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Singularly Perturbed Markov Chains with Two Small Parameters: A Matched Asymptotic Expansion

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This work is concerned with asymptotic properties of solutions to forward equations for singularly perturbed Markov chains with two small parameters. It is motivated by the model of a cost-minimizing firm involving production planning and capacity expansion and a two-level hierarchical decomposition. Our effort focuses on obtaining asymptotic expansions of the solutions to the forward equation. Different from previous work on singularly perturbed Markov chains, the inner expansion terms are constructed by solving certain partial differential equations. The methods of undetermined coefficients are used. The error bound is obtained.

Key Words: manufacturing system; singularly perturbed Markov chain; multiple scale; forward equation; asymptotic expansion.

1. INTRODUCTION

Much of the current interest stems from the motivation of various applications in manufacturing systems. In such applications, one aims to

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find the optimal production-planning strategies and to use Markov jump structures in the modeling of uncertainty (e.g., machine capacity). In many applications, one often needs to deal with large-scale systems. Even for a seemingly not so complex system, the optimal control can be very difficult to obtain (see [12, Sect. 5.2]), not to mention the added difficulty of a very large dimensional state space involved. In fact, not only is it an impossible task to obtain analytic solutions, but also the direct numerical approximation can be overwhelming and the straightforward use of numerical methods may not be adequate. Naturally, one seeks to break the large job into small pieces with the hope that certain decompositions and aggregations will lead to a simplification of the intractable systems; see Simon and Ando [13]. A viable alternative calls for taking advantage of the high contrast rates (some states vary an order of magnitude faster than the rest) of changes in the physical systems and using a singular perturbation approach as a tool to reduce the complexity of the underlying systems.

To have a thorough understanding of the problems, it is of the utmost importance to learn the intrinsic structures and asymptotic properties of the underlying Markov chains. Continuing effort has been devoted to studying singularly perturbed Markov chains; see [3, 9–12] among others. Recently, in [6], Khasminskii *et al.* used matched asymptotic expansion to establish the convergence of a sequence of the probability vectors. Singularly perturbed Markov chains with recurrent states, naturally divisible into a number of classes, are then treated in [7]. This line of work has been continued in [15], in which we have further derived asymptotic expansions for Markov chains with the inclusion of transient states and absorbing states, and asymptotic distributions for Markov chains having recurrent states.

In this work, we examine a continuous-time model with two small parameters. The use of multiple small parameters stems from consideration of hierarchical structures of various stochastic systems in manufacturing as well as in communication networks. The motivation comes from a production-marketing system (see [Chap. 11]). The underlying problem is concerned with a cost-minimizing firm involving production planning and capacity expansion and a two-level hierarchical decomposition. In [12], Sethi and Zhang have found the asymptotic optimal strategies by using the limit distribution of the Markov chains. In this work, our main effort is devoted to obtaining asymptotic properties of the solutions to the forward equation satisfied by the probability vector. In previous work on singularly perturbed Markov chains, the asymptotic expansions are obtained by solving appropriate algebraic and ordinary differential equations, whereas in this paper, partial differential equations are also involved. The results are useful in the subsequent studies on Markov decision processes and controlled Markovian systems involving multiple small parameters. To proceed, let us begin with an example of the manufacturing system.

EXAMPLE 1.1. Consider a production-marketing system that produces a single product type using a two-state production capacity $\alpha(t) \in \{0, 1\}$. Let $u(t)$ denote the production rate subject to the production constraints $0 \le u(t) \le \alpha(t)$. Let $x(t) \in \mathbb{R}^1$ denote the total surplus and let $z(t) \in \{z_1, z_2\}$ denote a two-state stochastic demand rate. The system is given by

$$\dot{x}(t) = u(t) - z(t), \quad x(0) = x, \tag{1.1}$$

where $x \in \mathbb{R}^1$ is the initial surplus. Let $w(t) \in \mathbb{R}^1$ denote the rate of advertising satisfying $0 \le w(t) \le K$, with K representing an upper bound on the advertising rate. The profit functional $J(\cdot)$ is defined by

$$\begin{aligned} &J(x, \alpha, z, u(\cdot), w(\cdot)) \\ &\quad = E\int_0^\infty e^{-\rho t}[\pi z(t) - (h_1(x(t)) + cu(t) + w(t))]\,dt, \end{aligned} \tag{1.2}$$

where $\rho > 0$ is the discount rate, π is the revenue per unit sale, $h_1(\cdot)$ is the surplus cost function, and $c < \pi$ is the unit production cost. The problem is to find a control $(u(t), w(t))$, $t \ge 0$, that maximizes $J(x, \alpha, z, u(\cdot), w(\cdot))$.

Let $\mathcal{M} = \{(0, z_1), (1, z_1), (0, z_2), (1, z_2)\}$ denote the state space of the pair $(\alpha(t), z(t))$. Then as in [12], the process $(\alpha(t), z(t))$ can be formulated as a Markov chain generated by

$$\begin{aligned} Q^{\varepsilon,\delta}(u, w) &= \frac{1}{\varepsilon}A(u) + \frac{1}{\delta}B(w) \\ &= \frac{1}{\varepsilon}\begin{pmatrix} -\mu_1 & \mu_1 & 0 & 0 \\ \lambda_1(u) & -\lambda_1(u) & 0 & 0 \\ 0 & 0 & -\mu_1 & \mu_1 \\ 0 & 0 & \lambda_1(u) & -\lambda_1(u) \end{pmatrix} \\ &\quad + \frac{1}{\delta}\begin{pmatrix} -\mu_2(w) & 0 & \mu_2(w) & 0 \\ 0 & -\mu_2(w) & 0 & \mu_2(w) \\ \lambda_2(w) & 0 & -\lambda_2(w) & 0 \\ 0 & \lambda_2(w) & 0 & -\lambda_2(w) \end{pmatrix}. \end{aligned}$$

Here ε and δ are small parameters signifying the frequency of jumps of α and z. Note here μ_1 represents the machine repair rate and hence is independent of production rate u. It is easily seen that

$$A(u) = \operatorname{diag}(\widetilde{Q}, \widetilde{Q}) \text{ with } \widetilde{Q} = \begin{pmatrix} -\mu_1 & \mu_1 \\ \lambda_1(u) & -\lambda_1(u) \end{pmatrix}$$

$$B(w) = \overline{Q}(w) \otimes I_2 \text{ with } \overline{Q}(w) = \begin{pmatrix} -\mu_2(w) & \mu_2(w) \\ \lambda_2(w) & -\lambda_2(w) \end{pmatrix},$$

where I_2 denotes the 2×2 identity matrix, and "$\otimes$" denotes the usual Kronecker product (i.e., with $X = (x_{ij})$ and $Y = (y_{ij})$, the ijth entry of $X \otimes Y$

is defined to be $(X \otimes Y)_{ij} = x_{ij} Y)$. In what follows, we shall generalize this model to include many states and weak and strong interactions.

The rest of the paper is arranged as follows. Section 2 gives the precise formulation of the problem. Section 3 proceeds with the construction of the asymptotic expansion. Section 4 concentrates on the error analysis. Finally, Section 5 concludes the paper with further remarks.

