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Essentially Finitely Generated Lie Algebras

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INTRODUCTION

In this paper we discuss the class of essentially finitely generated Lie algebras, defined by analogy with the module-theoretic concept found in Goodearl [6]. This concept proves to be an extremely useful tool for elucidating the structure of semisimple Lie algebras with Min-c. In particular it permits us to answer in the affirmative one part of an open question, namely, Question 4 of Aldosray and Stewart [2].

In Section 1 we establish some fundamental properties of the class $\mathscr{E}\mathscr{f}\mathscr{g}$ of essentially finitely generated Lie algebras, including the crucial property of E-closure. We also prove an L-module analogue of this result

In Section 2 we apply the concept of essentially finitely generated algebras to prime ideals of Lie algebras with Min-c. Our main result is an affirmative answer to part of Question 4 of Aldosray and Stewart [2]. Specifically, we prove that if a Lie algebra contains finitely many prime ideals with zero intersection, then L is essentially finitely generated. When L is semisimple, this implies that $L \in$ Min-c, and the answer to the open question follows.

Throughout this paper, all Lie algebras considered are of finite or infinite dimension over a field **k** of arbitrary characteristic, unless otherwise specified. Most notation used is standard, and may be found in Aldosray [1]. Aldosray and Stewart [2–4], or Amayo and Stewart [5]. Thus we write $I \leq L$ if I is a subalgebra of L and $I \lhd L$ if I is an ideal of L. The centralizer of I in L is written $C_L(I)$. The subalgebra generated by a subset $X \subseteq L$ is denoted $\langle X \rangle$, and the ideal generated by a subset $X \subseteq L$ is denoted $\langle X \rangle^L$. If $X = \{x\}$ is a singleton we write $\langle x \rangle^L$ in place of

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$\langle\{x\}\rangle^L$. Throughout this paper whenever we speak of an ideal of L being finitely generated we mean that it is finitely generated as an ideal of L, rather than finitely generated as a subalgebra of L. A Lie algebra L is *semisimple* if it has no nonzero abelian ideals.

An ideal $C \lhd L$ is a *centralizer ideal* if there exists an ideal $I \lhd L$ such that $C = C_L(I)$. An ideal $K \lhd L$ is a *complement ideal* if there exists an ideal $I \lhd L$ such that $K \cap I = 0$ and if $J \supseteq K$, $J \lhd L$, and $J \cap I = 0$, then $J = K$. That is, K is maximal subject to $K \cap I = 0$. We say that K is a *complement to* I. We require two related chain conditions: $L \in$ Min-c if L has the minimal condition for centralizer ideals (Aldosray and Stewart [2]) and $L \in$ Max-CI if L has the maximal condition for complement ideals (Aldosray and Stewart [3]). An ideal $E \lhd L$ is *essential* (written E ess L) if E intersects every nonzero ideal of L nontrivially. An element $x \in L$ is *regular* if $\langle x \rangle^L$ is an essential ideal of L.

Any other notation is defined as needed. The end (or absence) of a proof is signalled by a box (■).

1. ESSENTIALLY FINITELY GENERATED LIE ALGEBRAS

By analogy with ring theory [6], we make the following definition:

DEFINITION 1.1. (a) A Lie algebra L is *essentially finitely generated* if L contains a finitely generated essential ideal. We denote the class of all essentially finitely generated Lie algebras by $\mathscr{E}\mathscr{f}\mathscr{g}$.

(b) An L-module M is *essentially finitely generated over* L if M contains a finitely generated essential L-submodule.

Note that (b) is equivalent to M being essentially finitely generated as a $U(L)$-module, where $U(L)$ is the universal enveloping algebra of L.

EXAMPLE 1.2. The class $\mathscr{E}\mathscr{f}\mathscr{g}$ is neither Q-closed nor I-closed. To see this, let L be the infinite Heisenberg algebra with basis $\{z, x_i, y_i \mid i \in \mathbb{N}\}$ such that $[x_i, y_i] = z$ and all other elements commute. Then $L \in \mathscr{E}\mathscr{f}\mathscr{g}$ since $\langle z \rangle$ is essential. Let $I = \langle z, x_i \mid i \in \mathbb{N} \rangle \lhd L$. Then since both L/I and I are infinite-dimensional abelian, neither lies in $\mathscr{E}\mathscr{f}\mathscr{g}$.

