On the distance spectral radius of bipartite graphs

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ABSTRACT

In this paper, we determine the unique graph with minimum distance spectral radius among all connected bipartite graphs of order $n$ with a given matching number. Moreover, we characterize the graphs with minimal distance spectral radius in the class of all connected bipartite graphs with a given vertex connectivity.

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1. Introduction

Let $G$ be a connected simple graph on $n$ vertices with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices $u$, $v \in V$ is denoted by $d_{uv}$ and is defined as the length of the shortest path between $u$ and $v$ in $G$. The distance matrix of $G$ is denoted by $D(G)$ and defined by $D(G) = (d_{uv})_{u,v \in V}$. Since $D(G)$ is a real symmetric matrix, all its eigenvalues are real [8]. The distance spectral radius $\rho(G)$ of $G$ is the largest eigenvalue of its distance matrix $D(G)$. Since $D(G)$ is irreducible, by the Perron—Frobenius Theorem, the eigenvalue $\rho(G)$ is simple, and there exists a unique (up to multiples) positive eigenvector corresponding to $\rho(G)$. The unique normalized eigenvector corresponding to $\rho(G)$ is referred as the Perron vector of $D(G)$.

Balaban et al. [1] proposed the use of $\rho(G)$ as a molecular descriptor, while in [9] it was successfully used to infer the extent of branching and model boiling points of alkanes. Balasubramanian in [2,3] pointed out that the spectra of the distance matrices of many graphs such as the polyacenes, honeycomb...
and square lattice have exactly one positive eigenvalue, and he computed the spectrum of fullerenes $C_{60}$ and $C_{70}$.


Indulal in [11] has found sharp bounds on the distance spectral radius and the distance energy of graphs. In [10] Ilić characterized $n$-vertex trees with given matching number, which minimize the distance spectral radius. Liu has characterized graphs with minimal distance spectral radius in three classes of simple connected graphs with $n$ vertices: with fixed vertex connectivity, matching number and chromatic number, in [12]. Subhi and Powers in [15] proved that for $n \geq 3$ the path $P_n$ has the maximum distance spectral radius among all trees on $n$ vertices. Stevanović and Ilić in [14] generalized this result, and proved that among all trees with fixed maximum degree $\Delta$, the broom graph has maximum distance spectral radius and proved that the star $S_n$ is the unique graph with minimal distance spectral radius among all trees on $n$ vertices. In [6], Bose et al. have characterized the unique graph with minimal distance spectral radius among all graphs of order $n$ with $r$ pendant vertices. They have also determined the unique graph with maximal distance spectral radius among all graphs with $r$ pendant vertices for each $r \in \{2, 3, n - 3, n - 2, n - 1\}$.

A bipartite graph $G$ is a simple graph, whose vertex set $V(G)$ can be partitioned into two disjoint subsets $V_1$ and $V_2$ such that every edge of $G$ joins a vertex of $V_1$ with a vertex of $V_2$. If $G$ contains every edge joining a vertex of $V_1$ with a vertex of $V_2$, then $G$ is a complete bipartite graph and is denoted by $K_{m,n}$, where $m, n$ are the number of vertices in $V_1$ and $V_2$, respectively.

A vertex (edge) independent set of a graph $G$ is a set of vertices (edges) such that any two distinct vertices (edges) of the set are not adjacent (incident on a common vertex). The vertex (edge) independence number of $G$, denoted by $\alpha(G)$ ($\alpha'(G)$), is the maximum of the cardinalities of all vertex (edge) independent sets.

A vertex (edge) cover of a graph $G$ is a set of vertices (edges) such that each edge (vertex) of $G$ is incident with at least one vertex (edge) of the set. The vertex (edge) cover number of $G$, denoted by $\beta(G)$ ($\beta'(G)$), is the minimum of the cardinalities of all vertex (edge) covers.