2. FORMULATION

2.1. *Initial Formulation*

We work with a finite horizon $[0, T]$ for some $T > 0$. Recall that a generator $Q(t) \in \mathbb{R}^{m\times m}$ is said to be weakly irreducible (see [15]), if $f(t)Q(t) = 0$, and $\sum_{i=1}^{m} f_i(t) = 1$ has a unique nonnegative solution. The unique solution is termed a quasi-stationary distribution.

The structure of the generator $Q^{\varepsilon,\delta}$ to be considered is motivated by Example 1.1. To proceed, first generalize the setup in the example to include many states. For ease of presentation, consider the stationary case. Let $\alpha^{\varepsilon,\delta}(t)$ be a Markov chain with $\alpha^{\varepsilon,\delta}(t) = (\alpha_1^{\varepsilon}(t), \alpha_2^{\delta}(t))$ and state space $\mathcal{M} = \{(a_1, b_1), \dots, (a_{m_0}, b_1), \dots, (a_1, b_l), \dots, (a_{m_0}, b_l)\}$, where l and m_0 are some positive integers. Denote $m = lm_0$. Suppose the generator is given by $Q^{\varepsilon,\delta} = (1/\varepsilon)A + (1/\delta)B$, where

$$A = \operatorname{diag}\big(\widetilde{Q}, \dots \widetilde{Q}\big) \quad \text{and} \quad B = \overline{Q} \otimes I_{m_0}, \tag{2.1}$$

with $\widetilde{Q} = \big(\tilde{q}_{ij}\big) \in \mathbb{R}^{m_0\times m_0}$ being a generator and $\overline{Q} = (\bar{q}_{ij}) \in \mathbb{R}^{l\times l}$ being a generator and $I_{m_0} \in \mathbb{R}^{m_0\times m_0}$ is an $m_0 \times m_0$ dimensional identity matrix. Henceforth, we use the symbol diag to denote a diagonal block matrix with appropriate entries. Suppose that $\widetilde{Q}$ and $\overline{Q}$ are both weakly irreducible. Let $\tilde{\nu}$ denote the stationary distribution corresponding to $\widetilde{Q}$. Then it is easy to see that $\operatorname{diag}(\tilde{\nu}, \dots, \tilde{\nu})Q^{\varepsilon,\delta}\tilde{\mathbb{1}} = \overline{Q}/\delta$. This demonstrates that $\overline{Q}/\delta$ is the generator of the Markov chain $\alpha_2^{\delta}(t)$ with state space $\{b_1, \dots, b_l\}$. Similarly, we can show $\alpha_1^{\varepsilon}(t)$ is a Markov chain generated by $\widetilde{Q}/\varepsilon$ with state space $\{a_1, \dots, a_{m_0}\}$. We next generalize this idea further for nonstationary cases and to include weak and strong interactions.

2.2. *More General Formulation*

Suppose $\varepsilon > 0$ and $\delta > 0$ are small parameters, $\alpha^{\varepsilon,\delta}(t)$ is a finite state Markov chain with state space $\mathcal{M} = \{1, \dots, m\}$ generated by $Q^{\varepsilon,\delta}(t)$, and the row vector $p^{\varepsilon,\delta}(t) = \big(P(\alpha^{\varepsilon,\delta}(t) = 1), \dots, P(\alpha^{\varepsilon,\delta}(t) = m)\big) \in \mathbb{R}^{1\times m}$ denotes the probability distribution of the Markov chain at time t. It is well known that $p^{\varepsilon,\delta}(\cdot)$ is a solution to the forward equation

$$\frac{dp^{\varepsilon,\delta}(t)}{dt} = p^{\varepsilon,\delta}(t)Q^{\varepsilon,\delta}(t), \quad p^{\varepsilon,\delta}(0) = p^0, \tag{2.2}$$

where $p_i^0 \geq 0$ for each i and $\sum p_i^0 = 1$ is the initial probability distribution.

Assume that the generator $Q^{\varepsilon,\delta}(t) \in \mathbb{R}^{m\times m}$ has the form

$$Q^{\varepsilon,\delta}(t) = \frac{1}{\varepsilon}A(t) + \frac{1}{\delta}B(t) + \widehat{Q}(t). \tag{2.3}$$

Assume that $A(t)$ and $B(t)$ have the same partitioned form and the same form of Kronecker product as in (2.1) with $\widetilde{Q}$ and $\overline{Q}$ replaced by time dependent $\widetilde{Q}(t)$ and $\overline{Q}(t)$, respectively. The slowly changing motion is described by the generator

$$\widehat{Q}(t) = \begin{pmatrix} \hat{q}_{11}(t)I_{m_0} & \cdots & \hat{q}_{1l}(t)I_{m_0} \\ \vdots & & \vdots \\ \hat{q}_{l1}(t)I_{m_0} & \cdots & \hat{q}_{ll}(t)I_{m_0} \end{pmatrix}. \tag{2.4}$$

Throughout the rest of the paper, unless otherwise noted, we always work with (2.3) with time-varying generators. In addition, we often use the notion $\mathbb{1}_j = (1, \ldots, 1)' \in \mathbb{R}^{j\times 1}$, for some integer j. We make the following assumptions:

(A1) For each $t \in [0, T]$, $\widetilde{Q}(t)$ and $\overline{Q}(t) = (\bar{q}_{ij}(t))$ are weekly irreducible.

(A2) For some $n \geq 0$, $A(\cdot)$, $B(\cdot)$, and $\widehat{Q}(\cdot)$ are $(n+1)$-time continuously differentiable on [0, T]. In addition, $(d^{n+1}/dt^{n+1})A(\cdot)$, $(d^{n+1}/dt^{n+1})B(\cdot)$, and $(d^{n+1}/dt^{n+1})\widehat{Q}(\cdot)$ are Lipschitz on [0,T].

Define an operator $L^{\varepsilon,\delta}$ by

$$L^{\varepsilon,\delta}f = \frac{df}{dt} - f\left(\frac{1}{\varepsilon}A(t) + \frac{1}{\delta}B(t) + \widehat{Q}(t)\right) \tag{2.5}$$

for any smooth vector-valued function $f(\cdot)$. Then $L^{\varepsilon,\delta}f = 0$ if and only if f is a solution to the differential equation (2.2). We are now in a position to analyze the solution to (2.2).

3. ASYMPTOTIC EXPANSION

Using singular perturbation techniques, to approximate the solution to (2.2), we seek outer–inner expansions of the form

$$p^{\varepsilon,\delta}(t) = \varphi^{\varepsilon,\delta,n}(t) + \psi^{\varepsilon,\delta,n}\left(\frac{t}{\varepsilon}, \frac{t}{\delta}\right) + e^{\varepsilon,\delta,n}(t), \tag{3.1}$$

where $e^{\varepsilon,\delta,n}(t)$ is the remainder, and the outer (regular part) and initial layer correction terms are given by

$$\varphi^{\varepsilon,\delta,n}(t) = \sum_{i+j=0}^{n} \varepsilon^i \delta^j \varphi^{(i,j)}(t)$$

$$\text{and} \quad \psi^{\varepsilon,\delta,n}\Big(\frac{t}{\varepsilon},\frac{t}{\delta}\Big) = \sum_{i+j=0}^{n} \varepsilon^i \delta^j \psi^{(i,j)}\Big(\frac{t}{\varepsilon},\frac{t}{\delta}\Big), \tag{3.2}$$

respectively, such that $e^{\varepsilon,\delta,n}(t)$ is small and the error bound holds uniformly in t. The main theorem is recorded below.