In contrast to this example, we have:

PROPOSITION 1.3. *If $L \in \mathscr{E}\mathscr{f}\mathscr{g}$ and* 1 *is a complement ideal of L, then $L/I \in \mathscr{E}\mathscr{f}\mathscr{g}$.*

Proof. Let J be a finitely generated essential ideal of L. The $I + J$ ess L by Aldosray and Stewart [2, Lemma 2.1(e)]. Hence $(I + J)/I$ ess L/I by Aldosray and Stewart [3, Lemma 6.4(b)]. Since $(I + J)/I$ is clearly a finitely generated ideal of L/I, it follows that $L/I \in \mathscr{E}\mathscr{f}\mathscr{g}$. ■

PROPOSITION 1.4. (a). *If I and J are ideals of L, with I ess J ess L, then I ess L.*

(b) *If L is semisimple, I ess L, and $I \subseteq B \leq L$, then I ess B.*

(c) *If L is semisimple and I, J are ideals of L such that $I \subseteq J$ and I ess L, then I ess J and J ess L.*

Proof. (a) Suppose that $0 \neq K \lhd L$. Then $K \cap J \neq 0$. However, $K \cap J \lhd J$, so $(K \cap J) \cap I \neq 0$, whence $K \cap I \neq 0$. Therefore I ess L.

(b) Let $K \lhd B$ and suppose that $K \cap I = 0$. Then $K \subseteq C_B(I) = B \cap C_L(I)$. However, L is semisimple and I ess L, so $C_L(I) = 0$. Therefore $K = 0$ and I ess B.

(c) Lemma 2.1(e) of Aldosray and Stewart [3] implies that J ess L. Moreover, I ess J by the foregoing (b). ∎

COROLLARY 1.5. *Let I be an ideal of L such that I is essentially finitely generated. Then L is essentially finite generated if either of the following holds*:

(a) I ess L.

(b) $C_L(I) = 0$.

Proof. (a) Let $J = \langle x_1, \ldots, x_k \rangle^I$ be a finitely generated essential ideal of I and consider the ideal K of L generated by J. We have $J \subseteq K = \langle x_1, \ldots, x_k \rangle^L \subseteq I$, so K ess I. However, I ess L, so K ess L by Proposition 1.4(a). Therefore $L \in \mathscr{E}\mathscr{f}\mathscr{g}$.

(b) Since $C_L(I) = 0$ implies that I ess L, part (a) implies that $L \in \mathscr{E}\mathscr{f}\mathscr{g}$. ∎

We now prove that the class $\mathscr{E}\mathscr{f}\mathscr{g}$ is E-closed.

THEOREM 1.6. *Suppose that $I \lhd L$ and both I and $L/I \in \mathscr{E}\mathscr{f}\mathscr{g}$. Then $L \in \mathscr{E}\mathscr{f}\mathscr{g}$.*

Proof. Let $J = \langle x_1, \ldots, x_k \rangle^I$ ess I, and again define $K = \langle x_1, \ldots, x_k \rangle^L \subseteq I$. Then K is a finitely generated ideal of L. Moreover $J \subseteq K$, so K ess I. Let $N = \langle y_1, \ldots, y_t \rangle^L$ be such that $(N + I)/I$ ess L/I. This implies that $N + I$ ess L. We claim that $N + K$ ess L. Certainly $N + K$ is finitely generated. We complete the proof by showing that $N + K$ ess $N + I$, after which we can apply Proposition 1.4(a) to conclude that $N + K$ ess L. Consider the natural homomorphism α: $N + I \to I/N(\cap I)$ defined by $\alpha(n + i) = (N \cap I) + i$ for $n \in N$, $i \in I$. (That α is well defined is the content of the second isomorphism theorem.) Then $\alpha(K)$ess $\alpha(1)$ since K ess I. Therefore $\alpha^{-1}(\alpha(K))$ess $\alpha^{-1}(\alpha(I)) = N + I$. However, $\alpha^{-1}(\alpha(K)) = N + K$. ∎

COROLLARY 1.7. *$\mathcal{E}\!f\!g$ is closed under finite direct sums.*

There is an analogue of Theorem 1.6 for L-modules, which we shall require later:

THEOREM 1.8. *Suppose that $I \lhd L$, and both I and L/I, considered as L-modules, are essentially finitely generated over L. Then $L \in \mathcal{E}\!f\!g$.*

Proof. This follows exactly as in the proof of Theorem 1.6, replacing essential ideals by essential submodules and noting that α is an L-module homomorphism. There is a slight simplification because, in a notation analogous to that used in the proof of Theorem 1.6, we have $J = K$.

