity properties of the solution set of (E.$\beta$) with respect to perturbations of the data $z$ with relation to the regularization. As we shall discuss later, this work is related to that in [7, 20] which is based on the stability results of [3] from optimization theory. However, because of property (B) discussed below, we obtain a stronger result for regularizations of the form (1.10). For the weak case of (1.9) we obtain set-valued continuity properties in weak topologies that often arise in optimization problems (cf. [4]).

In the remainder of this section, we introduce necessary notations and assumptions, and indicate the existence of a solution to (E.$\beta$). We also introduce in Definition 1.7 the notion of stability with respect to the data for a problem (E.$\beta$)($z$) that we use. In Section 2 we obtain results for weak regularizations in which upper semicontinuity of the set-valued mapping $z \mapsto Q(z)$ is established and thereby obtain a stability result for the set $Q(z)$ of solutions of (E.$\beta$)($z$) with respect to a metric induced by the weak $Q$ topology. In Section 3 we examine the consequences of including a stronger regularizing term. We obtain the upper semicontinuity of the set-valued mapping $z \mapsto Q(z)$ or certain of its subsets and establish conditions that imply the stability in the sense of Definition 1.7 for the metric associated with the norm topology. Also, conditions are obtained implying there are at most finitely many solutions to (E.$\beta$). These conditions are justified by the nondegenerate nature of the physical problem. Finally, in Section 4 we consider the set of stationary points of (E.$\beta$) under the condition that the constraints are inactive (again justified by the physical nondegeneracy of the problem). Sufficient conditions for uniqueness are obtained that are related to the sufficiency conditions from optimization theory. These conditions should be useful in the design and implementation of experiments.

In formulating our problems we delineate the following properties for $Q$, $Q_{ad}$, and $N(\cdot)$.

(Q1) $Q$ is a Hilbert space that embeds compactly in $L^2$.

The set of admissible coefficients $Q_{ad}$ satisfies

(Q2) $Q_{ad} \subset Q \cap \{a \in L^\infty: a \text{ satisfies } (1.4)\}$.

To prove existence, we stipulate the following assumptions

(Q3) $\tilde{Q}_K = \{a \in Q_{ad}: N(a) \leqq K\}$ is bounded in $Q$ for any $K > 0$.

(N) $a \mapsto N(a)$ is a nonnegative functional that is lower semicontinuous with respect to the weak topology on $Q$.

The following properties involving the observation space $Z$ and the observation operator $C$ are useful.

(A) $C$ is a continuous linear mapping from $W_0^{1,2}$ into $Z$, and

(B) $z \in Z_0 \subset Z$ where $Z_0$ is a Hilbert space that embeds compactly into $Z$.

*Remark* 1.1. Property (B) merits discussion. Indeed, in many treatments it is assumed that, for example, $Z = L^2$ and the observations are elements in $L^2$. However, in applications the data $z$ are obtained by means of a finite, say $N_o$, set of measurements. If one wishes to take $Z$ to be a function space, these data are then interpolated in some manner to obtain $z$ as a function. The result, although certainly in $Z$, is, in fact, in a finite-dimensional subspace $Z_0$ of $Z$. On the other hand, if one does not interpolate the data reside in $Z = \mathbb{R}^{N_o}$ and $Z_0 = Z$. Clearly, in both cases (B) is satisfied.

It is well known that Eq. (1.1) is associated with the continuous bilinear form on $W_0^{1,2}$

$$l(\varphi, \psi; a) = \int_\Omega a \nabla\varphi \cdot \nabla\psi \, dx$$

that may be associated with a continuous linear operator $A : W_0^{1,2} \mapsto W^{-1,2}$.

If we denote by $A_k$ the linear operators associated with coefficient $a_k \in Q_{ad}$, the following two results are true.

LEMMA 1.2. *If* $a_k \to a$ *in* $L^1$ *with* $a_k$ *satisfying* (1.4), *then*

$$\|A_k u - Au\|_{-1} \to 0$$

*as* $k \to \infty$ *for every* $u \in W_0^{1,2}$.

LEMMA 1.3. *Let* $\{A_k\}_{k=1}^{\infty}$ *be a sequence of continuous linear operators from* $W_0^{1,2}$ *into* $W^{-1,2}$ *such that for every* $\varphi \in W_0^{1,2}$

$$\|A_{k_1}\varphi - A_{k_2}\varphi\|_{-1} \to 0$$

*as* $k_1, k_2 \to \infty$ *and there is a positive constant* $\mu_0$ *for which*

$$\langle A_k \varphi, \varphi \rangle \geqq \mu_0 \|\nabla \varphi\|^2 \tag{1.7}$$

*for all* $k$. *Then there exists a continuous linear operator* $A$ *from* $W_0^{1,2}$ *into* $W^{-1,2}$ *such that* $A_k\varphi \to A\varphi$ *in* $W^{-1,2}$ *for every* $\varphi \in W_0^{1,2}$ *and that satisfies inequality* (1.7). *If* $f \in W^{-1,2}$, *then for each* $k$ *there exists a unique* $u_k \in W_0^{1,2}$ *and there exists a unique* $u \in W_0^{1,2}$ *such that* $A_k u_k = f$ *and* $Au = f$, *respectively. Moreover,* $u_k \to u$ *in* $W_0^{1,2}$ *as* $k \to \infty$.

We refer the reader to [9] for proofs.

*Remark* 1.4. Since the measure of $\Omega$ is finite, we note that convergence in $L^2$ implies convergence in $L^1$.

PROPOSITION 1.5. *Under* (Q1) *and* (Q2), *if* $\{a_k\}_{k=1}^{\infty}$ *is a sequence in* $Q_{ad}$ *such that* $a_k \to a$ *weakly in* $Q$, *then* $a \in Q_{ad}$ *and* $u(a_k) \to u(a)$ *in* $W_0^{1,2}$.

*Proof.* Since $a_k \to a$ weakly in $Q$, it follows that there exists a subsequence $\{a_{k_i}\}_{i=1}^{\infty}$ such that $a_{k_i} \to a$ in $L^2$ and $a_{k_i} \to a$ almost everywhere in $\Omega$. Thus, $a \in Q_{ad}$. In fact, by a subsequence of a subsequence argument, we see that $a_k \to a$ in $L^2$ implies the convergence of the sequence $\{u(a_k)\}_{k=1}^{\infty}$ to $u(a)$ in $W_0^{1,2}$ from Lemmas 1.2, 1.3, and Remark 1.4. Hence, the mapping $a \to u(a)$ is continuous from $Q_{ad}$ with the weak $Q$ topology to $W_0^{1,2}$ with the norm topology. ∎

PROPOSITION 1.6. *Under assumptions* (Q1)–(Q3), (N), *and* (A), *there exists a solution to* (E.$\beta$) *for any* $\beta > 0$.

*Proof.* We set

$$d = \inf\{J(a) : a \in Q_{ad}\},$$

where $d \geqq 0$ from (N). Let $\varepsilon > 0$ and consider the set

$$\hat{Q} = \{a \in Q_{ad}: d + \varepsilon \geqq J(a)\}.$$

It is clear that

$$d = \inf\{J(a): a \in \hat{Q}\}. \tag{1.8}$$

Since $\beta > 0$, we see that for $a \in \hat{Q}$

$$\varepsilon + d \geqq J(a) \geqq \beta N(a);$$

$\hat{Q}$ is contained in a closed ball $B_Q$ in $Q$. That $\hat{Q}$ is closed in the weak $Q$ topology follows from (Q3) and (N). But $Q$ is reflexive so $B_Q$ is weakly compact. Hence, $\hat{Q}$ must be compact in the weak $Q$ topology.