An edge independent set (edge independence number) is usually called a matching (matching number). For a connected graph $G$ of order $n$, its matching number $\alpha'(G)$ satisfies $1 \leq \alpha'(G) \leq \lfloor \frac{n}{2} \rfloor$. When we consider an edge cover of a graph, we always assume that the graph contains no isolated vertex. It is known that for a graph $G$ of order $n$, $\alpha(G) + \beta(G) = n$; and if in addition $G$ has no isolated vertex, then $\alpha'(G) + \beta'(G) = n$. For a bipartite graph, $\alpha'(G) = \beta(G)$, and $\alpha(G) = \beta'(G)$.

Recall that the vertex connectivity of a graph $G$, denoted by $\kappa(G)$, is the minimum number of vertices whose deletion yields the resulting graph disconnected. It is a very important parameter in characterizing graph connectivity.

Let $B_n^k$ be the class of all bipartite graphs of order $n$ with matching number $k$, and $B_n^s$ be the class of all bipartite graphs of order $n$ with vertex connectivity $s$.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we determine the unique graph with minimum distance spectral radius in $B_n^k$. As a corollary we find the unique graph with minimum distance spectral radius among all connected bipartite graphs on $n$ vertices with a given vertex independence number or vertex cover number or edge cover number. Finally in Section 4, we characterize the graphs with minimal distance spectral radius in $B_n^s$.

2. Preliminaries

For a simple graph $G(V, E)$ and a subset $S$ of $V$, $G[S]$ denotes the induced subgraph on $S$ (i.e., the maximal subgraph of $G$ with vertex set $S$). By $d_G(v)$ we denote the degree of the vertex $v$ in $G$. The star $S_n$ is a tree on $n$ vertices with one vertex of degree $n - 1$ and the remaining of degree 1. The path and the cycle of order $n$ are denoted by $P_n$ and $C_n$, respectively.

Let $e$ be an edge of a connected graph $G$ joining vertices $u$ and $v$ such that $G - e = G'(\text{say})$, is also connected, and let $D'$ be the distance matrix of $G'$. As observed already in [14], the removal of $e$ does not create shorter paths than the ones in $G$, and therefore, $d_{ij} \leq d'_{ij}$ for all $i, j \in V$. Moreover,
1 = d_{uv} < d'_{uv} and by the Perron–Frobenius theorem, one can conclude that

\[ \rho(G) < \rho(G - e). \]

In particular, for any spanning tree \( T \) of \( G \), we have that

\[ \rho(G) \leq \rho(T). \]  \hfill (2.1)

Similarly, adding a new edge \( f \) between two non-adjacent vertices \( s \) and \( t \) of \( G \) does not increase distances, while it does decrease at least one distance. Again by the Perron–Frobenius theorem,

\[ \rho(G + f) < \rho(G). \]  \hfill (2.2)

Inequality (2.2) tells us immediately that the complete graph \( K_n \) has the minimum distance spectral radius among the connected graphs on \( n \) vertices, while the inequality (2.1) shows that the maximum distance spectral radius will be attained for a particular tree.

Let \( x(G) = (x_1, x_2, \ldots, x_n)^t \) be the Perron vector of \( D(G) \) corresponding to \( \rho(G) \). Then

\[ \rho(G) x_i = \sum_{j \in V(G)} d_{ij} x_j. \]

3. The graph with minimum distance spectral radius in \( B_n^k \)

Here we find the unique graph with minimum distance spectral radius in \( B_n^k \).

**Theorem 3.1.** \( K_{k,n-k} \) is the unique graph that minimizes the distance spectral radius in \( B_n^k \).

**Proof.** Let \( G \) be a graph in \( B_n^k \) with minimum distance spectral radius. For \( k = \lfloor \frac{n}{2} \rfloor \) the discussion is trivial.

Let \( (U, W) \) be the bipartition of the vertex set of \( G \) such that \( |W| \geq |U| \geq k \), and let \( M \) be a maximal matching of \( G \). Since the distance spectral radius of a graph decreases with addition of edges so for \( |U| = k \), \( G = K_{k,n-k} \).