REMARK 1.9. *If $L \in \mathcal{E}\!f\!g$, then L need not satisfy* Min-c. *For if L is the McLain algebra $L_F(\mathbb{Z})$ over any field F, then $L \in \mathcal{E}\!f\!g \setminus$* Min-c.

PROPOSITION 1.10. *$L \in \mathcal{E}\!f\!g$ if and only if L does not contain an infinite direct sum of ideals.*

Proof. Consider L as a $U(L)$-module and apply the arguments of Goodearl [6, Proposition 3.13]. ∎

COROLLARY 1.11. (a) *$L \in \mathcal{E}\!f\!g$ if and only if $L \in$* Max-CI.

(b) *If L is semisimple, then $L \in \mathcal{E}\!f\!g$ if and only if $L \in$* Max $-$ c.

2. SEMISIMPLE ALGEBRAS WITH Min-c

Next, we apply our results to give a new characterisation of semisimple Lie algebras with Min-c.

THEOREM 2.1. *Let L be a semisimple Lie algebra. Then $L \in$* Min-c *if and only if every prime ideal is essentially finitely generated over L.*

Proof. First we assume $L \in$ Min-c and prove that every prime ideal P is essentially finitely generated. If P is not a minimal prime ideal, then P ess L by Aldosray and Stewart [3, Proposition 2.12]. Therefore P contains a regular element p of L by Corollary 2.3 of Aldosray [1]. Hence $\langle p\rangle^L$ is a finitely generated essential ideal of L, and since $\langle p\rangle^L \subseteq P$ it follows that P is essentially finitely generated over L.

Otherwise we may assume that P is a minimal prime ideal, so that $P + C_L(P)$ess L by Aldosray and Stewart [3, Proposition 2.6(a)]. Therefore $P + C_L(P)$ contains a regular element x of L by Corollary 2.3 of Aldosray [1], so that $\langle x\rangle^L$ ess L. However, L is semisimple, so $P \cap C_L(P) = 0$ and $P + C_L(P)$ is a direct sum. Therefore we can write x uniquely as $x = a + b$, where $a \in P$, $b \in C_L(P)$. Now $\langle x\rangle^L = \langle a + b\rangle^L \subseteq \langle a\rangle^L + \langle b\rangle^L$, and $\langle a\rangle^L \cap \langle b\rangle^L = 0$ by directness of the sum.

We claim that $\langle a\rangle^L$ is an essential ideal of P. Suppose that $J \lhd P$ and $J \cap \langle a\rangle^L = 0$. Since $J \subseteq P$ we also have $J \cap \langle b\rangle^L = 0$. Therefore $[J, \langle a\rangle^L] = 0$ and $[J, \langle b\rangle^L] = 0$, so that $[J, \langle a+b\rangle^L] = 0$ and hence $[J, \langle x\rangle^L] = 0$. Therefore $J \subseteq C_L(\langle x\rangle^L)$. However, L is semisimple and $\langle x\rangle^L$ ess L, so $C_L(\langle x\rangle^L) = 0$, implying that $J = 0$. We deduce that $\langle a\rangle^L$ is essential in P, and hence that P is essentially finitely generated over L. ∎

Now we assume that every prime ideal P is essentially finitely generated and prove that $L \in$ Min-c. Let P be a prime ideal of L. If P is not minimal prime, then P ess L by Aldosray and Stewart [3, Proposition 2.12]. However, $P \in \mathscr{E}\mathscr{f}\mathscr{g}$, so $L \in \mathscr{E}\mathscr{f}\mathscr{g}$ by Corollary 1.5.

If on the other hand P is a minimal prime ideal of L, then we claim that P is a maximal centralizer ideal of L. This is proved for $L \in$ Min-c in Aldosray and Stewart [2, Lemma 4.2], but here we do not know that $L \in$ Min-c so we must use a different argument, as follows. Since $C_L(P) \neq 0$ and P is a prime ideal, it follows that $P = C_L(C_L(P))$ and P is a centralizer ideal. Now suppose that $P = C_L(I)$ and that there exists an ideal $J \neq 0$ such that $C_L(J) > P$. Then $[C_L(J), J] = 0$, therefore $[C_L(J), J] \subseteq P$. Since $C_L(J) \nsubseteq P$, the primeness of P implies that $J \subseteq P$. However, $[J, P] = 0$, so $J \subseteq \zeta_1(P) = 0$ by semisimplicity, a contradiction.