The mapping $a \to \|Cu(a) - z\|_Z^2$ is continuous from the weak $Q$ topology on $\hat{Q}$ to $\mathbb{R}$ by Proposition 1.5 and assumption (A). Hence, from assumption (N), it follows that the mapping $a \to J(a)$ is lower semicontinuous with respect to the weak $Q$ topology on $\hat{Q}$ into $\mathbb{R}$. Therefore, we see that

$$d + \varepsilon \geq \underline{\lim}\, J(a_{k_i}) \geq J(a)$$

and $a \in \hat{Q}$. Since the mapping a $\mapsto J(a)$ is lower semicontinuous with respect to the weak $Q$ topology and $\hat{Q}$ is compact in this topology, it follows that the functional $J(\cdot)$ assumes the value $d$ on $\hat{Q}$. ▌

Of interest are the cases in which the function $N(\cdot)$ is the seminorm

$$N(a) = [a]_1^2, \tag{1.9}$$

where

$$[\varphi]_1^2 = \int_\Omega |\nabla\varphi|^2\, dx$$

with $|\nabla\varphi| = (\nabla\varphi \cdot \nabla\varphi)^{1/2}$ and $Q = W^{1,2}$ and the seminorm

$$N(a) = [a]_2^2 \tag{1.10(i)}$$

where

$$[\varphi]_2^2 = [\varphi]_1^2 + \int_\Omega \sum_{i,j=1}^{n} (\varphi_{x_i x_j})^2 \, dx$$

and $Q = W^{2,2}$. We also use the notation

$$[\varphi, \psi]_1 = \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx$$

and

$$[\varphi, \psi]_2 = [\varphi, \psi]_1 + \int_\Omega \left\{ \sum_{i,j=1}^{n} (\varphi_{x_i x_j}) \, \psi_{x_i x_j} \right\} dx.$$

For $Q = W^{2,2}$ we also consider

$$N(a) = \|a\|_2^2. \tag{1.10)(ii}$$

Since $W^{2,2}$ embeds in $C^0(cl(Q))$ for $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, we may weaken the constraints on the admissible set by taking

$$Q_{ad} = \{a \in Q: a \geqq \mu_0 > 0 \text{ in } \Omega\}. \tag{1.11}$$

For the most part results for (1.10)(i) and (ii) are much the same. However, there are some important differences that we shall note in Sections 3 and 4.

We view the regularization embodied by the inclusion of a term of the form (1.9) as a basic case in the sense that $W^{1,2}$ seems to be the lowest integer order Sobolev space that is a Hilbert space in which existence of a solution to (E.$\beta$) holds. Further, there apparently is no increase in regularity of the optimal estimator due to regularization even for those optimal estimators strictly satisfying the constraints. For (1.10)(i) and (1.10)(ii) with $\Omega \subset \mathbb{R}^n$ and $n = 2, 3$, $W^{2,2}$ is the lowest order integer Sobolev space that is a Hilbert space embedding compactly into $C^0(cl(\Omega))$. Hence, the inequalities in the definition of $Q_{ad}$ hold pointwise in $\Omega$ and not just almost everywhere. This embedding allows us to weaken the constraints on $Q_{ad}$ as we indicated above. Accordingly, estimation problems with space dimension 2 or 3 are often formulated in this space.

To consider the continuous dependence of (E.$\beta$) on $z$, we introduce notation to emphasize the dependence of our problems on the data $z$.

(E.$\beta$)($z$) Find $a_z \in Q_{ad}$ such that $J(a_z, z) = \inf\{J(a, z): a \in Q_{ad}\}$

where

$$J(a, z) = \|Cu(a) - z\|_Z^2 + \beta N(a)$$

and

$$Q_{ad} = \{a \in Q: \mu_0 \leqq a \leqq \mu_1 \quad a.e. \text{ in } \Omega\}.$$

Although above we have shown that this problem has a solution under the assumptions (Q1)–(Q3), (A), and (N), there is no guarantee that there is a unique solution. Hence, we introduce the notation

$$Q(z) = \{a: a \text{ is a solution to E.}\beta)(z)\}.$$

In [10] problems (E.$\beta$) under strong regularization assumptions and under the assumption

$$\text{there is a unique } \tilde{a} \in Q_{ad} \text{ such that } u(\tilde{a}) = \tilde{z} \tag{1.12}$$

demonstrate that for every $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for $\delta < \delta_0$ and for $\beta$ sufficiently small (determined by $\delta$)

$$\|z - \tilde{z}\|_Z \leqq \delta \Rightarrow \|a_z - \tilde{a}\|_Q \leqq \varepsilon.$$

In [7, 20] under strong regulation but without assumption (1.12), it is shown that while there may be multiple solutions to an estimation problem, for certain $\beta$ there is a number $R$ such that if there is a local minimizer $a_z$ of $J(a, z)$ within distance $R$ of $a_{z_0}$, then

$$\|a_z - a_{z_0}\|_Q \leqq K\|z - z_0\|_Z^{1/2}.$$

These results on the stability of (E.$\beta$)($z$) depend on results of Alt [3] in minimization theory that rely on sufficiency conditions for optimality [16, 22]. For the case (E.$\beta$) with $\beta = 0$, that is, without the benefit of regularization, these conditions can be verified for certain examples [20]. These results do not depend on the data being in the attainable set to the uniqueness of the optimal solution.

For this paper we use the following as our definition of stability with respect to data. It should be noted that this definition applies to solutions of (E.$\beta$)($z$) and not just local minimizers as above.

Definition 1.7. The problem (E.$\beta$)($z$) is stable at $z_0 \in Z_0 \subset Z$ if there exists a monotone increasing function $\rho: \mathbb{R} \mapsto \mathbb{R}$ with $\rho(0) = 0$ and metrics $d_Q(\cdot, \cdot)$ on $Q_{ad}$ and $d_0(\cdot, \cdot)$ on $Z_0$ such that if $a_z$ is a solution of (E.$\beta$)($z$), that is, $a_z \in Q(z)$ with $z \in Z_0$, then there is a solution $a_{z_0}$ of (E.$\beta$)($z_0$) with the property that

$$d_Q(a_z, a_{z_0}) \leqq \rho(d_0(z, z_0)).$$

## 2. Continuity Results in Weak Topologies

In this section we investigate the continuity properties of the set-valued mapping $z \mapsto Q(z)$. It is found that with assumption (B), in addition to assumptions (Q1)–(Q3), (N), and (A) that are needed for existence, the set-valued mapping $z \mapsto Q(z)$ is upper semicontinuous from $\mathscr{Z}_0$ defined below with weak $Z_0$ topology to $Q_{ad}$ with the weak $Q$ topology. To this end, we recall the following definition [4].

Definition 2.1. Let $X$ and $Y$ be Hausdorff topological spaces and let $x \mapsto F(x)$ be a set-valued mapping form $X$ into subsets of $Y$. The mapping $F$ is said to be upper semicontinuous at $x_0$ if for every neighborhood $N$ of $F(x_0)$ there exists a neighborhood $M$ of $x_0$ such that $F(M) \subset N$.

For our application we let $Z_0$ be a subspace of $Z$ satisfying assumption (B), and let $Y$ be $Q_{ad}$ with the weak $Q$ topology. Let $\tilde{a}$ belong to $Q_{ad}$ and let $\eta$ be a positive number. Consider $\mathscr{Z}_0$ in $Z_0$ defined by

$$\mathscr{Z}_0 = \{z \in Z_0 : \|z - z_0\|_{Z_0} \leqq \eta\}.$$

We take $X = \mathscr{Z}_0$ with the weak $Z_0$ topology. From the compact embedding of $Z_0$ into $Z$ there exists a positive number $k_0$ such that for any $z \in Z_0$

$$\|z\|_Z \leqq k_0 \|z\|_{Z_0}.$$

Since $\beta > 0$, it follows that

$$J(\tilde{a}, z) \geqq \beta N(a_z)$$

and for $z \in \mathscr{Z}_0$

$$2(\|Cu(\tilde{a})\|_Z^2 + k_0^2(\|z_0\|_{Z_0} + \eta)^2) + \beta N(\tilde{a}) \geqq \beta N(a_z). \tag{2.1}$$

Define $\tilde{Q}_\eta = \bigcup_{z \in \mathscr{Z}_0} Q(z)$. We have the following.

Lemma 2.2. *Under assumptions* (Q1)–(Q3), (N), (A), *and* (B), *the map-*

*ping $z \mapsto Q(z)$ is a set-valued mapping such that for each $z \in \mathfrak{X}_0$ the set $Q(z)$ is compact in the weak $Q$ topology. Further, the set $\tilde{Q}_\eta$ is compact in the weak $Q$ topology.*

*Proof.* For $z \in \mathfrak{X}_0$ it follows from (2.1) that $Q(z)$ is norm bounded. Further, $\tilde{Q}_\eta \subset Q_{ad}$ is bounded in $Q$. Now it follows that if $\tilde{Q}_\eta$ is weakly closed and weakly sequentially compact, then $\tilde{Q}_\eta$ is weakly compact [8, 19]. We show that $\tilde{Q}_\eta$ is sequentially compact since the arguments demonstrating that the set $\tilde{Q}_\eta$ is closed are similar. Moreover, the arguments for $Q(z)$ follow in an obvious manner.