Let us assume that \( |U| > k \) and \( U_M, W_M \) be the sets of vertices of \( U, W \) which are incident to the edges of \( M \), respectively. Therefore, \( |U_M| = |W_M| = k \). Note that \( G \) contains no edges between the vertices of \( U \setminus U_M \) and the vertices of \( W \setminus W_M \), otherwise any such edge may be united with \( M \) to produce a matching of cardinality greater than that of \( M \), violating the maximality of \( M \).

Adding all possible edges between the vertices of \( U_M \) and \( W_M \), \( U_M \) and \( W_M \), \( U \setminus U_M \) and \( W_M \) we get a graph \( G' \) with \( \rho(G) > \rho(G') \). We now form a complete bipartite graph \( G'' = K_{k,n-k} \) from \( G' \) with the bipartition \((U_M, W \cup (U \setminus U_M))\).

Let \( |U \setminus U_M| = n_1, |W \setminus W_M| = n_2 \). So \( n_2 \geq n_1 \). We partition \( V(G') = V(G'') \) into \( U_M \cup W_M \cup (U \setminus U_M) \cup (W \setminus W_M) \) as shown in Fig. 1. If the distance matrices \( D(G') \) and \( D(G'') \) are partitioned according to \( U_M, W_M, (U \setminus U_M), \) and \( (W \setminus W_M) \), then their difference is

\[
D(G') - D(G'') = \begin{bmatrix}
0 & 0 & J_{k \times n_1} & 0 \\
0 & 0 & -J_{k \times n_1} & 0 \\
J_{n_1 \times k} & -J_{n_1 \times k} & 0 & J_{n_1 \times n_2} \\
0 & 0 & J_{n_2 \times n_1} & 0
\end{bmatrix}
\]

where \( J \) is the matrix of appropriate order, all of whose entries are 1. We denote \( \rho(G') \) by \( \rho \) and \( \rho(G'') \) by \( \rho_1 \). Let \( x \) be the Perron vector of \( D(G'') \). Then by symmetry, components of \( x \) have the same value, say \( x_1 \) for the vertices in \( U_M \) and \( x_2 \) for the vertices in \( W \cup (U \setminus U_M) \). Then \( x \) can be written as,
We have
\[
\frac{1}{2}(\rho - \rho_1) \geq \frac{1}{2}x^t(D(G') - D(G''))x = n_1x_2[2kx_1 + n_2x_2 - kx_2].
\] (3.3)

Now from \(D(G'')x = \rho_1x\), we have
\[
\rho_1x_1 = 2(k - 1)x_1 + (n - k)x_2,
\]
\[
\rho_1x_2 = kx_1 + 2(n - k - 1)x_2.
\]

Thus,
\[
(\rho_1 + 2 - k)(x_1 - x_2) = (2k - n)x_2 = -(n_1 + n_2)x_2.
\] (3.4)

Following [7], the distance spectral radius of the complete bipartite graph \(K_{p,q}\) is \(p + q - 2 + \sqrt{p^2 - pq + q^2}\), so \(\rho_1 > n + k - 1\).

Again, \(k(x_1 - x_2) + n_2x_2 = -\frac{(n_1 + n_2)kx_2}{\rho_1 + 2 - k} + n_2x_2\) [By (3.4)]
\[
> \frac{[-(n_1 + n_2)k + n_2(n + 1)]x_2}{\rho_1 + 2 - k} \quad \text{[Since } \rho_1 > n + k - 1 \text{ ]}
\]
\[
= \frac{[-(n_1 + n_2)k + n_2(n_1 + n_2 + 2k + 1)]x_2}{\rho_1 + 2 - k}
\]
\[
> \frac{n_2^2 + n_2n_1 + n_2x_2}{\rho_1 + 2 - k} > 0.
\]

Thus from (3.3) we get \(\rho > \rho_1\), and so \(\rho(G) > \rho(G'')\), a contradiction. Therefore \(|U| = k\). □

From the above theorem we have the following corollary.
Corollary 3.2. The graph \( K_{\sigma,n-s} \) is the unique graph that minimizes the distance spectral radius among all connected bipartite graphs of order \( n \) with vertex cover number or vertex independence number or edge cover number \( \sigma \).