THEOREM 3.1. *Assume* (A1) *and* (A2). *For small parameters* $\varepsilon > 0$ *and* $\delta > 0$, *denote the unique solution to* (2.2) *by* $p^{\varepsilon,\delta}(\cdot)$. *Then for* $0 \le i + j \le n$, *we can construct* $\varphi^{(i,j)}(\cdot)$ *and* $\psi^{(i,j)}(\cdot,\cdot)$ *such that*

(a) $\varphi^{(i,j)}(\cdot)$ *is twice differentiable on* $[0, T]$.

(b) *For each* i, *there are* $\kappa_1 > 0$ *and* $\kappa_2 > 0$ *such that*

$$\Big|\psi^{(i,j)}\Big(\frac{t}{\varepsilon},\frac{\varepsilon}{\delta}\Big)\Big| \le K_1 \exp\Big(-\frac{\kappa_1 t}{\varepsilon}\Big) + K_2 \exp\Big(-\frac{\kappa_2 t}{\delta}\Big).$$

(c) *Suppose there exist constants* $h_1 > 0$ *and* $h_2 > 0$ *such that* $h_1\varepsilon \le \delta \le h_2\varepsilon$. *Then the following estimate holds*

$$\sup_{t\in[0,T]} \Big| p^{\varepsilon,\delta}(t) - \sum_{i+j=0}^{n} \varepsilon^i\delta^j\varphi^{(i,j)}(t) - \sum_{i+j=0}^{n} \varepsilon^i\delta^j\psi^{(i,j)}\Big(\frac{t}{\varepsilon},\frac{t}{\delta}\Big)\Big|$$
$$\le K\big(\varepsilon^{n+1} + \delta^{n+1}\big). \tag{3.3}$$

When $i = j = 0$ in the above theorem, we obtain the limit of $p^{\varepsilon,\delta}(t)$ for $t > 0$.

COROLLARY 3.2. *Suppose* $Q^{\varepsilon,\delta}(\cdot)$ *is continuously differentiable on* $[0, T]$, *which satisfies* (A1), *and* $(d/dt)Q(\cdot)$ *is Lipschitz on* $[0, T]$. *Then for all* $t > 0$, $\lim_{\varepsilon,\delta\to 0} p^{\varepsilon,\delta}(t) = \varphi^{(0,0)}(t)$.

To obtain the desired result, we obtain the outer and inner expansion terms by direct construction in what follows. They involve solutions to algebraic-differential equations.

3.1. *Outer Expansion*

We begin with the construction of $\varphi^{\varepsilon,\delta,n+1}(\cdot)$ in the asymptotic expansion. Consider the differential equation $L^{\varepsilon,\delta}\varphi^{\varepsilon,\delta,n+1}(t) = 0$, where $L^{\varepsilon,\delta}$ is given by (2.5).

Equating coefficients of $\varepsilon^i\delta^j$, we have for $i+j=0$,

$$\begin{aligned}
\varepsilon^{-1}\delta^0 &: \quad 0=\varphi^{(0,0)}(t)A(t),\\
\varepsilon^{0}\delta^{-1} &: \quad 0=\varphi^{(0,0)}(t)B(t),\\
\varepsilon^{0}\delta^{0} &: \quad \frac{d\varphi^{(0,0)}(t)}{dt}=\varphi^{(1,0)}(t)A(t)+\varphi^{(0,1)}(t)B(t)\\
&\qquad\qquad +\varphi^{(0,0)}(t)\widehat{Q}(t),\\
\varepsilon^{1}\delta^{-1} &: \quad 0=\varphi^{(1,0)}(t)B(t),\\
\varepsilon^{-1}\delta^{1} &: \quad 0=\varphi^{(0,1)}(t)A(t),
\end{aligned} \tag{3.4}$$

and for $1\le i+j\le n$,

$$\begin{aligned}
\varepsilon^{2}\delta^{-1} &: \quad 0=\varphi^{(2,0)}(t)B(t),\\
\varepsilon^{1}\delta^{0} &: \quad \frac{d\varphi^{(0,1)}(t)}{dt}=\varphi^{(2,0)}(t)A(t)+\varphi^{(1,1)}(t)B(t)\\
&\qquad\qquad +\varphi^{(1,0)}(t)\widehat{Q}(t),\\
\varepsilon^{0}\delta^{1} &: \quad \frac{d\varphi^{(0,1)}(t)}{dt}=\varphi^{(1,1)}(t)A(t)+\varphi^{(0,2)}(t)B(t)\\
&\qquad\qquad +\varphi^{(0,1)}(t)\widehat{Q}(t),\\
\varepsilon^{-1}\delta^{2} &: \quad 0=\varphi^{(0,2)}(t)A(t).\\
&\cdots\cdot
\end{aligned} \tag{3.5}$$

We construct the solutions to (3.4). For construction of solutions to systems with higher order, the approach is similar, only the notation is more involved. The main idea is as follows. We separate the equations in (3.4) into two groups. The first group consists of the first two equations. We show that they share a common solution. In fact, this common solution is the limit as $\varepsilon\to 0$ and $\delta\to 0$. The second group includes the rest of the three equations. To solve the equations in the second group, we use the methods of undetermined coefficients. First, assume the solution to the last two equations is of particular form with certain functions ("multiplier") to be determined. Then our task reduces to finding these functions by using the third equation. Such an idea is used in the construction of initial layer corrections as well as higher order terms.

3.1.1. *Construction of* $\varphi^{(0,0)}(t)$

Based on (A1), $\widetilde{Q}(t)$ and $\overline{Q}(t)$ are irreducible. Let $\pi(t)$ and $\lambda(t)$ be their quasi-stationary distributions, respectively. Denote $\pi_\lambda(t) = \big(\lambda_1(t)\pi(t),\dots,\lambda_l(t)\pi(t)\big)\in\mathbb{R}^m$. According to assumption (A2), the vector-valued functions $\lambda(t)$, $\pi(t)$ and $\pi_\lambda(t)$ are $(n+1)$-times continuously differentiable. We claim that the vector-valued function $\varphi^{(0,0)}(t)=\pi_\lambda(t)$ solves the first two equations of the system (3.4). Note that $\pi(t)\widetilde{Q}(t)=0$, so

$$\begin{aligned}
\varphi^{(0,0)}(t)A(t) &= \big(\lambda_1(t)\pi(t),\dots,\lambda_l(t)\pi(t)\big)\operatorname{diag}\big(\widetilde{Q}(t),\dots,\widetilde{Q}(t)\big)\\
&= \big(\lambda_1(t)\pi(t)\widetilde{Q}(t),\dots,\lambda_l(t)\pi(t)\widetilde{Q}(t)\big)=0,
\end{aligned}$$

where $A(t) = \mathrm{diag}(\widetilde{Q}(t), \ldots, \widetilde{Q}(t))$. As for the second equation, $\lambda(t)\overline{Q}(t) = 0$; that is, $\sum_{i=1}^{l} \lambda_i(t)\bar{q}_{ij}(t) = 0$, for all $1 \le j \le l$. Thus,

$$\begin{aligned}
\varphi^{(0,0)}(t)B(t) &= \big(\lambda_1(t)\pi(t), \ldots, \lambda_l(t)\pi(t)\big) \begin{pmatrix} \bar{q}_{11}(t)I_{m_0} & \ldots & \bar{q}_{1l}(t)I_{m_0} \\ \vdots & & \vdots \\ \bar{q}_{l1}(t)I_{m_0} & \ldots & \bar{q}_{ll}(t)I_{m_0} \end{pmatrix} \\
&= \left(\sum_{i=1}^{l} \lambda_i(t)\pi(t)\bar{q}_{i1}(t)I_{m_0}, \ldots, \sum_{i=1}^{l} \lambda_i(t)\pi(t)\bar{q}_{il}(t)I_{m_0} \right) \\
&= \left(\left(\sum_{i=1}^{l} \lambda_i(t)\bar{q}_{i1}(t) \right) \pi(t), \ldots, \left(\sum_{i=1}^{l} \lambda_i(t)\bar{q}_{il}(t) \right) \pi(t) \right) = 0.
\end{aligned}$$

In addition, since $\sum_{i=1}^{l} \lambda_i(t) = 1$ and $\sum_{i=1}^{m_0} \pi_i(t) = 1$, $\varphi^{(0,0)}(t)\mathbb{1}_m = \sum_{i=1}^{m} \varphi_i^{(0,0)}(t) = 1$.