Let $\{a_k\}_{k=1}^\infty$ be a sequence in $\tilde{Q}_\eta$. Then there exists a sequence $\{z_k\}_{k=1}^\infty$ belonging to $\mathfrak{X}_0$ such that $a_k \in Q(z_k)$ for each $k$. Now from (2.1) and since $z_k \in \mathfrak{X}_0$, it follows that there exist subsequences $\{a_{k_i}\}_{i=1}^\infty$ and $\{z_{k_i}\}_{i=1}^\infty$ such that $a_{k_i} \to a$ weakly in $Q$ and $z_{k_i} \to z$ strongly in $Z$ from assumption (B). Since $Q$ embeds compactly into $L^2$ from (Q1), there is a subsequence $\{a_{k_i}\}_{i=1}^\infty$ such that

$$a_{k_i} \to a \text{ in } L^2 \text{ and almost everywhere in } \Omega.$$

Hence, $a$ belongs to $Q_{ad}$ and from Proposition 1.5

$$u(a_{k_i}) \to u(a) \qquad \text{in } W_0^{1,2}.$$

From assumptions (N), (A), and (B) with

$$J(a_{k_i}, z_{k_i}) = \|u(a_{k_i}) - z_{k_i}\|_Z^2 + \beta N(a_{k_i})$$

we see that

$$\underline{\lim}\, J(a_{k_i}, z_{k_i}) \geqq J(a, z).$$

Furthermore, if $\tilde{a} \in Q_{ad}$ is an arbitrary element, then from (B)

$$J(\tilde{a}, z) = \lim J(\tilde{a}, z_{k_i}) \geqq \underline{\lim}\, J(a_{k_i}, z_{k_i}),$$

and we conclude that $a \in Q(z) \in \tilde{Q}_\eta$. ∎

LEMMA 2.3. *Under assumptions* (Q1)–(Q3), (N), (A), and (B), *the mapping $z \mapsto Q(z)$ is a closed mapping of $\mathfrak{X}_0$ with the weak $Z_0$ topology into $\tilde{Q}_\eta \subset Q_{\text{ad}}$ with the weak $Q$ topology.*

*Proof.* The graph of $Q$ is given by

$$G(Q) = \{(z, a): a \in Q(z)\}$$

so that $G(Q) \subset \mathfrak{X}_0 \times \tilde{Q}_\eta$. To show that $Q$ is closed, we show that $G(Q)$ is closed in $\mathfrak{X}_0 \times \tilde{Q}_\eta$ wiith the weak topology inherited from $Z_0$ and $Q$. Since $\mathfrak{X}_0 \times \tilde{Q}_\eta$ is bounded in $Z_0 \times Q$, it is a metric space with respect to the weak topology. Hence, to show that $G(Q)$ is closed it suffices to show that the limit points of sequences in $G(Q)$ are in $G(Q)$. Let $\{(z_k, a_k)\}_{k=1}^{\infty}$ be a sequence converging weakly to $(z, a)$ in $Z_0 \times X$. In this case from the compactness of the embeddings of $Z_0$ into $Z$ and $Q$ into $L^2$, there exists a subsequence $(z_{k_i}, a_{k_i})$ such that

$$a_{k_i} \to a \text{ in } L^2 \text{ and almost everywhere in } \Omega.$$

and

$$z_{k_i} \to z \qquad \text{in } Z.$$

The arguments are now essentially the same as in the previous lemma. ∎

Since $z \mapsto Q(z)$ is a closed mapping from $Z_0$ to a compact space $\tilde{Q}_\eta \subset Q_{ad}$, we have the following [4, p. 42].

THEOREM 2.4. *Under assumptions* (Q1)–(Q3), (N), (A), *and* (B), *the mapping* $z \mapsto Q(z)$ *of* $\mathfrak{X}_0$ *with its weak* $Z_0$ *topology into* $\tilde{Q}_\eta \subset Q_{ad}$ *with the weak* $Q$ *topology is upper semicontinuous.*

COROLLARY 2.5. *Let assumptions* (Q1)–(Q3), (N), (A), *and* (B) *hold. Let* $d_{Z_0}(\cdot, \cdot)$ *and* $d_Q(\cdot, \cdot)$ *be the metrics on* $\mathfrak{X}_0$ *and* $\tilde{Q}_\eta$, *respectively, with the weak topologies. Then given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *with the property that if* $d_{Z_0}(z, z_0) < \delta$ *there is* $a_{Z_0} \in Q(z_0)$ *such that* $d_Q(a_z, a_{z_0}) < \varepsilon$.

*Proof.* Since $Q(z_0)$ is compact in the weak topology, it is totally bounded. Hence, given $\varepsilon > 0$, $Q(z_0)$ may be covered by finitely many balls $Q_i = \{(a: d_Q(a, a^i_{z_0}) < \varepsilon\}$ where $a^i_{z_0} \in Q(z_0)$ for $i = 1 \ldots n$. The set $N = \bigcup_{i=n} Q_i$ is a neighborhood of $Q(z_0)$. From the upper semicontinuity there exists a neighborhood $M$ of $z_0$ such that $Q(M) \subset N$. Thus, there is a $\delta > 0$ such that $\{z: d_{Z_0}(z, z_0) < \delta\} \cap \mathfrak{X}_0$ is contained in the neighborhood $M$. The result now follows. ∎

COROLLARY 2.6. *Under the assumptions above let* $d_{Z_0}(\cdot, \cdot)$ *and* $d_Q(\cdot, \cdot)$ *be the metrics on* $\mathfrak{X}_0$ *and* $\tilde{Q}_\eta$, *respectively, with the weak topologies. There is a number* $\delta_0$ *and a monotone increasing function* $\varepsilon$: $[0, \delta_0) \mapsto \mathbb{R}^+$ *for which* $\varepsilon(0) = 0$ *such that for any* $a_z \in Q(z)$ *where* $z \in \mathfrak{X}_0$ *there is* $a_{z_0} \in Q(z_0)$ *such that*

$$d_Q(a_z, a_{z_0}) \leqq \varepsilon(d_{Z_0}(z, z_0)).$$

*Remark* 2.7. The above results hold for example taking $N(a)$ to be the seminorm in (1.9) with $Z = L^2$ and $Z_0 = W_0^{1,2}$.

*Remark* 2.8. We may take $N(a)$ to be of the form in (1.10)(i) or (1.10)(ii) with $Z = L^2$ and $Z_0 = W_0^{1,2}$.

*Remark* 2.9. If we set

$$J(a) = \|\nabla(u(a) - z)\|_Z^2 + \beta N(a)$$

then the results follow for $Z = W_0^{1,2}$ and $Z_0 = W_0^{1,2} \cap W^{2,2}$.

## 3. Continuity Properties of Subsets of $Q(z)$ in Stronger Topologies

In the previous section we saw that under assumption (B) the set-valued mapping $z \mapsto Q(z)$ of $\mathcal{Z}_0$ into $\tilde{Q}_\eta$ is upper semicontinuous in the weak topology. In this section we consider set-valued mappings of the form $z \mapsto \hat{Q}(z)$ where $\hat{Q}(z) \subset Q(z)$. Our interest here is to obtain conditions implying this mapping is upper semicontinuous from $\mathcal{Z}_0$ with weak $Z_0$ topology to $\tilde{Q}_\eta$ equipped with the metric that is induced by the norm on $Q$. These results allow us to obtain stronger stability results than those based only on optimizatoin theory. They are, in fact, a consequence of the regularity properties the optimal estimators enjoy for problems regularized by functionals of the form (1.10)(i) and (ii). We also study some of the properties of certain subsets $\tilde{Q}(z)$ of $Q(z)$. For example, we obtain conditions implying that there are finitely many elements in $\tilde{Q}(z)$. This property is significant for the application of algorithms such as simulated annealing [1].

Since $Q = W^{2,2}$ embeds compactly into $C^0(cl(\Omega))$ for $\mathbb{R}^n$ with $n = 2$ or 3, we take $Q = W^{2,2}$ with

$$Q_{ad} = \{a \in Q: \mu_1 \geqq a \geqq \mu_0 > 0\} \tag{3.1)(i}$$

and

$$Q_{ad} = \{a \in Q: a \geqq \mu_0 > 0\} \tag{3.1)(ii}$$

for (1.10)(ii).

We note that for $a \in Q_{ad}$, the problem

$$\begin{aligned} -\nabla\cdot(a\nabla w) &= g \qquad \text{in } \Omega \\ w &= 0 \qquad \text{on } \Gamma \end{aligned}$$

with $\Gamma$ Lipschitz and $g \in L^2$ satisfies the estimate

$$\|w\|_2 \leqq C(\mu_0, \|a\|_2)\|g\| \tag{3.2}$$

(see [2]) where $C(\mu_0, \|a\|_2)$ indicates a positive constant depending only on the lower bound $\mu_0$ and the $W^{2,2}$-norm of the coefficient $a$.