4. The graphs in \( B_n^* \) with minimal distance spectral radius

In this section, we characterize the graphs with minimal distance spectral radius in \( B_n^* \). It is shown in [16] that \( K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} \) has minimum distance spectral radius among all connected bipartite graphs. This result also says that for vertex connectivity \( s = \lceil \frac{n}{2} \rceil \), \( K_{s,n-s} \) is the unique graph with minimum distance spectral radius in the class \( B_n^* \).

Clearly \( B_2^* = \{ P_4, S_4 \} \) and \( B_3^* = \{ P_5, S_5, C_5 \} \), where \( C_5 \) is the graph with a single pendant attached to a vertex of \( C_4 \). It can be easily verified that \( S_4 \) and \( C_4 \) are the graphs with minimal distance spectral radius in \( B_4^* \) and \( B_5^* \), respectively. Thus for \( 3 \leq n \leq 5 \), the discussion is over. From now onwards we will assume that \( n \geq 6 \).

To prove the main result in this section, we need to define some notations and prove some lemmas.

In \( K_{p,q} \), we assume that \( p \geq q \) and by \( K_{p,0}, \ p \geq 1 \), we mean \( pK_1 \). We define two graphs \( O_s \lor 1 \ (K_{n_1,n_2} \cup K_{m_1,m_2}) \) and \( O_s \lor 2 \ (K_{n_1,n_2} \cup K_{m_1,m_2}) \), where \( \cup \) is the union of two graphs, \( O_s \ (s \geq 1) \) is an empty graph of order \( s \) and \( \lor \) is a graph operation that joins all the vertices in \( O_s \) to the vertices belonging to the partitions of cardinality \( n_1 \) in \( K_{n_1,n_2} \) and \( m_1 \) in \( K_{m_1,m_2} \) respectively; whereas \( \lor \) is a graph operation that joins all the vertices in \( O_s \) to the vertices belonging to the partitions of cardinality \( n_2 \) in \( K_{n_1,m_2} \) and \( m_2 \) in \( K_{m_1,m_2} \) respectively. Note that \( \lor \) is defined only when \( n_2 \geq 1 \) and \( m_2 \geq 1 \).

Lemma 4.1. If \( s + q \geq p + 1 \) and \( p \geq s \), then \( \rho(O_s \lor 1 \ (K_1 \cup K_{p,q})) > \rho(O_s \lor 1 \ (K_1 \cup K_{p+1,q-1})) \).

Proof. Let us denote \( O_s \lor 1 \ (K_1 \cup K_{p,q}) \) by \( G \) and \( O_s \lor 2 \ (K_1 \cup K_{p+1,q-1}) \) by \( G' \). We partition \( V(G) = V(G') \) into \( \{ v \} \cup C \cup A \cup B \cup \{ b_q \} \), where \( C = \{ c_1, c_2, \ldots, c_s \} \), \( A = \{ a_1, a_2, \ldots, a_p \} \) and \( B = \{ b_1, b_2, \ldots, b_{q-1} \} \) as in Fig. 2.