3.1.2. *Construction of* $\varphi^{(1,0)}(t)$

To obtain the desired solutions, we use the methods of undetermined coefficients. We postulate that the fourth and fifth equations of (3.4) are given by

$$\begin{aligned}
&\varphi^{(1,0)}(t) = (\lambda_1(t)x(t), \ldots, \lambda_l(t)x(t)), \text{ and} \\
&\varphi^{(0,1)}(t) = (y_1(t)\pi(t), \ldots, y_l(t)\pi(t)),
\end{aligned}$$

where $\pi(t)$ and $\lambda(t)$ are the quasi-stationary distributions of $\widetilde{Q}(t)$ and $\overline{Q}(t)$, respectively, and where the vector-valued functions $x(t) = (x_1(t), \ldots, x_{m_0}(t)) \in \mathbb{R}^{1\times m_0}$ and $y(t) = (y_1(t), \ldots, y_l(t)) \in \mathbb{R}^{1\times l}$ are to be determined. In addition, $\varphi^{(1,0)}(t)\mathbb{1}_m = \sum_{i=1}^{m} \varphi_i^{(1,0)}(t) = 0$, and $\varphi^{(0,1)}(t)\mathbb{1}_m = \sum_{i=1}^{m} \varphi_i^{(0,1)}(t) = 0$.

The fourth and fifth equations of (3.4) are not enough to determine $x(t)$ and $y(t)$. We also need to use the third equation of (3.4), which can be written as

$$\varphi^{(1,0)}(t)A(t) + \varphi^{(0,1)}(t)B(t) = \frac{d\varphi^{(0,0)}(t)}{dt} - \varphi^{(0,0)}(t)\widehat{Q}(t).$$

The right-hand side of the above equation can be expressed as

$$\begin{aligned}
\frac{d\varphi^{(0,0)}(t)}{dt} - \varphi^{(0,0)}(t)\widehat{Q}(t) &= \frac{d\pi_\lambda(t)}{dt} - \pi_\lambda(t)\widehat{Q}(t) \\
&= \left(\lambda_1(t)\frac{d\pi(t)}{dt}, \ldots, \lambda_l(t)\frac{d\pi(t)}{dt} \right)
\end{aligned}$$

$$+\left(\frac{d\lambda_1(t)}{dt}\pi(t),\ldots,\frac{d\lambda_l(t)}{dt}\pi(t)\right)$$
$$-\left(\sum_{i=1}^{l}\lambda_i(t)\hat{q}_{i1}(t)\pi(t),\ldots,\sum_{i=1}^{l}\lambda_i(t)\hat{q}_{il}(t)\pi(t)\right).$$

Two equations may be derived from the third equation of (3.4). They are

$$\varphi^{(1,0)}(t)A(t)=\left(\lambda_1(t)\frac{d\pi(t)}{dt},\ldots,\lambda_l(t)\frac{d\pi(t)}{dt}\right), \tag{3.6}$$

and

$$\varphi^{(0,1)}(t)B(t)=\left(\frac{d\lambda_1(t)}{dt}\pi(t),\ldots,\frac{d\lambda_l(t)}{dt}\pi(t)\right)$$
$$-\left(\sum_{i=1}^{l}\lambda_i(t)\hat{q}_{i1}(t)\pi(t),\ldots,\sum_{i=1}^{l}\lambda_i(t)\hat{q}_{il}(t)\pi(t)\right). \tag{3.7}$$

Consider (3.6). In view of the block diagonal form of $A(t)$, (3.6) is equivalent to

$$\lambda_1(t)x(t)\widetilde{Q}(t)=\lambda_1(t)\frac{d\pi(t)}{dt},$$
$$\cdots$$
$$\lambda_l(t)x(t)\widetilde{Q}(t)=\lambda_l(t)\frac{d\pi(t)}{dt}.$$

It suffices to show that the following system has a solution

$$x(t)\widetilde{Q}(t)=\frac{d\pi(t)}{dt} \quad \text{and} \quad \sum_{i=1}^{m_0}x_i(t)=0. \tag{3.8}$$

Denote the null space of $\widetilde{Q}(t)$ by $N(\widetilde{Q}(t))$. The weak irreducibility of $\widetilde{Q}(t)$ implies that rank $(\widetilde{Q}(t))=m_0-1$, and hence $\dim(N(\widetilde{Q}(t)))=1$. Note that $N(\widetilde{Q}(t))$ is spanned by $\mathbb{1}_{m_0}$. By virtue of the well-known Fredholm alternative, the first equation in (3.8) has a solution only if its right-hand side is orthogonal to $N(\widetilde{Q}(t))$. Since $N(\widetilde{Q}(t))$ is spanned by $\mathbb{1}_{m_0}$, $\pi(t)\mathbb{1}_{m_0}=1$ and $\frac{d\pi(t)}{dt}\mathbb{1}_{m_0}=\frac{d(\pi(t)\mathbb{1}_{m_0})}{dt}=0$, so the orthogonality is verified. Next we show that the system (3.8) has a unique solution. To this end, let us rewrite the system (3.8) as

$$\begin{aligned}
&\tilde{q}_{11}(t)x_1(t)+\cdots+\tilde{q}_{m_0,1}(t)x_{m_0}(t)=\frac{d\pi_1(t)}{dt},\\
&\qquad\cdots\\
&\tilde{q}_{1,m_0}(t)x_1(t)+\cdots+\tilde{q}_{m_0m_0}(t)x_{m_0}(t)=\frac{d\pi_{m_0}(t)}{dt},\\
&x_1(t)+\cdots+x_{m_0}(t)=0.
\end{aligned} \tag{3.9}$$

The existence and uniqueness of the solution to (3.9) follows from the weak irreducibility and Fredholm alternative. Denote $\widetilde{Q}_c(t) = (\mathbb{1}_{m_0} \vdots \widetilde{Q}(t))$. Then the solution can be expressed as

$$x(t) = \frac{d\pi(t)}{dt} \widetilde{Q}'_c(t)[\widetilde{Q}_c(t)\widetilde{Q}'_c(t)]^{-1}.$$

Moreover $x(t)$ is n-times continuously differentiable on $[0, T]$. Consequently, (3.6) has a unique solution $\varphi^{(1,0)}(t) = (\lambda_1(t)x(t), \ldots, \lambda_l(t)x(t))$ with the condition $\sum_{i=1}^{m} \varphi^{(1,0)}(t) = 0$. In addition, $\varphi^{(1,0)}(t)$ is n-times continuously differentiable.