In applications, observation functions are often obtained by the interpolation of pointwise measurements using, for example, tensor products of linear or cubic splines. We assume then that

$$Z = L^2 \qquad \text{and} \qquad Z_0 = W_0^{1,2} \tag{3.3)(i}$$

and

$$Z = W_0^{1,2} \qquad \text{and} \qquad Z_0 = W^{1,2} \cap W^{2,2} \tag{3.3)(ii}$$

with $C$ = identity. In the first case

$$J(a) = \|u(a) - z\|^2 + \beta[a]_2^2. \tag{3.4)(i}$$

or

$$J(a) = \|u(a) - z\|^2 + \beta\|a\|_2^2 \tag{3.4)(ii}$$

and in the second

$$J(a) = \|\nabla(u(a) - z)\|^2 + \beta[a]_2^2. \tag{3.4)(iii}$$

Recall that

$$Q_{ad} = \{a \in Q: \mu_1 \geqq a \geqq \mu_0 > 0\} \tag{3.5)(i}$$

and

$$Q_{ad} = \{a \in Q: a \geqq \mu_0 > 0\}, \tag{3.5)(ii}$$

and the estimation problem is given as the following minimization problem.

$$\text{Find } a_0 \in Q_{ad} \text{ such that} \quad J(a_0, z) = \inf\{J(a, z): a \in Q_{ad}\}. \tag{3.6}$$

Here we again may define the mapping of $Z_0$ into the collection of subsets of $Q_{ad}$ by

$$z \mapsto Q(z) = \{a: a \text{ solves (3.6) for data } z\}.$$

By using the regularity of the solution of (3.6), we obtain conditions implying upper semicontinuity of the set-valued mapping $z \mapsto Q(z)$ from $Z_0$ with its weak topology to $Q_{ad}$ with the strong $Q$ topology. Further, by applying the results from optimization, we can determine conditions under which $Q(z)$ or certain subsets $\hat{Q}(z)$ have at most finitely many elements. Moreover, these subsets are stable with respect to the perturbation of data in the sense defined in Definition 1.1. We focus our analysis on (3.6) with (3.4)(i) and (3.5)(i), and we will point out the consequences of having $N(a)$ be a norm.

We utilize regularity properties of solutions of (3.6) (cf. [13, 21]) to demonstrate compactness of $Q(z)$. Our approach is to use the Kuhn–Tucker Theorem to obtain an Euler–Lagrange equation giving a necessary condition for the solution of (3.6). To this end, set $Y = W^{\nu,2} \times W^{\nu,2}$ for $\nu \in (1, 2]$ and define the function

$$G: Q \mapsto Y$$

by

$$G(a) = \begin{bmatrix} \mu_0 - i\,a \\ i\,a - \mu_1 \end{bmatrix},$$

where $i$ represents the embedding mapping from $W^{2,2}$ into $W^{\nu,2}$. We note that $Y$ is a Hilbert space and has a positive cone with a nonempty interior. It is obvious that if $a \in Q_{ad}$, then $a$ is a regular point for the constraint $G(a) \leqq 0$ in the sense of [15] (see also [16, 22]). That is, for any $a \in Q_{ad}$ there is an $h \in Q$ such that

$$G(a) + DG(a)\,(h) = \begin{bmatrix} \mu_0 - i\,a - i\,h \\ i\,a - \mu_1 + i\,h \end{bmatrix} < 0.$$

Clearly, choosing $h = -a + (\mu_1 + \mu_0)/2$ satisfies the condition.

The Kuhn–Tucker Theorem [15] implies the following.

PROPOSITION 3.1. *There exists a Lagrange multiplier* $\lambda = (\lambda_0, \lambda_1)$ *belonging to* $Y^* = (W^{\nu,2})^* \times (W^{\nu,2})^*$ *such that* $\lambda \geqq 0$ *and such that every solution* $a_0$ *of* (3.6) *satisfies*

$$\langle G(a_0), \lambda\rangle = 0, \tag{3.7}$$

*and is a stationary point of the Lagranian associated with* $\lambda$. *That is, if* $a_0$ *is a solution of* (3.6), *then*

$$DL(a_0, \lambda)(h) = 0 \tag{3.8}$$

*for every* $h \in Q$ *where*

$$L(a_0, \lambda) = J(a_0) + \langle G(a_0), \lambda\rangle. \tag{3.9}$$

*Remark* 3.2. Since $W^{2,2}$ is dense in $W^{\nu,2}$, it follows from (3.8) and the regular point condition that each solution $a_0$ of (3.6) has a unique Lagrange multiplier.

*Remark* 3.3. The Lagrange multipliers $\lambda_i$, $i = 0, 1$, belong to $(W^{\nu,2})^*$ the dual space of $W^{\nu,2}$. Since $W^{\nu,2}$ is a closed subspace of $W^{\nu,2}$, there exist elements $\hat{\lambda}_0, \hat{\lambda}_1 \in W^{-\nu,2}$, the dual space of $W_0^{\nu,2}$, such that $\langle \lambda_i, h\rangle = \langle \hat{\lambda}_i, h\rangle$, with $i = 0, 1$, for all $h \in W_0^{\nu,2}$.

The derivatives of $J$ can easily be seen to be given by

$$DJ(a)(h) = 2(\nabla(u(a) - z), \nabla v) + 2\beta[a, h]_2 \tag{3.10}$$

and

$$D^2J(a)(h,h) = 2(\nabla(u(a) - z), \nabla w) + 2\|\nabla v\|^2 + 2\beta[h]_2^2. \tag{3.11}$$

From (3.8), (3.9), and (3.10), we have

$$(\nabla(u(a_0) - z), \nabla v) + \beta[a_0, h]_2 - \langle\hat{\lambda}_0, h\rangle + \langle\hat{\lambda}_1, h\rangle = 0 \tag{3.12}$$

for all $h \in W_0^{2,2}$. Recalling the weak formulation of (1.1)′, we have with $\varphi = u$

$$\mu_0\|\nabla u\| \leqq \|f\|_{-1}. \tag{3.13}$$

The Fréchet derivative of $u(a_0)$, $v = Du(a_0)(h)$, must satisfy

$$\int_\Omega a\,\nabla v\cdot\nabla\varphi\,dx = -\int_\Omega h\,\nabla u(a)\cdot\nabla\varphi\,dx \tag{3.14}$$

for any $\varphi \in W_0^{1,2}$ (cf. [13]). Further, the second derivative is given by $w = D^2u(a_0)\,(h, h)$ and satisfies the equation

$$\int_\Omega a\,\nabla w\cdot\nabla\varphi\,dx = -2\int_\Omega h\,\nabla v\cdot\nabla\varphi\,dx \tag{3.15}$$

for every $\varphi \in W_0^{1,2}$ where $v = Du(a_0)\,(h)$ satisfies Eq. (3.14). Using inequality (3.13), we may obtain the following estimates on the solutions $v$ and $w$ of (3.14) and (3.15), respectively. Thus, we find the estimates

$$\|\nabla v\| \leqq \kappa_1\,\mu_0^{-2}\,\|h\|_2\,\|f\|_{-1} \tag{3.16}$$

and

$$\|\nabla w\| \leqq 2\,\kappa_1^2\,\mu_0^{-3}\,\|h\|_2^2\,\|f\|_{-1}. \tag{3.17}$$

Introduce the adjoint equation

$$-\nabla\cdot(a\,\nabla p) = -\Delta(u(a) - z) \qquad \text{in } \Omega$$

with boundary conditions

$$p = 0 \qquad \text{on } \Gamma.$$

The weak formulation is given by

$$\int_\Omega a\,\nabla\rho\cdot\nabla\varphi\,dx = \int_\Omega \nabla(u(a_0) - z)\cdot\nabla\varphi\,dx \tag{3.18}$$

for every $\varphi \in W_0^{1,2}$. Setting $\varphi = \rho$ in Eq. (3.14) and $\varphi = v$ in Eq. (3.18), we have

$$\int_\Omega \nabla(u(a_0) - z)\cdot\nabla v\,dx = -\int_\Omega h\,\nabla u(a_0)\cdot\nabla\rho\,dx. \tag{3.19}$$

By substitution of Eq. (3.19) into Eq. (3.12), we see that any optimal estimator $a_0$ is a solution of a variational formulation of a Neumann boundary value problem. Hence, we may use Eq. (3.12) and interpolation theory to deduce smoothness of the optimal estimators [2, 14]. In this way we obtain the following result (cf. [13, 21]).

PROPOSITION 3.4. *Let $\Omega$ have a $C^4$ boundary. Then any solution $a_0$ of the minimization problem* (3.6) *belongs to $W^{4-\nu,2}$ for $\nu > 1$. Further, the following estimate holds*

$$\|a_0\|_{4-\nu} \leqq C(\beta, f, z, \mu_0) + \|\lambda\|_{Y^*},$$

*where $C(\beta, f, z, \mu_0)$ is a positive constant depending on the indicated parameters.*

*Remark* 3.5. The same result holds for the functional (3.4)(i). If a solution $a_0$ of (3.6) satisfies $\mu_1 > a_0 > \mu_0$, then by (3.7) $\lambda = 0$. It follows that in the cases of (3.6)(i) and (3.4)(ii) $a_0 \in W^{4,2}$ while the regularity results for (3.4)(iii) are unchanged.