As we pass from \( G \) to \( G' \), the distance of \( b_q \) is decreased by 1 with \( \{ v \} \cup C \cup A \cup B \) and the distance of \( b_q \) is increased by 1 with \( A_1 \); the distances within any other pairs of vertices remain unaltered. If the distance matrices are partitioned according to \( \{ v \}, C, A, B \) and \( \{ b_q \} \), then their difference is

\[
D(G) - D(G') = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & e_C \\
0 & 0 & 0 & -e_A \\
0 & 0 & 0 & e_B \\
1 & e_C^t & -e_A^t & e_B^t & 0
\end{bmatrix}
\]

where \( e_i = (1, \ldots, 1)^t \) and \( i = A, B, C \). We denote \( \rho(G) \) by \( \rho \) and \( \rho(G') \) by \( \rho_1 \). Let \( x \) be the Perron vector of \( D(G') \). Then by symmetry, components of \( x \) have the same value, say \( a \) for the vertices in \( A \cup \{ b_q \} \), \( b \) for the vertices in \( B \), \( c \) for the vertices in \( C \), and \( x_v \) for \( v \). Then \( x \) can be written as,

\[
x = \left( x_1, c, \ldots, c, a, \ldots, a, b, \ldots, b, a \right)^t.
\]

We now have

\[
\frac{1}{2} (\rho - \rho_1) \geq \frac{1}{2} x^t (D(G) - D(G')) x = a [x_v + cs - pa + b(q - 1)].
\] (4.5)
Now from $D(G')x = \rho_1 x$, we have
\[
\begin{align*}
\rho_1 x_1 &= sc + 3(q - 1)b + 2(p + 1)a, \\
\rho_1 c &= x_1 + 2(s - 1)c + 2(q - 1)b + (p + 1)a, \\
\rho_1 a &= 2x_1 + sc + (q - 1)b - qa, \\
\rho_1 b &= 3x_1 + 2sc + 2(q - 2)b + (p + 1)a.
\end{align*}
\]
From which we get,
\[
\begin{align*}
(\rho_1 + 1)(x_1 - c) &= c + (p + 1)a + (q - 1)b - sc, \\
(\rho_1 + 1)(c - a) &= -2x_1 + (s - 1)c - qa + (q - 1)b, \\
(\rho_1 + 2)(x_1 - a) &= 2(q - 1)b + a > 0 \Rightarrow x_1 > a.
\end{align*}
\]
Since distance matrix is nonnegative and irreducible, its spectral radius is bounded below by the minimum row sum and thus we have $\rho_1 > 3p \geq 3s$.

Again by the given condition $q - 1 \geq p - s = k$ (say). Therefore from (4.8), we get
\[
\begin{align*}
(\rho_1 + 1)(s - a) &> -x_1 + pc - qa + (s - 1 - p)c + (p - s)b \\
\Rightarrow (\rho_1 + 1 - p)(c - a) &> -x_1 + (s - 1)c + kb \\
\Rightarrow (\rho_1 + 1 - p)(c - a) &> -x_1 + kb - c \\
\Rightarrow (c - a) &> \frac{1}{(\rho_1 + 1 - p)}[ -c + x_1 + kb - c].
\end{align*}
\]
Using (4.9) in (4.7), we get
\[
\begin{align*}
(\rho_1 + 1)(x_1 - c) &= c + (p + 1)a - sc + \frac{(\rho_1 + 2)(x_1 - a)}{2} \\
\Rightarrow (\rho_1 + 1)(x_1 - c) &> \frac{1}{2}[2c + 2(p + 1)a - 2sc + (\rho_1 + 2)(x_1 - a)] \\
\Rightarrow (\rho_1 + 1)(x_1 - c) &> \frac{1}{2}[2c + 2(p + 1)a - 2sc + 2s(x_1 - a)] \quad [\text{Since } \rho_1 \geq 3s] \\
\Rightarrow (\rho_1 + 1 - s)(x_1 - c) &> \frac{1}{2}[2c + 2(p + 1 - s)a] > 0 \\
\Rightarrow x_1 > c.
\end{align*}
\]

Fig. 2. The graphs $G$ and $G'$ in Lemma 4.1.
Proof. Let \( \text{Lemma 4.3.} \)

\[
\text{If } s \text{ distance matrices are partitioned according to } \{ \}
\]

\[
\text{Corollary 4.2.} \text{ If } q \]

\[
\text{p} = \text{V} \text{ We partition } \]

\[
\text{By the above lemma we have the following corollary.} \]

\[
\text{Thus by (4.5), } \rho > \rho_1. \quad \square
\]

By the above lemma we have the following corollary.