3.1.3. *Construction of* $\varphi^{(0,1)}(t)$

Examine (3.7). Rewrite Eq. (3.7) in its component form

$$\sum_{i=1}^{l} y_i(t)\bar{q}_{ij}(t)\pi(t) = \frac{d\lambda_j(t)}{dt}\pi(t) - \sum_{i=1}^{l} \lambda_i(t)\hat{q}_{ij}(t)\pi(t) \quad \text{for } j = 1, \ldots, l.$$

To establish the existence of a solution, it suffices to show that

$$y(t)\overline{Q}(t) = \frac{d\lambda(t)}{dt} - \lambda(t)Q^h(t) \tag{3.10}$$

has a solution, where $Q^h(t) = (\hat{q}_{ij}) \in \mathbb{R}^{l\times l}$. Since $\sum_{i=1}^{l} \lambda_i(t) = 1$, $((d/dt)\lambda(t))\mathbb{1}_l = (d/dt)(\lambda(t)\mathbb{1}_l)) = 0$. Owing to $Q^h(t)\mathbb{1}_l = 0$ $\left(\frac{d\lambda(t)}{dt} - \lambda(t)Q^h(t)\right)\mathbb{1}_l = 0$; i.e., the right-hand side of (3.10) is orthogonal to $\mathbb{1}_l$. By (A1), $\text{rank}(\overline{Q}(t)) = l - 1$. An argument similar to that of (3.8) confirms that

$$y(t)\overline{Q}(t) = \frac{d\lambda(t)}{dt} - \lambda(t)Q^h(t), \quad \sum_{i=1}^{l} y_i(t) = 0, \tag{3.11}$$

has a unique solution $y(t) = (y_1(t), \ldots, y_l(t))$. Hence $\varphi^{(0,1)}(t) = (y_1(t)\pi(t), \ldots, y_l(t)\pi(t))$ solve (3.7). It is easy to see that $\varphi^{(0,1)}(t)$ is n-times continuously differentiable. In addition, we have $\sum_{i=1}^{m} \varphi_i^{(0,1)}(t) = \sum_{j=1}^{l} y_j(t)(\sum_{k=1}^{m_0} \pi_k(t)) = \sum_{j=1}^{l} y_j(t) = 0$.

3.1.4. *Construction of* $\varphi^{(i,j)}(t)$ *for* $2 \le i + j \le n + 1$

In essentially the same way, we can construct $\varphi^{(i,j)}(t)$ for $2 \le i + j \le n + 1$. The details are thus omitted. To summarize, we state the following theorem.

THEOREM 3.3. *Assume* (A1) *and* (A2) *hold. Then there exist* $[(n+1)-(i+j)]$*-times continuously differentiable solutions* $\varphi^{(i,j)}(t)$, $0 \le i+j \le n+1$, *for* (3.4) *and* (3.5). *In addition, the following conditions are satisfied*:

$$\sum_{k=1}^{m} \varphi_k^{(0,0)}(t) = 1, \qquad \sum_{k=1}^{m} \varphi_k^{(i,j)}(t) = 0, \quad \textit{for } 1 \le i+j \le n+1. \tag{3.12}$$

Remark 3.4. The conditions in (3.12) imply that, for $1 \le i+j \le n+1$, $\varphi^{(i,j)}(0)\mathbb{1}_m = 0$; i.e., $\varphi^{(i,j)}(0)$ is orthogonal to $\mathbb{1}_m$. This fact will help us to obtain the desired exponential decay property of the initial layer terms.

3.2. *Inner Expansion*

In this section, we construct the initial layer terms $\psi^{(i,j)}(\cdot,\cdot)$. It consists of two parts. First, we obtain the sequence $\psi^{(i,j)}(\cdot,\cdot)$ by solving a system of partial differential equations. Then we present the exponential decay property of the solutions.

3.2.1. *Construction of* $\psi^{(i,j)}(\cdot,\cdot)$

Following the idea in singular perturbation, define the stretched time variables as follows:

$$\tau = \frac{t}{\varepsilon}, \quad \mu = \frac{t}{\delta}. \tag{3.13}$$

Note that $\tau \to \infty$ as $\varepsilon \to 0$, and $\mu \to \infty$ as $\delta \to 0$ for any $t > 0$.

Consider the differential equation $L^{\varepsilon,\delta}\psi^{\varepsilon,\delta,n+1}(\tau,\mu) = 0$, where the operator $L^{\varepsilon,\delta}$ is defined in (2.5) and $\psi^{\varepsilon,\delta,n+1}(\tau,\mu)$ is defined in (3.2). Thus the above equation can be written as

$$\left(\frac{1}{\varepsilon}\frac{\partial}{\partial\tau} + \frac{1}{\delta}\frac{\partial}{\partial\mu}\right)\psi^{\varepsilon,\delta,n+1}(\tau,\mu) = \psi^{\varepsilon,\delta,n+1}(\tau,\mu)Q^{\varepsilon,\delta}(\tau,\mu). \tag{3.14}$$

Taking Taylor expansions of $Q^{\varepsilon,\delta}(t)$ about $t=0$,

$$\begin{aligned}
Q^{\varepsilon,\delta}(t) &= \sum_{k=0}^{n+1} \frac{t^k}{k!}\frac{d^k Q(0)}{dt^k} + R^{(n+1)}(t) \\
&= \sum_{k=0}^{n+1}\left(\varepsilon^{-1}\frac{t^k}{k!}\frac{d^k A(0)}{dt^k} + \delta^{-1}\frac{t^k}{k!}\frac{d^k B(0)}{dt^k} + \frac{t^k}{k!}\frac{d^k \widehat{Q}(0)}{dt^k}\right) + R^{(n+1)}(t) \\
&= \sum_{k=0}^{n+1}\left(\varepsilon^{k-1}\frac{(t/\varepsilon)^k}{k!}\frac{d^k A(0)}{dt^k} + \delta^{k-1}\frac{(t/\delta)^k}{k!}\frac{d^k B(0)}{dt^k}\right. \\
&\qquad \left. + \delta^k \frac{(t/\delta)^k}{k!}\frac{d^k \widehat{Q}(0)}{dt^k}\right) + R^{(n+1)}(t),
\end{aligned}$$

where $R^{(n+1)}(t) = O(t^{n+2})$ uniformly for $t \in [0, T]$. Using τ and μ, we can write the above equation as

$$Q^{\varepsilon,\delta}(\tau, \mu) = \sum_{k=0}^{n+1} \left(\varepsilon^{k-1} \frac{\tau^k}{k!} \frac{d^k A(0)}{dt^k} + \delta^{k-1} \frac{\mu^k}{k!} \frac{d^k B(0)}{dt^k} + \delta^k \frac{\mu^k}{k!} \frac{d^k \widehat{Q}(0)}{dt^k} \right) + R^{(n+1)}(\tau, \mu).$$