The above regularity result holds for any solution of (3.6). The only restriction is the smoothness of the boundary of $\Omega$ that is necessary in order to apply regularity theory for elliptic operators. Denote the set of Lagrange multiplers for the minimization problem (2.6) by $\Lambda$. The embedding $W^{4-\nu,2} \mapsto W^{2,2}$ for $\nu \in (1, 2)$ is compact. Hence, if the set $\Lambda$ is bounded in $Y^*$, then we may deduce compactness of $Q(z)$. We have the following from Proposition 3.4.

THEOREM 3.6. *Let $\Omega$ have a $C^4$ boundary and let the set $\Lambda$ of Lagrange multiplers of* (3.6) *be bounded in $Y^*$. Then the set $Q(z)$ is compact in $Q$.*

*Proof.* It suffices to show that $Q(z)$ is closed in $Q$ and sequentially compact. We show that $Q(z)$ is closed since sequential compactness follows from Proposition 3.4 and arguments similar to those of the closedness of the set $Q(z)$. To this end, let $\{a_k\}_{k=1}^{\infty}$ be a sequence of solutions of (3.6) such that $a_k \to a_0$ in $Q$ as $k \to \infty$. Then it follows from Lemmas 1.1 and 1.2 that the corresponding sequence of solution $\{u_k\}_{k=1}^{\infty}$ to (1.1)′ converges in $V$ to $u = u(a_0)$. Also, since convergence in $Q$ implies convergence in $C^0(cl(\Omega))$, it follows that $a_0 \in Q_{ad}$. Hence, for

$$J(a_k, z) = d = \inf\{J(a, z) : a \in Q_{ad}\},$$

we see that

$$d = J(a_k, z) \to J(a_0, z)$$

as $k \to \infty$. Thus, $J(a_0, z) = d$ and $a_0 \in Q(z)$. ∎

We consider the set

$$\tilde{Q}(z) = \{a \in Q(z): \mu_1 > a(x) > \mu_0 \text{ for all } x \in cl(\Omega)\}$$

and denote the condition

(PND) $\tilde{Q}(z) \neq \varnothing$.

If $a \in \tilde{Q}(z)$, then from (3.7) the corresponding Lagrange multiplier is zero. Hence, if (PND) holds, there are solutions in $Q(z)$ that have a zero Lagrange multiplier. Moreover, from Remark 3.5 members of $\tilde{Q}(z)$ belong to $W^{\nu,2}$ for $\nu \in (2, 4)$ and thus are continuous in $cl(\Omega)$. It follows that for sufficiently large $N$ the set

$$\tilde{Q}_N(z) = \{a \in Q(z): \mu_1 - 1/N \geqq a \geqq \mu_0 + 1/N \text{ in } cl(\Omega)\}$$

is nonempty. Set $\tilde{Q}(z) = \tilde{Q}_N(z)$. It is easy to see that $\tilde{Q}(z)$ is compact in $Q$.

*Remark* 3.7. The condition (PND) may be justified on physical grounds in that it embodies an assumption that the problem is nondegenerate. That is, the "actual coefficient $\bar{a}$" (which may not even belong to the space $Q$) is bounded away from zero. Indeed, the specification of $\mu_0$ is the quantification of the physical nondegeneracy of the problem for the purpose of its mathematical formulation. As much, it may be too large. It seems reasonable that if the lower bound $\mu_0$ is reduced sufficiently, then there are solutions to the estimation problem that strictly satisfy the inequalities. If this were not the case, then no matter how small $\mu_0$ there would exist solutions of (3.6) assuming the value $\mu_0$. This would contradict the nondegeneracy of the physical system. If $\mu_0$ does reflect some knowledge of the physical properties of the system, reducing $\mu_0$ only enlarges the admissible set and thus does not exclude the physically meaningful parameters. For example, in the case of porous media, nondegeneracy implies that there is nonzero permeability everywhere and the material contains no impermeable blocks. In fact, if a block were impermeable, we could simply exclude it from the domain $\Omega$.

We now determine conditions under which the mapping $z \mapsto Q(z)$ (or to subsets of $Q(z)$) is upper semicontinuous as a set-valued mapping from the weak topology of $Z_0$ into $Q$. To this end let $\eta > 0$ and set

$$\mathfrak{X}_0 = \{z \in Z_0: \|z - z_0\|_{Z_0} \leqq \eta\}.$$

PROPOSITION 3.8. *If the Lagrange multipliers $\lambda_z$ for* (3.6) *are uniformly bounded in $(W^{\nu,2})^*$ with respect to $z \in \mathfrak{X}_0$, then the set*

$$\tilde{Q}_\eta = \bigcup_{z \in \mathfrak{X}_0} Q(z)$$

*is a bounded set in* $W^{\nu,2}$ *for* $\nu \in (2, 4)$. *Hence,* $\tilde{Q}_\eta$ *is compact in* $W^{2,2}$.

*Proof.* This follows from the regularity of the optimal estimators and the estimate of Proposition 3.4. ▌

PROPOSITION 3.9. *The set-valued mapping* $z \mapsto Q(z)$ *from* $\mathfrak{X}_0$ *with the weak* $Z_0$ *topology to* $\tilde{Q}_\eta$ *with the metric induced by the* $Q$*-norm is a closed mapping.*

*Proof.* Let $(z_k, a_{z_k})$ be a sequence in $Z_0 \times Q$ such that $z_k \to z$ weakly in $Z_0$ and $a_{z_k} \to a$ in $Q$. Then, it is clear that $u(a_{z_k}) \to u(a)$ in $W_0^{1,2}$ and $a \in Q_{ad}$. Thus, it follows that

$$J(a_{z_k}, z_k) \to J(a, z)$$

as $k \to \infty$. If $\mathbf{a} \in Q_{ad}$ is arbitrary, then it also follows that

$$J(\mathbf{a}, z_k) \geqq J(a_{z_k}, z_k)$$

and

$$J(\mathbf{a}, z_k) \to J(\mathbf{a}, z)$$

as $k \to \infty$. Hence, we see that

$$J(\mathbf{a}, z) \geqq J(a, z)$$

for an arbitrary $\mathbf{a} \in Q_{ad}$. Therefore, $a \in Q(z)$. ▌

In conclusion we have the following result.

THEOREM 3.10. *Let the set of Lagrange multipliers*

$$\Lambda(\eta) = \{\lambda_z : z \in \mathfrak{X}_0\}$$

*be bounded in* $(W^{\nu,2})^*$, $\nu \in (2, 4)$. *Then the mapping* $z \mapsto Q(z)$ *from* $\mathfrak{X}_0$ *with the weak* $Z_0$ *topology into* $\tilde{Q}_\eta$ *is upper semicontinuous.*

A sufficient condition for the Lagrange multipliers to be uniformly bounded is that (PND) be satisfied uniformly with respect to $z \in \mathfrak{X}_0$.

(UPND) For every $z \in \mathfrak{X}_0$, the set

$$\tilde{Q}(z) = \{a \in Q(z) : \mu_1 > a(x) > \mu_0 \text{ for all } x \in c\ell(\Omega)\} \neq \emptyset.$$

If $a \in \tilde{Q}(z)$, then the corresponding Lagrange multiplier is zero. Hence, under assumption (UPND), for each $z \in \mathfrak{X}$ there are solutions in $Q(z)$ that have a zero Lagrange multiplier. Thus, under (UPND) for each $z \in \mathfrak{X}_0$ there is a nonempty subset of $Q(z)$ with zero Lagrange multiplier. Moreover, from Remark 3.5 members of $\tilde{Q}(z)$ belong to $W^{\nu,2}$ and thus are continuous in $c\ell(\Omega)$. Hence, for sufficiently large $N_z$, $z \in \mathfrak{X}_0$, the set

$$\tilde{Q}_N(z) = \{a \in Q(z) : \mu_1 - 1/N \geqq a \geqq \mu_0 + 1/N \text{ in } cl(\Omega)\} \neq \emptyset$$

for $N \geqq N_z$. Set $\hat{Q}(z) = \tilde{Q}_{N_z}(z)$. It is easy to see that $\hat{Q}(z)$ is compact in $Q$ and $\hat{Q}_\eta = \bigcup_{z \in \mathfrak{X}_0} \hat{Q}(z)$ is compact as well. From the above we have the following.