**Corollary 4.2.** If \( q \geq 1 \), then \( \rho(O_5 \lor 1 (K_1 \cup K_{p,q})) \geq \rho(O_5 \lor 1 (K_1 \cup K_{p,q})) \); equality holds only when \( p = q \).

**Lemma 4.3.** If \( s + q + 4 \leq p \), then \( \rho(O_5 \lor 1 (K_1 \cup K_{p,q})) > \rho(O_5 \lor 1 (K_1 \cup K_{p-1,q+1})) \).

**Proof.** Let \( p = s + q + k \), \( k \geq 4 \). Let us denote \( O_5 \lor 1 (K_1 \cup K_{p,q}) \) by \( G \) and \( O_5 \lor 1 (K_1 \cup K_{p-1,q+1}) \) by \( G' \). We partition \( V(G) = V(G') \) into \( \{v\} \cup C \cup A \cup B \cup \{a_p\} \), where \( C = \{c_1, c_2, \ldots, c_s\} \), \( A = \{a_1, a_2, \ldots, a_{p-1}\} \) and \( B = \{b_1, b_2, \ldots, b_q\} \) as in Fig. 3.

As we pass from \( G \) to \( G' \), the distance of \( a_p \) is increased by 1 with \( \{v\} \cup C \cup B \) and the distance of \( a_p \) is decreased by 1 with \( A \); the distances within any other pair of vertices remain unaltered. If the distance matrices are partitioned according to \( \{v\}, C, A, B \) and \( \{a_p\} \), then their difference is

\[
D(G) - D(G') = \begin{bmatrix}
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -e_{C} & e_{A} \\
0 & 0 & 0 & -e_{A} & -e_{B} \\
1 & -e_{C} & e_{A} & -e_{B} & 0
\end{bmatrix}
\]
Thus we now have
\[ 1292 \text{ is bounded below by the minimum row sum and thus we have} \]
\[ \text{for the vertices in } A, \]
\[ B \text{ for the vertices in } B \cup \{q\}, \]
\[ c \text{ for the vertices in } C, \]
\[ \text{and } x_1 \text{ for } v. \]
Then \( x \) can be written as,
\[
x = \left( x_1, c, \ldots, c, a, \ldots, a, b, \ldots, b \right)_x.
\]
We now have
\[
\frac{1}{2}(\rho - \rho_1) \geq \frac{1}{2}x^T(D(G) - D(G'))x = b[-x_1 - sc - bq + ap(p - 1)]. \tag{4.11}
\]
Now from \( D(G')x = \rho_1x, \) we have
\[ (\rho_1 + 2)(x_1 - a) = 2(q + 1)b > 0, \]
\[ (\rho_1 + 2)(b - c) = 2x_1 > 0. \]
Thus \( x_1 > a \) and \( b > c. \) We also have,
\[ (\rho_1 + 2)(2a - b) = x_1 + 3(p - 1)a > 0 \implies 2a > b, \tag{4.12} \]
and \( (\rho_1 + 4)(2a - x_1) = sc + 2(p + 1)a - (q + 1)b > 0 \] [By (4.12)]
\[ \implies 2a > x_1. \tag{4.13} \]
Again, \( (\rho_1 + 1)(a - b) = -x_1 - sc + (p - 1)a - a - qb. \tag{4.14} \]
If \( a \geq b, \) then from (4.14), we have \( -x_1 - sc + (p - 1)a - qb \geq a; \) and by (4.11), we get \( \rho > \rho_1. \)
Let us assume that \( a < b. \) Since distance matrix is nonnegative and irreducible, its spectral radius is bounded below by the minimum row sum and thus we have
\[ \rho_1 > p + 2q + 2s. \]
Now, \( (q + 1)a - qb = qa - b + a \]
\[ \implies (\rho_1 + 1)[(q + 1)a - qb] = [-qx_1 - sqc + (p - 2)qa - q^2b] + (\rho_1 + 1)a \] [By (4.14)]
\[ > q(2a - x_1) + p(q + 1)a + (s - q)a + a - sc - q^2b \]
\[ > q(2a - x_1) + p(q + 1)a + (s - q)a + a - qsb - q^2b \] [Since \( b > c \)]
\[ = q(2a - x_1) + p(q + 1)a + (s - q)a + a - qb(s + q) \]
\[ = q(2a - x_1) + p(q + 1)a + (s - q)a + a - qbp - k \]
\[ \implies (\rho_1 + 1 - p)[(q + 1)a - qb] > q(2a - x_1) + 2(s - q)a + a + qbk. \tag{4.15} \]
If \( s \geq q, \) then by (4.13) and (4.15), \( (q + 1)a > qb. \)
Otherwise, let \( t = q - s \) and again by (4.15),
\[ (\rho_1 + 1 - p)[(q + 1)a - qb] > q(2a - x_1) + a + (kqb - 2ta) \]
\[ > q(2a - x_1) + (4qb - 2ta) \]
\[ > q(2a - x_1) + (4tb - 2ta) \] [Since \( q > t \)]
\[ > 0 \] [Since \( 2a > x_1 \) and \( b > a \).]
Thus we can conclude that \((q + 1)a > qb\).