Dropping the term $R^{(n+1)}(\tau, \mu)$ in $Q^{\varepsilon,\delta}(t)$, it follows from (3.14)

$$\left(\varepsilon^{-1} \frac{\partial}{\partial \tau} + \delta^{-1} \frac{\partial}{\partial \mu} \right) \psi^{\varepsilon,\delta,n+1}(\tau, \mu) = \psi^{\varepsilon,\delta,n+1}(\tau, \mu) \sum_{k=0}^{n+1} \left(\varepsilon^{k-1} \frac{\tau^k}{k!} \frac{d^k A(0)}{dt^k} + \delta^{k-1} \frac{\mu^k}{k!} \frac{d^k B(0)}{dt^k} + \delta^k \frac{\mu^k}{k!} \frac{d^k \widehat{Q}(0)}{dt^k} \right).$$

That is,

$$\sum_{i+j=0}^{n+1} \left(\varepsilon^{-1} \frac{\partial}{\partial \tau} + \delta^{-1} \frac{\partial}{\partial \mu} \right) \varepsilon^i \delta^j \psi^{(i,j)}(\tau, \mu) = \sum_{k=0}^{n+1} \varepsilon^i \delta^j \psi^{(i,j)}(\tau, \mu) \times \sum_{i+j=0}^{n+1} \left(\varepsilon^{k-1} \frac{\tau^k}{k!} \frac{d^k A(0)}{dt^k} + \delta^{k-1} \frac{\mu^k}{k!} \frac{d^k B(0)}{dt^k} + \delta^k \frac{\mu^k}{k!} \frac{d^k \widehat{Q}(0)}{dt^k} \right).$$

Equating the coefficients of $\varepsilon^i \delta^j$, we have for $i + j = 0$,

$$\begin{aligned}
\varepsilon^{-1}\delta^0 &: \quad \frac{\partial \psi^{(0,0)}(\tau, \mu)}{\partial \tau} = \psi^{(0,0)}(\tau, \mu) A(0), \\
\varepsilon^0\delta^{-1} &: \quad \frac{\partial \psi^{(0,0)}(\tau, \mu)}{\partial \mu} = \psi^{(0,0)}(\tau, \mu) B(0), \\
\varepsilon^0\delta^0 &: \quad \frac{\partial \psi^{(1,0)}(\tau, \mu)}{\partial \tau} + \frac{\partial \psi^{(0,1)}(\tau, \mu)}{\partial \mu} \\
&\qquad = \psi^{(0,0)}(\tau, \mu) D(\tau, \mu) + \psi^{(1,0)}(\tau, \mu) A(0) \\
&\qquad\quad + \psi^{(0,1)}(\tau, \mu) B(0) \\
\varepsilon^1\delta^{-1} &: \quad \frac{\partial \psi^{(1,0)}(\tau, \mu)}{\partial \mu} = \psi^{(1,0)}(\tau, \mu) B(0), \\
\varepsilon^{-1}\delta^1 &: \quad \frac{\partial \psi^{(0,1)}(\tau, \mu)}{\partial \tau} = \psi^{(0,1)}(\tau, \mu) A(0),
\end{aligned} \tag{3.15}$$

and for $1 \le i + j \le n$,

$$\begin{aligned}
\varepsilon^2\delta^{-1} \quad &: \quad \frac{\partial \psi^{(2,0)}(\tau,\mu)}{\partial \mu} = \psi^{(2,0)}(\tau,\mu)B(0),\\
\varepsilon^1\delta^{0} \quad &: \quad \frac{\partial \psi^{(2,0)}(\tau,\mu)}{\partial \tau} + \frac{\partial \psi^{(1,1)}(\tau,\mu)}{\partial \mu}\\
&\qquad = \psi^{(0,0)}(\tau,\mu)\frac{d^2A(0)}{dt^2}\tau^2 + \psi^{(1,0)}(\tau,\mu)D(\tau,\mu)\\
&\qquad\quad + \psi^{(2,0)}(\tau,\mu)A(0) + \psi^{(1,1)}(\tau,\mu)B(0)\\
\varepsilon^0\delta^{1} \quad &: \quad \frac{\partial \psi^{(1,1)}(\tau,\mu)}{\partial \tau} + \frac{\partial \psi^{(0,2)}(\tau,\mu)}{\partial \mu}\\
&\qquad = \psi^{(0,0)}(\tau,\mu)\frac{d^2B(0)}{dt^2}\mu^2 + \psi^{(0,1)}(\tau,\mu)D(\tau,\mu)\\
&\qquad\quad + \psi^{(1,1)}(\tau,\mu)A(0) + \psi^{(0,2)}(\tau,\mu)B(0)\\
\varepsilon^{-1}\delta^{2} \quad &: \quad \frac{\partial \psi^{(0,2)}(\tau,\mu)}{\partial \tau} = \psi^{(0,2)}(\tau,\mu)A(0),\\
&\dots,
\end{aligned} \tag{3.16}$$

where

$$D(\tau,\mu) = \tau\frac{dA(0)}{dt} + \mu\frac{dB(0)}{dt} + \widehat{Q}(0).$$

We will construct $\psi^{(i,j)}(\cdot,\cdot)$ by solving (3.15) and (3.16).

THEOREM 3.5. *Assume* (A1) *and* (A2). *Then there exist continuously differentiable solutions* $\psi^{(i,j)}(t)$. $0 \le i + j \le n$ *for the systems* (3.15) *and* (3.16). *The solution to* (3.15) *can be expressed as*

$$\begin{aligned}
\psi^{(0,0)}(\tau,\mu) &= \big(p^0 - \varphi^{(0,0)}(0)\big)\exp(A(0)\tau + B(0)\mu),\\
\psi^{(1,0)}(\tau,\mu) &= H^{(1,0)}(\tau)\exp(A(0)\tau + B(0)\mu),\\
\psi^{(0,1)}(\tau,\mu) &= H^{(0,1)}(\mu)\exp(A(0)\tau + B(0)\mu),
\end{aligned} \tag{3.17}$$

where the row vectors $H^{(1,0)}(\tau)$ *and* $H^{(0,1)}(\mu)$ *are given by*

$$\begin{aligned}
H^{(1,0)}(\tau) &= (p^0 - \varphi^{(0,0)}(0))\int_0^\tau \exp(A(0)s)\frac{dA(0)}{dt}\\
&\qquad\qquad \times \exp(-A(0)s)s\,ds - \varphi^{(1,0)}(0),\\
H^{(0,1)}(\mu) &= \frac{1}{2}\mu^2\big(p^0 - \varphi^{(0,0)}(0)\big)\frac{dB(0)}{dt}\\
&\qquad + \mu(p^0 - \varphi^{(0,0)}(0))\widehat{Q}(0) - \varphi^{(0,1)}(0).
\end{aligned} \tag{3.18}$$

For $2 \le i + j \le n + 1$, *the functions* $\psi^{(i,j)}(\cdot,\cdot)$ *can be expressed similarly.*

Proof. Construction of $\psi^{i,j}(\tau,\mu)$ for $0 \le i+j \le 1$: Let us determine $\psi^{(0,0)}(\tau,\mu)$ from the first two equations of (3.15). First, $\psi^{(0,0)}(\tau,\mu)$ can be taken as

$$\psi^{(0,0)}(\tau,\mu) = C^{(0,0)} \exp(A(0)\tau + B(0)\mu), \tag{3.19}$$

where $C^{(0,0)}$ is a constant vector to be determined by the initial value. From the first equation of (3.15), we have

$$\psi^{(0,0)}(\tau,\mu) = C_1(\mu) \exp(A(0)\tau), \tag{3.20}$$

where $C_1(\mu)$ is a vector-valued function. To determine $C_1(\mu)$, substituting (3.20) into the second equation of (3.15), we obtain that

$$\frac{dC_1(\mu)}{d\mu} \exp(A(0)\tau) = C_1(\mu) \exp(A(0)\tau) B(0).$$

Since $A(0)$ and $B(0)$ commute, $\exp(A(0)\tau)B(0) = B(0)\exp(A(0)\tau)$, and

$$\frac{dC_1(\mu)}{d\mu} \exp(A(0)\tau) = C_1(\mu) B(0) \exp(A(0)\tau). \tag{3.21}$$