COROLLARY 3.11. *Let* (UPND) *hold. Then the mapping* $z \mapsto \hat{Q}(z)$ *is upper semicontinuous from* $\mathfrak{X}_0$ *with the weak* $Z_0$ *topology to* $\hat{Q}_\eta \subset Q_{ad}$ *with the strong* $Q$ *topology.*

Having determined conditions under which $Q(z)$ and certain of its subsets are compact and the mapping $z \mapsto \hat{Q}(z)$ is upper semicontinuous, we now obtain a stability result. As a side observation we also determine conditions such that there are, in fact, finitely many elements in $Q(z)$ or $\hat{Q}(z)$. We emphasize (3.4)(i) and (3.4)(iii) with (3.5)(i) as an admissible set since the case (3.4)(ii) and (3.5)(ii) is straightforward. We begin by emphasizing the natural condition (if we are to observe anything at all the forcing term must be nonzero).

(C) $f \in W^{-1,2}$ and $f \neq 0$.

Define the functional for $i = 0, 1$

$$M_i(h) = \|v\|_i^2,$$

where $v$ is the solution of Eq. (3.14).

LEMMA 3.12. *If $f$ satisfies condition* (C), *then the mapping* $h \mapsto M_i(h)$ *for* $i = 0, 1$ *is a continuous function on* $W^{2,2}$ *and has the property that if* $h = $ *constant, then* $M_i(h) = 0$.

*Proof.* The continuity follows from the continuity properties of the mapping $h \mapsto v$ defined by means of Eq. (3.14) and from the estimate (3.16). Further, from (3.14) we observe that

$$\int_\Omega a \nabla v \cdot \nabla u \, dx = -\int_\Omega h |\nabla u|^2 \, dx.$$

Now if $M_i(h) = 0$ for $i = 0$ or 1, it follows from Poincaré's inequality that $v = 0$ almost everywhere in $\Omega$. Hence, from the above, we see that

$$\int_\Omega h|\nabla u|^2 \, dx = 0.$$

Now $|\nabla u| > 0$ on a set of positive measure. Otherwise, Eq. (1.1)′ implies that $\langle f, \varphi \rangle = 0$ for every $\varphi \in W_0^{1,2}$, contradicting condition (C). We conclude that if $h$ is a constant, then $h = 0$. ∎

Under condition (C) the following holds (cf. [17, p. 27]).

COROLLARY 3.13. *Let* (C) *hold. The functional defined by* $i = 0, 1$

$$h \mapsto (M_i(h) + [h]_2^2)^{1/2}$$

*is equivalent to the* $W^{2,2}$*-norm* $\|h\|_2$. *That is, there exist positive constants* $c_0$ *and* $c_1$ *such that*

$$c_0\|h\|_2 \leqq (M_i(h) + [h]_2^2)^{1/2} \leqq c_1\|h\|_2.$$

We now introduce the condition

(D) $\nu\mu_0^3 c_0 - 2\tilde{\kappa}_0\kappa_1^2\|u(a) - z\|_i\|f\|_{-1} > 0$ for $i = 0$ or 1, where $\tilde{\kappa}_0 = \kappa_0$ if $i = 0$ and $\tilde{\kappa}_0 = 1$ if $i = 1$, and where $\nu = \min\{\beta, 1\}$.

*Remark* 3.14. Condition (D) is a condition on how well the data and the model fit, the size of $\|u(a) - z\|_i$, and the size of $\|f\|_{-1}$. For experimental design with fixed $\beta$ and an a priori bound on the term $\|u(a) - z\|_i$, (D) may be satisfied by taking $\|f\|_{-1}$ to be sufficiently small. We note that at worst we have the estimate

$$\|u(a) - z\|_i \leqq \|u(a)\|_i + \|z\|_i$$

so that for $i = 1$

$$\|u(a) - z\|_1 \leqq \frac{1}{\mu_0}\|f\|_{-1} + \|z\|_1$$

and for $i = 0$

$$\|u(a) - z\|_0 \leqq (\kappa_0/\mu_0)\|f\|_{-1} + \|z\|_0.$$

In either case $\|u(a) - z\|_i \leqq C(f, z)$ which is bounded.

PROPOSITION 3.15. *Let $\beta > 0$ and let conditions* (C) *and* (D) *hold. Then there exists a positive constant $\delta$ such that*

$$D^2J(a)(h,h) \geqq \delta\|h\|_2^2 \tag{3.20}$$

*for any $h \in Q$.*

*Proof.* The second Fréchet derivative of $J$ is given by (3.11). Hence, we have the estimate with $\beta > 0$ for $i = 0$ or 1

$$D^2J(a)(h,h) \geqq 2(\|v\|_i^2 - \|u(a) - z\|_i\|w\|_i + \beta[h]_2^2).$$

From Poincaré's inequality we see that

$$D^2J(a)(h,h) \geqq 2(\|v\|_i^2 - \tilde{\kappa}_0\|u(a) - z\|_i\|\nabla w\| + \beta[h]_2^2).$$

Finally, it follows from (3.17) that

$$D^2J(a)(h,h) \geqq 2\left(\nu c_0^2 - 2\tilde{\kappa}_0\kappa_1^2\|u(a) - z\|_i\left(\frac{1}{\mu_0}\right)^3\|f\|_{-1}\right)\|h\|_2^2,$$

where $\nu = \min(\beta, 1)$, and the result follows by selecting $\delta > 0$ such that

$$\delta < 2\left(\nu c_0^2 - 2\tilde{\kappa}_0\kappa_1^2\|u(a) - z\|_i\left(\frac{1}{\mu_0}\right)^3\|f\|_{-1}\right). \quad \blacksquare$$

*Remark* 3.16. We observe that the result of Proposition 3.15 implies that a solution $a_0$ of (3.6) satisfying (D) is isolated.

Define the following subsets of $Q(z)$:

$$Q_\delta(z) = \{a_0 \in Q(z)\colon a_0 \text{ satisfies } (3.20)\}.$$

From (3.11), (3.20), and Proposition 1.1, we observe the following.

*Remark* 3.17. In the norm case (3.4)(ii), condition (D) may be replaced by

$$\delta < 2\left(\beta - 2\tilde{\kappa}_0\kappa_1^2\|u(a) - z\|_i\left(\frac{1}{\mu_0}\right)^3\|f\|_{-1}\right) \tag{3.21}$$

since the estimates of Corollary 3.13 are no longer necessary.

The following is easily proved.

LEMMA 3.18. *$Q_\delta(z)$ is closed in $Q(z)$.*

THEOREM 3.19. *Let $\Omega$ have a $C^4$ boundary, let conditions* (C) *and* (D) *hold, and suppose that the set of Lagrange multipliers $\Lambda$ are bounded in $Y^*$. Then for each $\delta > 0$, $Q_\delta(z)$ has at most finitely many elements.*

*Proof.* From Theorem 3.6 and Lemma 3.12 it follows that $Q_\delta(z)$ is a compact set in $Q$. Further, from Remark 3.16 the elements of $Q_\delta(z)$ are isolated. If there were infinitely many elements in $Q_\delta(z)$, then from the compactness there must be a cluster point $a_0 \in Q_\delta(z)$. But then $a_0$ is not isolated giving a contradiction. ▌

COROLLARY 3.20. *Let $\Omega$ have a $C^4$ boundary and let* (C) *hold. If there exists $\delta_0 > 0$ such that*

$$\nu c_0\mu_0^3 - 2\tilde{\kappa}_0\kappa_1^2\|u(a) - z\|\|f\|_{-1} > \delta_0 \tag{3.22}$$

*holds for each $a \in Q(z)$ and if $Q(z) = \tilde{Q}(z)$ holds, then $Q(z)$ has only finitely many elements.*

We now establish sufficient conditions that depend only on $\mu_0$ $\mu_1$, $z$, and $f$ for (3.22) to hold for the cases (3.4)(i) or (3.4)(iii) in which $N(a)$ is a seminorm. To this end let $a = \alpha$ be a constant such that $\alpha \in Q_{ad}$. Then for any $a_0 \in Q(z)$, $J(\alpha) \geqq J(a_0)$. Since $N(\alpha) = 0$, it follows that

$$\|u(\alpha) - z\|_i \geqq \|u(a_0) - z\|_i. \tag{3.23}$$

Consider now the problem

$$\text{find } \psi \in W_0^{1,2} \text{ such that}$$

$$\int_\Omega \nabla\psi\cdot\nabla\varphi = \langle f, \varphi\rangle$$

for every $\varphi \in W_0^{1,2}$. It is clear that $u(\alpha) = \alpha^{-1}\psi$, and we formulate the minimization problem

$$\text{find } \alpha_0 \in [\mu_0, \mu_1] \text{ such that}$$

$$j(\alpha_0) = \|\alpha_0^{-1}\psi - z\|_i^2 = \inf\{j(\alpha): \alpha \in Q_{ad}\}, \tag{3.24}$$

where $j(\alpha) = \alpha^{-2}\|\psi\|_i^2 - 2\alpha^{-1}(\psi, z)_i + \|z\|_i^2$. Under condition (C), $\|\psi\|_i \neq 0$ and set $\hat{\alpha}^{-1} = (\psi, z)_i/\|\psi\|_i^2$. Thus $\alpha_0^{-1} = \hat{\alpha}^{-1}$ if

$$\mu_0^{-1} \geqq \hat{\alpha}^{-1} \geqq \mu_1^{-1},$$
$$\alpha_0^{-1} = \mu_0^{-1} \qquad \text{if } \hat{\alpha}^{-1} > \mu_0^{-1}, \tag{3.25}$$

and

$$\alpha_0^{-1} = \mu_1^{-1} \qquad \text{if } \hat{\alpha}^{-1} < \mu_1^{-1}.$$

PROPOSITION 3.21. *Let $\Omega$ have a $C^4$ boundary, let* (C) *hold, and suppose that $Q(z) = \tilde{Q}(z)$. If*

$$\nu\mu_0^3 - 2\tilde{\kappa}_0\kappa_1^2\|\alpha_0^{-1}\psi - z\|_i\|f\|_{-1} > 0, \tag{3.26}$$