Finally, \((\rho_1 + 2)(a - c) = x_1 - sc + (p - 1)a - (q + 1)b\)
\[= (x_1 - a) - sc + (q + s + k)a - (q + 1)b\]
\[= (x_1 - a) + s(a - c) + ((q + 1)a - qb) + (k - 1)a - b\]
\[\Rightarrow (\rho_1 + 2 - s)(a - c) = (x_1 - a) + ((q + 1)a - qb) + (k - 1)a - b\]
\[\geq (x_1 - a) + ((q + 1)a - qb) + (3a - b)\]
\[> 0 \text{ [Since } x_1 > a, (q + 1)a > qb \text{ and } 2a > b]\]
\[\Rightarrow a > c.\]

Therefore, \(-x_1 - sc - bq + a(p - 1) = -x_1 - sc - bq + a(q + s + k - 1)\)
\[= ((q + 1)a - qb) + s(a - c) + (k - 2)a - x_1\]
\[\geq ((q + 1)a - qb) + s(a - c) + (2a - x_1)\]
\[> 0 \text{ [Since } (q + 1)a > qb, a > c \text{ and } 2a > x_1]\].

Therefore from (4.11), we get \(\rho > \rho_1.\) □

Similar to the above lemma we have the following lemma.

**Lemma 4.4.** If \(n \geq 6\) and \(1 \leq s < \lfloor \frac{n - 1}{2} \rfloor\), then \(\rho(K_{s,n-s}) > \rho(O_5 \cup K_{1 \cup K_{n-s-2,1}}).\)

**Lemma 4.5.** If \(G \in B_n^s\) and \(U\) is a vertex cut-set of order \(s\) in \(G\) such that \(G \setminus U\) has two nontrivial components, then \(G\) cannot be a graph with minimal distance spectral radius in \(B_n^s\).

**Proof.** Let \(G_1\) and \(G_2\) be the nontrivial components of \(G \setminus U\) with bipartitions \((A, B)\) and \((C, D)\) respectively. Let \(U = U_1 \cup U_2\) be the bipartition of \(U\) induced by the bipartition of \(G\). Now joining all possible edges between the vertices of \(A\) and \(B\), \(C\) and \(D\), \(U_1\) and \(U_2\) we get a graph \(\overline{G}\) in \(B_n^s\) such that \(\rho(G) \geq \rho(\overline{G})\). Therefore we suppose that \(G = \overline{G}\).