Note that $\exp(A(0)\tau)$ is invertible for any matrix $A(0)$ and its inverse is $\exp(-A(0)\tau)$. Postmultiplying $\exp(-A(0)\tau)$ to both sides of (3.21), we obtain $\frac{dC_1(\mu)}{d\mu} = C_1(\mu)B(0)$. It follows that $C_1(\mu) = C^{(0,0)}\exp(B(0)\mu)$, where $C^{(0,0)}$ is a constant vector. Substituting $C_1(\mu)$ into (3.20), $\psi^{(0,0)}(\tau,\mu) = C^{(0,0)}\exp(B(0)\mu)\exp(A(0)\tau)$. Note that $\exp(B(0)\mu)\exp(A(0)\tau) = \exp(A(0)\tau + B(0)\mu)$. It follows that $\psi^{(0,0)}(\tau,\mu) = C^{(0,0)} \exp(A(0)\tau + B(0)\mu)$. Thus (3.19) is obtained.

To solve for $\psi^{(1,0)}(\tau,\mu)$ and $\psi^{(0,1)}(\tau,\mu)$, start with the fourth and fifth equations in (3.15). Assume that the solutions can be written as

$$\begin{aligned} \psi^{(1,0)}(\tau,\mu) &= H^{(1,0)}(\tau)\exp(A(0)\tau + B(0)\mu) \\ \psi^{(0,1)}(\tau,\mu) &= H^{(0,1)}(\mu)\exp(A(0)\tau + B(0)\mu), \end{aligned}$$

where $H^{(1,0)}(\cdot)$ and $H^{(0,1)}(\cdot)$ are vector-valued functions to be determined later. As in the construction of the outer expansion terms, $H^{(1,0)}(\cdot)$ and $H^{(0,1)}(\cdot)$ need to be determined by the use of the third equation in (3.15). In fact,

$$\begin{aligned} &\left(\frac{dH^{(1,0)}(\tau)}{d\tau} + \frac{dH^{(0,1)}(\mu)}{d\mu}\right)\exp(A(0)\tau + B(0)\mu) \\ &\quad = \tau C^{(0,0)} \exp(A(0)\tau + B(0)\mu)\frac{dA(0)}{dt} + \mu C^{(0,0)}\exp(A(0)\tau \\ &\qquad + B(0)\mu)\frac{dB(0)}{dt} + C^{(0,0)}\exp(A(0)\tau + B(0)\mu)\widehat{Q}(0). \end{aligned}$$

We separate the above equation into two equations

$$\frac{dH^{(1,0)}(\tau)}{d\tau}\exp(A(0)\tau+B(0)\mu)=\tau C^{(0,0)}\exp(A(0)\tau+B(0)\mu)\frac{dA(0)}{dt},$$
$$\frac{dH^{(0,1)}(\mu)}{d\mu}\exp(A(0)\tau+B(0)\mu)=\mu C^{(0,0)}\exp(A(0)\tau+B(0)\mu)\frac{dB(0)}{dt}$$
$$+C^{(0,0)}\exp(A(0)\tau+B(0)\mu)\widehat{Q}(0).$$

Note that $\exp(A(0)\tau+B(0)\mu)$ is invertible and its inverse is $\exp(-A(0)\tau-B(0)\mu)$. By postmultiplying $\exp(-A(0)\tau-B(0)\mu)$ to both sides of the above two equations, we have

$$\frac{dH^{(1,0)}(\tau)}{d\tau}=\tau C^{(0,0)}\exp(A(0)\tau+B(0)\mu)\frac{dA(0)}{dt}\exp(-A(0)\tau-B(0)\mu),$$
$$\frac{dH^{(0,1)}(\mu)}{d\mu}=\mu C^{(0,0)}\exp(A(0)\tau+B(0)\mu)\frac{dB(0)}{dt}\exp(-A(0)\tau-B(0)\mu)$$
$$+C^{(0,0)}\exp(A(0)\tau+B(0)\mu)\widehat{Q}(0)\exp(-A(0)\tau-B(0)\mu).$$

In addition, the following communitivity holds

$$\frac{dA(0)}{dt}\exp(B(0)\mu)=\exp(B(0)\mu)\frac{dA(0)}{dt},$$
$$\frac{dB(0)}{dt}\exp(A(0)\tau+B(0)\mu)=\exp(A(0)\tau+B(0)\mu)\frac{dB(0)}{dt},$$
$$\widehat{Q}(t)\exp(A(0)\tau+B(0)\mu)=\exp(A(0)\tau+B(0)\mu)\widehat{Q}(t).$$

It follows that

$$\frac{dH^{(1,0)}(\tau)}{d\tau}=\tau C^{(0,0)}\exp(A(0)\tau)\frac{dA(0)}{dt}\exp(-A(0)\tau),$$
$$\frac{dH^{(0,1)}(\mu)}{d\mu}=\mu C^{(0,0)}\frac{dB(0)}{dt}+\ C^{(0,0)}\widehat{Q}(0).$$

The solutions to the above equations are

$$\begin{aligned}H^{(1,0)}(\tau)&=C^{(0,0)}\int_0^\tau\exp(A(0)s)\frac{dA(0)}{dt}\exp(-A(0)s)s\,ds+C^{(1,0)},\\ H^{(0,1)}(\mu)&=\frac{1}{2}\mu^2C^{(0,0)}\frac{dB(0)}{dt}+\mu C^{(0,0)}\widehat{Q}(0)+C^{(0,1)},\end{aligned}\tag{3.22}$$

where $C^{(1,0)}$ and $C^{(0,1)}$ are constant row vectors to be determined by the initial data. Choosing matched (between outer and inner terms) initial conditions leads to

$$\varphi^{(0,0)}(0)+\psi^{(0,0)}(0,0)=p^0,\quad \varphi^{(1,0)}(0)+\psi^{(1,0)}(0,0)=0,$$
$$\varphi^{(0,1)}(0)+\psi^{(0,1)}(0,0)=0.$$

The first equation together with (3.19) gives $C^{(0,0)} = p^0 - \varphi^{(0,0)}(0)$ and the second and third equations together with (3.22) imply that $C^{(1,0)} = -\varphi^{(1,0)}(0)$, and $C^{(0,1)} = -\varphi^{(0,1)}(0)$. Substituting $C^{(0,0)}$, $C^{(1,0)}$, and $C^{(0,1)}$ into (3.22), we get

$$\begin{aligned} H^{(1,0)}(\tau) &= \big(p^0 - \varphi^{(0,0)}(0)\big) \int_0^\tau \exp(A(0)s) \frac{dA(0)}{dt} \\ &\quad \times \exp(-A(0)s)s\,ds - \varphi^{(1,0)}(0), \\ H^{(0,1)}(\mu) &= \frac{1}{2}\mu^2 \big(p^0 - \varphi^{(0,0)}(0)\big) \frac{dB(0)}{dt} + \mu\big(p^0 - \varphi^{(0,0)}(0)\big)\widehat{Q}(0) \\ &\quad - \varphi^{(0,1)}(0). \end{aligned}$$

This completes the construction of $\psi^{(i,j)}(\cdot,\cdot)$ for $0 \le i+j \le 1$.