*where $\alpha_0$ is the solution of* (3.24), *then $Q(z)$ has only finitely many elements.*

*Proof.* Let $\delta_0 > 0$ be such that

$$\nu\mu_0^3 - 2\tilde{\kappa}_0\kappa_1^2\|\alpha_0^{-1}\psi - z\|_i\|f\|_{-1} > \delta_0.$$

From (3.23) inequality (3.25) holds for any $a_0 \in Q(z)$, and the result follows from Corollary 3.2. ▌

*Remark* 3.22. We note that $\hat{\alpha}^{-1}$ depends only on the relation of $z$ with $f$ (or its solution $\psi$). Suppose data $z$ and $\psi$ are such that

(E) $(\psi, z)_i > 0$

and condition (C) holds. By choosing $\mu_1$ sufficiently large we may satisfy the right inequality in (3.25). The constraint involving $\mu_0$ is one that implies that the problem is nondegenerate. It is not unreasonable to use the solution of (3.23) without constraints but under condition (E) to specify $\mu_0$. Let us consider the case in which $\mu_0^{-1} = \hat{\alpha}^{-1}$. In this case we have

$$j(\alpha_0) = \|u(\alpha_0) - z\|^2 = (\|z\|_i^2\|\psi\|_i^2 - (z, \psi)_i^2)/\|\psi\|_i^2.$$

Furthermore, if kcondition (E) holds, $\alpha_0 = \mu_0$, and

$$1 > 2c_0^{-1}\tilde{\kappa}_0\kappa_1^2(\psi, z)_i^3(\|z\|_i^2\|\psi\|_i^2 - (z, \psi)_i^2)^{1/2}\|f\|_{-1}/\|\psi\|_i^7 \tag{3.27}$$

holds, then there exists $\beta \in (0, 1)$ such that (3.25) holds.

*Remark* 3.23. For the case $i = 1$, it is clear that (E) holds if and only if

$$\langle f, z\rangle > 0. \tag{E)$'$}$$

Also, noting that $\|f\|_{-1} = \|\nabla\psi\|$ and $(\nabla\psi, \nabla z)_V = \langle f, z\rangle$, inequality (3.27) becomes

$$1 > 2c_0^{-1}\kappa_1^2\langle f, z\rangle^3(\|z\|_V^2\|f\|_{-1}^2 - \langle f, z\rangle^2)^{1/2}/\|f\|_{-1}^6.$$

*Remark* 3.24. We can give an interpretation of (E)′. Let $\rho$ be given by $\rho = u(\tilde{a}) - z$ *for* $\tilde{a} \in Q_{ad}$. Then we may think of $\rho$ as a measurement error. If the data and the model equation are reasonable, then there is some $\tilde{a}$ in $Q_{ad}$ such that $\rho$ is small. Note in [10] it is assumed that there is a coefficient $\tilde{a}$ such that $\rho = 0$. Now it follows that for any $\varphi \in V$

$$\int_\Omega \tilde{a}\,\nabla u(\tilde{a})\cdot\nabla\varphi\,dx = \langle f, \varphi\rangle.$$

Hence, we have

$$\int_\Omega \tilde{a}\,\nabla(z + \rho)\cdot\nabla\varphi\,dx = \langle f, \varphi\rangle,$$

and setting $\varphi = z$,

$$\int_\Omega \tilde{a}\,\nabla(z + \rho)\cdot\nabla z\,dx = \langle f, z\rangle.$$

If (E)′ does not hold, then

$$\int_\Omega \tilde{a}|\nabla z|^2\,dx \leqq -\int_\Omega \tilde{a}\,\nabla\rho\cdot\nabla z\,dx.$$

Hence, we find that

$$\|\nabla z\| \leqq (\mu_1/\mu_0)\|\nabla\rho\|.$$

This inequality implies the measurements are dominated by the error of the measurement. It follows that if

$$\|\nabla z\| > (\mu_1/\mu_0)\|\nabla\rho\|,$$

then condition (E)′ holds.

In order to apply the results on stability of Alt [3] (see also [7, 20, 21]), it is necessary for the fit-to-data functional and the constraint function to satisfy a Lipschitz condition. We state this as the following proposition since it is straightforward to check.

LEMMA 3.25. *There exist constants $K_1$ and $K_2$ that may depend on $a_1$ and $z_1$ such that*

$$|J(a_1, z_1) - J(a_2, z_2)| \leqq K_1(\|a_1 - a_2\|_2 + \|z_1 - z_2\|)$$

*and*

$$\|G(a_1) - G(a_2)\|_\nu \leqq K_2\|a_1 - a_2\|_2.$$

*We now have the following result* (*cf.* [7, 20, 21]).

PROPOSITION 3.26. *Let* (D) *hold. There is a positive number r such that if there is a solution $a_z \in Q(z)$ such that*

$$a_z \in Q(z_0) \cap B(a_0, r),$$

*where $B(a_0, r) = \{a \in Q: \|a - a_0\|_Q \leqq r\}$ then*

$$\|a_z - a_0\|_Q \leqq \kappa\|z - z_0\|_Z^{1/2},$$

*where $\kappa$ is a constant dependent on $a_0$, r, and the Lipschitz constant C.*

*Remark* 3.27. The above result gives stability with respect to the data $z$ if there exists a local minimum solution of (3.6) within a certain neighborhood of $a_0$.

We now use the results on the upper semicontinuity of the mapping $z \mapsto Q(z)$ to obtain the following.

THEOREM 3.28. *Let $\partial\Omega$ be $C^4$, let the set of Lagrange multipliers $\Lambda(\mathfrak{X}_0)$ be bounded in $(W^{\nu,2})^*$, and let* (C) *and* (D) *hold. Then there exists a neighborhood $M \subset Z_0$ of $z_0$ such that for any $z \in M$ each $a_z \in Q(z)$ satisfies*

$$\|a_z - a_0\|_Q \leqq C\|z - z_0\|_{Z_0}^{1/2}$$

*for some element $a_0 \in Q(z_0)$.*

*Remark* 3.29. Recall that if $\partial\Omega$ is $C^4$ and $Q(z) = \tilde{Q}(z)$ holds, then the mapping $z \mapsto Q(z)$ is upper semicontinuous.

Hence, we have the following for the nondegenerate case.

THEOREM 3.30. *Suppose that $\partial\Omega$ is $C^4$,* (C) *and* (D), *and $Q(z) = \tilde{Q}(z)$. Then there exists a neighborhood $M \subset Z_0$ of $z_0$ such that for any $z \in M$ each $a_z \in Q(z)$ satisfies*

$$\|a_z - a_0\|_Q \leqq C\|z - z_0\|_{Z_0}^{1/2}$$

*for some* $a_0 \in Q(z_0)$.

Finally we observe the following in addition to Theorem 3.30.

COROLLARY 3.31. *Let* $\Omega$ *have a* $C^4$ *boundary. Let* (C) *and* (D) *hold, and suppose that* $Q(z) = \tilde{Q}(z)$. *Then there is a neighborhood* $M \subset Z_0$ *of* $z_0$ *such that for each* $z \in M$, *the set* $Q(z)$ *has only finitely many elements.*

## 4. UNIQUENESS AND CONTINUOUS DEPENDENCE

We consider the functional (1.5) with (E.$\beta$) and $N(a)$ satisfying (1.10(ii) and (1.11). We assume that there is a physically nondegenerate (PND) solution $a > \mu_0$. Hence, the associated Lagrange multiplier is zero, and such a solution $a$ satisfies the system of equations

$$\begin{aligned} -\nabla\cdot(a\,\nabla u) &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.1}$$

$$\begin{aligned} -\nabla\cdot(a\,\nabla p) &= u - z && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.2}$$

and the variational equation

$$\beta(a, h)_2 = \int_\Omega (\nabla u \cdot \nabla p) h\, dx \tag{4.3}$$

for all $h \in W^{2,2}$. In fact, the system (4.1)–(4.3) characterizes all stationary points $a_0$ of $J(\cdot)$ such that $a_0 > \mu_0$. Hence, we define the following set

$$Q_s(z) = \{a \in W^{2,2}\colon a \text{ is a solution of (4.1)–(4.3) and } a \geqq \mu_0\}.$$

It is clear that if $Q(z) = \tilde{Q}(z)$, then $Q(z) \subset Q_s(z)$.