If there exists some vertex \(w\) in \(G \setminus U\) such that \(d_G(w) = s\), then forming a complete bipartite graph within the vertices of \(G \setminus \{w\}\) we would get a graph in \(B_n^s\) with smaller distance spectral radius. Thus we may assume that each vertex in \(G \setminus U\) has degree greater than \(s\).

Let \(|A| = m_1, |B| = m_2, |C| = n_1, |D| = n_2, |U_1| = t, |U_2| = k\).
Thus $2a > c$ and therefore by (4.18), we have $\rho > \rho_1$. □
Let $G_1^s$, $G_2^s$, $G_3^s$ and $G_4^s$ be the graphs described in Fig. 5. The following is the main result in this section.

**Theorem 4.6.** Let $G$ be a graph in $B_n^s$, with minimal distance spectral radius; where $1 \leq s \leq \lfloor \frac{n-1}{2} \rfloor$. Then $G \in \{G_1^s, G_3^s\}$, if $n$ is odd and $G \in \{G_2^s, G_4^s\}$, if $n$ is even.

**Proof.** Let $G$ be a graph with minimal distance spectral radius in $B_n^s$. Let $U$ be a vertex cut-set of $G$ containing $s$ vertices, whose deletion yields the components $G_1, G_2, \ldots, G_t$ of $G \setminus U$, where $t \geq 2$. If some component $G_i$ of $G \setminus U$ has at least two vertices, then it must be complete bipartite. Again if some component $G_i$ of $G \setminus U$ is a singleton, say $G_i = \{u\}$, then $u$ is adjacent to all the vertices of $U$ otherwise $\kappa(G) < s$; hence the subgraph $G[U]$ induced by $U$ contains no edges, and belongs to the same partition of $G$. We now have the following cases.

**Case 1:** All the components of $G \setminus U$ are singletons. Then $G = K_{s,n-s}$. For $s = \lfloor \frac{n-1}{2} \rfloor$ we have $K_{s,n-s} \cong G_1^s$, if $n$ is odd and $K_{s,n-s} \cong G_2^s$, if $n$ is even; and thus the result.

Let us assume that $1 \leq s < \lfloor \frac{n-1}{2} \rfloor$. Then by Lemma 4.4, $\rho(K_{s,n-s}) > \rho(O_s \lor K_{1 \cup K_{n-s-2}})$, which contradicts the minimality of $G$. Therefore not all the components of $G \setminus U$ can be singletons.

**Case 2:** One component of $G \setminus U$, say $G_1$, contains at least two vertices. Then $G \setminus U$ contains exactly two components; otherwise, forming a complete bipartite graph within the vertices of $G_1 \cup G_2 \cup \ldots \cup G_t$ we obtain a new graph $\hat{G}$ from $G$ with smaller distance spectral radius such that $\hat{G} \in B_n^s$, a contradiction. Let $G_1, G_2$ be the components of $G \setminus U$. By Lemma 4.5, either $G_1 = K_1$ or $G_2 = K_1$. Without loss of generality assume that $G_2 = K_1 = \{u\}$. Then $u$ joins all vertices of $U$, and each vertex of $G_1$ joins every vertex of $G_1$ which are in the same partition as $u$. Since $G$ is a graph with minimal distance spectral radius then by Corollary(4.2), $G = O_s \lor (K_1 \cup K_{q,p})$ for some $p$ and $q$. We note that $p \geq s$, otherwise $s$ cannot be the vertex connectivity of $G$. If $q + s \leq p \leq q + s + 3$, then the result follows. Again if $q + s > p$, then by repeated application of Lemma 4.1, $G = G_1^s$, if $n$ is odd and $G = G_2^s$, if $n$ is even. Finally if $p \geq q + s + 4$, then by using Lemma 4.3 repeatedly, we have $G$ is either $G_2^s$ or $G_3^s$ according as $n$ is odd or even. □

References