By Theorem 3.4 of Aldosray [1] we have $P = C_L(\langle x\rangle^L)$ for some nonzero uniform element $x \in L$. Hence $P \cap \langle x\rangle^L = C_L(\langle x\rangle^L) \cap \langle x\rangle^L = 0$ by semisimplicity. Therefore the sum $P + \langle x\rangle^L$ is direct. Now let I be a finitely generated essential ideal of P and let $\pi\colon P + \langle x\rangle^L \to P$ be a projection. Then $\pi^{-1}(I)$ is a finitely generated essential ideal of $P + \langle x\rangle^L$, and $P + \langle x\rangle^L$ is an essentially finitely generated ideal of L. Furthermore, $P + \langle x\rangle^L$ ess L, for if $K \lhd L$ and $K \cap (P + \langle x\rangle^L) = 0$, then $K \subseteq C_L(P + \langle x\rangle^L) = C_L(P) \cap C_L(\langle x\rangle^L) = C_L(P) \cap P = 0$ by semisimplicity. Therefore $L \in \mathscr{E}\mathscr{f}\mathscr{g}$ by Corollary 1.5. However, for semisimple L, this implies that $L \in \text{Min} - \text{c}$ by Corollary 1.11(b). ∎

Finally we apply the machinery developed in this paper and in Aldosray [1] to answer, in the affirmative, part of Question 4 of Aldosray and Stewart [2]. This asks whether a semisimple Lie algebra is in Min-c if the intersection of finitely many maximal centralizers is zero. In fact we prove a slightly more general result. Theorem 2.3: a Lie algebra is in $\mathscr{E}\mathscr{f}\mathscr{g}$ if the intersection of finitely many prime ideals is zero. (Recall that a Lie algebra L is *prime* if 0 is a prime ideal of L.) This result implies the answer to Question 4 because $\mathscr{E}\mathscr{f}\mathscr{g}$ = Max-CI by Corollary 1.11(a). Moreover, for semisimple Lie algebras Max-CI = Max-c = Min-c by Theorem 4.1 of Aldosray and Stewart [4] and Lemma 2.1 of Aldosray and Stewart [2], and a maximal centralizer is prime by Lemma 4.1 of Aldosray and Stewart [2].

LEMMA 2.2. *Let L be a prime Lie algebra. Then every nonzero ideal of L is essential.*

Proof. Let $0 \neq I \lhd L$ and suppose that $I \cap J = 0$, where $J \lhd L$. Then $[I, J] = 0$. Since L is prime and $I \neq 0$ we must have $J = 0$. Therefore I ess L. ∎

THEOREM 2.3. *Suppose that a Lie algebra L contains finitely many prime ideals $P_1, \dots, P_k$ such that $P_1 \cap \cdots \cap P_k = 0$. Then $L \in \mathscr{E}\mathscr{f}\mathscr{g}$.*

Proof. Consider the descending chain

$$L \supseteq P_1 \supseteq P_1 \cap P_2 \supseteq \cdots \supseteq P_1 \cap \cdots \cap P_k = 0$$

and, by omitting redundant P_i if necessary, assume that k is the first positive integer for which $P_1 \cap \cdots \cap P_k = 0$. We claim that every factor

$$(P_1 \cap \cdots \cap P_t)/(P_1 \cap \cdots \cap P_{t+1})$$

is essentially finitely generated when considered as an L-module. If so, then Theorem 1.8 and induction on t implies that $L \in \mathscr{E}\mathscr{f}\mathscr{g}$.

To prove the claim, let $P_1 \cap \cdots \cap P_t = Q$. Then $Q/(Q \cap P_{t+1})$ is isomorphic as an L-module to $(Q + P_{t+1})/P_{t+1}$, which is an L-submodule of L/P_{t+1}, a prime Lie algebra. By the choice of k we may assume that $Q/(Q \cap P_{t+1})$ is nonzero. By Lemma 2.2, every nonzero L-submodule X of $(Q + P_{t+1})/P_{t+1}$ is essential in L/P_{t+1}. Choose X to be any nonzero one-generator submodule: then $(Q + P_{t+1})/P_{t+1}$ is an essentially finitely generated L-module.

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