In this section we examine the continuity properties of the mapping $z \mapsto Q_s(z)$ and obtain upper semicontinuity. In addition we obtain a condition similar to but more stringent than that for sufficiency (D) implying uniqueness for the solution of (4.1)–(4.3). If $Q(z) = \tilde{Q}(z)$, then there exists a unique solution of the estimation problem and that solution is Lipschitz continuous with respect to perturbations of the data.

We begin by showing that the set of solutions of (4.1)–(4.3) is bounded in $W^{2,2}$.

PROPOSITION 4.1. *If $a \in Q_s(z)$, then*

$$\|a\|_2 \leqq \frac{1}{\beta} \kappa_1 \mu_0^{-1} \|f\|_{-1} (\kappa_0^2 \mu_0^{-2} \|f\|_{-1} + \kappa_0 \mu_0^{-1} \|z\|).$$

*Proof.* From standard elliptic estimates we see that

$$\|\nabla u\| \leqq \frac{1}{\mu_0} \|f\|_{-1}$$

and

$$\|\nabla p\| \leqq \kappa_0^2 \mu_0^{-2} \|f\|_{-1} + \kappa_0 \mu_0^{-1} \|z\|.$$

From (4.3) we have

$$\|a_0\|_2 \leqq \frac{1}{\beta} \kappa_1 \|\nabla u\| \|\nabla p\|$$

and the estimate follows. ∎

The following shows that $Q_s(z)$ is weakly compact.

PROPOSITION 4.2. *If $\{a_k\}_{k=1}^{\infty}$ is a sequence in $W^{2,2}$ such that $a_k \in Q_s(z)$ for each $k$, then there is a subsequence $a_{k_i} \to a$ weakly in $W^{2,2}$ where the limit $a$ satisfies Eqs.* (4.1)–(4.3).

*Proof.* Since $W^{2,2}$ embeds compactly in $C^0(c\ell(\Omega))$, it follows from the previous lemma that there is a subsequence $\{a_{k_i}\}_{i=1}^{\infty}$ such that

$$a_{k_i} \to a \text{ weakly} \qquad \text{in } W^{2,2} \text{ and } C^0(c\ell(\Omega)).$$

Hence, $a \geqq \mu_0$. Further, it is also clear that the subsequence may be chosen such that

$$u_{k_i} \to u \text{ weakly} \qquad \text{in } W_0^{1,2}$$

and

$$p_{k_i} \to p \text{ weakly} \qquad \text{in } W_0^{1,2} \cap W^{2,2}$$

from elliptic regularity estimates [2, 14]. This convergence is sufficiently strong to pass to the limit in (4.1)–(4.3). Hence, the limiting functions $a$, $u$, and $p$ form a solution of the system (4.1)–(4.3). ∎

*Remark* 4.3. From the above propositions it follows that if $Q(z) = \tilde{Q}(z)$, then $Q_s(z)$ is nonempty and weakly compact in $W^{2,2}$.

Let $\mathfrak{X} = \{z: \|z - z_0\| \leqq \eta\}$ denote the closed ball in $L^2$ of radius $\eta$ centered at $z_0$ and $Q_s^\eta = \bigcup_{z \in \mathfrak{X}} Q_s(z)$. From elliptic regularity theory we have the following.

PROPOSITION 4.4. *Let $Q(z) = \tilde{Q}(z)$. Then the set $Q_s^\eta$ is nonempty and compact in $W^{2,2}$.*

*Proof.* From regularity theory we have seen that if $a \in Q_s^\eta$, then

$$\|a\|_{W^{\nu,2}} \leqq C(z_0, \eta, f)$$

for $\nu \in [2, 4)$. It follows that if $Q_s^\eta$ is closed and sequentially compact then it is compact. This is straightforward and is shown in a manner similar to the proof of Proposition 4.2. ∎

By a similar proof, we have the following.

PROPOSITION 4.5. *Let $Q(z) = \tilde{Q}(z)$. The mapping $z \mapsto Q_s(z)$ from $\mathfrak{X}$ with the weak $L^2$ topology into $Q_s^\eta$ with the strong $W^{2,2}$ topology is closed.*

The above results imply the following.

THEOREM 4.6. *Let $Q(z) = \tilde{Q}(z)$. The mapping $z \mapsto Q_s(z)$ from $B$ with the weak $L^2$ topology into $Q_s^\eta$ with the $W^{2,2}$ topology is upper semicontinuous.*

We now wish to determine conditions such that there is a unique solution to the system (4.1)–(4.3). These results require a weaker norm for the data term than in [12] although we determine uniqueness if the constraints are inactive. To this end, suppose that $(a_1, u_1, p_1)$ and $(a_2, u_2, p_2)$ are both solutions of (4.1)–(4.3). Define $\alpha = a_1 - a_2$, $v = u_1 - u_2$, $\rho = p_1 - p_2$, and $\zeta = z_1 - z_2$. We see that $(\alpha, v, \rho)$ is a solution of the system

$$\begin{aligned} -\nabla \cdot (a_1 \nabla v) &= \nabla \cdot (\alpha \nabla u_2) && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.4}$$

$$\begin{aligned} -\nabla \cdot (a_1 \nabla \rho) &= \nabla \cdot (\alpha \nabla p_2) + v + \zeta && \text{in } \Omega \\ \rho &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.5}$$

$$\beta(\alpha, h)_2 = \int_\Omega (\nabla v \cdot \nabla p_1 + \nabla u_2 \cdot \nabla \rho) h \, dx \tag{4.6}$$

for any $h \in W^{2,2}$. The following estimates clearly hold

$$\|\nabla v\| \leqq \mu_0^{-2}\kappa_1\|\alpha\|_2\|f\|_{-1} \tag{4.7}$$

$$\|\nabla \rho\| \leqq \mu_0^{-2}\kappa_0\kappa_1\|\alpha\|_2(\|u_1 - z\| + \kappa_0\|f\|) + \kappa_0\mu_0^{-1}\|\zeta\| \tag{4.8}$$

$$\beta\|\alpha\|_2 \leqq \kappa_1(\|\nabla v\|\|\nabla p_1\| + \|\nabla u_2\|\|\nabla \rho\|). \tag{4.9}$$

From (4.7)–(4.9) we see that

$$\begin{aligned}\|\alpha\|_2 \leqq \frac{\kappa_0}{\beta}\kappa_1^2\mu_0^{-3}\|f\|^{-1}\{2\|u_1 - z\| + \kappa_0\|f\|_{-1}\}\|\alpha\|_2 \\ + \frac{\kappa_0}{\beta}\kappa_1\mu_0^2\|f\|_{-1}\|\zeta\|.\end{aligned} \tag{4.10}$$

Accordingly, we specify the condition

$$\begin{aligned}C &= C(\|u_1 - z_1\|, f, \mu_0, \beta) \\ &= \frac{\kappa_0}{\beta}\kappa_1^2\mu_0^{-3}\|f\|_{-1}\{2\|u_1 - z\| + \kappa_0\|f\|_{-1}\} < 1.\end{aligned} \tag{F}$$

Thus, inequality (4.10) yields the inequality

$$\|\alpha\|_2 \leqq \{\kappa_0\kappa_1\|f\|_{-1}/((1 - C)\beta\mu_0)\}\|\zeta\|$$

and implies the result.

THEOREM 4.7. *If conditions $Q(z) = \tilde{Q}(z)$ and* (F) *hold, then system* (4.1)–(4.3) *has a unique solution.*

COROLLARY 4.8. *If $Q(z) = \tilde{Q}(z)$ and condition* (F) *holds, the mapping $z \mapsto Q_s(z)$ is single valued and Lipschitz continuous.*

*Remark* 4.9. We note that (F) is similar to condition (D) and the inequality in the proof of Proposition 3.15. Hence, a comment similar to that of Remark 3.14 holds indicating that, for fixed $\beta$, condition (F) prescribes a condition on the relation of model output to the data. Further, condition (F) may be satisfied as well by choosing $\|f\|_{-1}$ sufficiently small.

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