The construction of the solutions for $2 \le i+j \le n+1$ is similar. We omit the details. ∎

3.2.2. *Exponential Decay of* $\psi^{(i,j)}(\cdot,\cdot)$

We obtained the asymptotic series $\psi^{(i,j)}(\cdot,\cdot)$ in Theorem 3.5. In this section, we will verify the exponential decay properties of the solutions. Similar to Lemma A.2 in [15, p.300], we have the following result.

LEMMA 3.6. *Suppose that assumptions* (A1) *and* (A2) *are satisfied.*

(a) *Suppose $P(s)$ is the solution to the differential equation*:

$$\frac{dP(s)}{ds} = P(s)A(0), \quad P(0) = I. \tag{3.23}$$

Then $P(s) \to \overline{P}_A$ as $s \to \infty$ and

$$\big|\exp(A(0)s) - \overline{P}_A\big| \le K_A \exp(-\kappa_A s), \quad \text{for some } \kappa_A > 0, \tag{3.24}$$

where $\overline{P}_A = \operatorname{diag}(\mathbb{1}_{m_0}\nu(0), \dots, \mathbb{1}_{m_0}\nu(0))$ and $\nu(0) = (\nu_1(0), \dots, \nu_{m_0}(0)) \in \mathbb{R}^{1\times m_0}$ is the quasi-stationary distribution corresponding to the generator $\widetilde{Q}(0)$.

(b) *Suppose $P(s)$ is the solution to the differential equation*:

$$\frac{dP(s)}{ds} = P(s)B(0), \quad P(0) = I. \tag{3.25}$$

Then $P(s) \to \overline{P}_B$ as $s \to \infty$ and

$$\big|\exp(B(0)s) - \overline{P}_B\big| \le K_B \exp(-\kappa_B s), \quad \text{for some } \kappa_B > 0, \tag{3.26}$$

where

$$\overline{P}_B = \begin{pmatrix} \lambda_1(0)I_{m_0} & \dots & \lambda_l(0)I_{m_0} \\ \vdots & & \vdots \\ \lambda_1(0)I_{m_0} & \dots & \lambda_l(0)I_{m_0} \end{pmatrix} \tag{3.27}$$

and $\lambda(0) = (\lambda_1(0), \dots, \lambda_l(0))$ is the quasi-stationary distribution of the generator $\overline{Q}(0) = (\bar{q}_{ij}(0))$.

Proof. To prove (a), note that $\mathbb{1}_{m_0}\nu(0)$ has identical rows $(\nu_1(0), \dots, \nu_{m_0}(0))$. In view of the block-diagonal structure of $A(0)$, we have

$$P(s) = \exp(A(0)s) = \mathrm{diag}(\exp(\widetilde{Q}(0)s), \dots, \exp(\widetilde{Q}(0)s)).$$

By using Lemma A.2 in [15, p. 300], it is readily seen that there exist constants $K_A > 0$ and $\kappa_A > 0$ such that $|\exp(\widetilde{Q}(0)s) - \mathbb{1}_{m_0}\nu(0)| \le K_A \exp(-\kappa_A s)$. Thus $P(s) \to P_A$ as $s \to \infty$ and $|\exp(\widetilde{A}(0)s) - \overline{P}_A| \le K_A \exp(-\kappa_A s)$. The first result is proved.

To prove (b), note that after some rows and columns are interchanged, $B(0)$ becomes the block diagonal form $B_*(0) = \mathrm{diag}(\overline{Q}(0), \dots, \overline{Q}(0))$. That is, there is an invertible constant matrix $R \in \mathbb{R}^{m\times m}$ such that $RB(0)R^{-1} = B_*(0)$. Similar to the first part of this lemma, we have

$$\exp(B_*(0)s) \to \mathrm{diag}(\mathbb{1}_l\lambda(0), \dots, \mathbb{1}_l\lambda(0)) \in \mathbb{R}^{m\times m} \overset{\text{def}}{=} \overline{P}_B, \quad \text{as } s \to \infty,$$

where $\mathrm{diag}(\mathbb{1}_l\lambda(0), \dots, \mathbb{1}_l\lambda(0))$ is a block diagonal matrix with m_0 blocks. Consequently,

$$R^{-1}\exp(B_*(0)s)R \to R^{-1}\mathrm{diag}(\mathbb{1}_l\lambda(0), \dots, \mathbb{1}_1\lambda(0))R, \quad \text{as } s \to \infty.$$

Note that

$$R^{-1}\exp(B_*(0)s)R = \exp(R^{-1}B_*(0)Rs) = \exp(B(0)s)$$

and $\overline{P}_B$ has the form given by (3.27). Thus we have $\exp(B(0)s) \to \overline{P}_B$ as $s \to \infty$ with the desired convergence rate. This completes the proof. ∎

LEMMA 3.7. *Suppose that assumptions* (A1) *and* (A2) *are satisfied. Let P_A and P_B be defined as in Lemma* 3.6. *Then there exist constants $K_A > 0$, $K_B > 0$, $\kappa_A > 0$, and $\kappa_B > 0$ such that* $|\exp(A(0)s + B(0)t) - \overline{P}_A\overline{P}_B| \le K_A \exp(-\kappa_A s) + K_B \exp(-\kappa_B t)$.

Proof. Notice the fact that $A(t)$ and $B(t)$ commute. We start with the inequality:

$$\begin{aligned}
&\left|\exp(A(0)s + B(0)t) - \overline{P}_A\overline{P}_B\right| \\
&\quad = \left|\exp(A(0)s + B(0)t) - \exp(A(0)s)\overline{P}_B + \exp(A(0)s)\overline{P}_B - \overline{P}_A\overline{P}_B\right| \\
&\quad = \left|\exp(A(0)s)(\exp(B(0)t) - \overline{P}_B) + (\exp(A(0)s) - \overline{P}_A)\overline{P}_B\right| \\
&\quad \le \left|\exp(A(0)s)\right|\left|\exp(B(0)t) - \overline{P}_B\right| + \left|\exp(A(0)s) - \overline{P}_A\right|\left|\overline{P}_B\right|.
\end{aligned}$$

By Lemma 3.6, there exist constants $K'_A > 0$, $K'_B > 0$, $\kappa_A > 0$, and $\kappa_B > 0$ such that

$$\begin{aligned}
\left|\exp(A(0)s) - \overline{P}_A\right| &\le K'_A \exp(-\kappa_A s), \quad \text{and} \\
\left|\exp(B(0)t) - \overline{P}_B\right| &\le K'_B \exp(-\kappa_B t).
\end{aligned}$$

Since $|\overline{P}_B| \leq M_B$ for some constant $M_B > 0$ and $|\exp(A(0)s| \leq M_A$ for some constant $M_A > 0$, we have

$$\begin{aligned}\left|\exp(A(0)s + B(0)t) - \overline{P}_A\overline{P}_B\right| &\leq M_A K'_B \exp(-\kappa_B t) + M_B K'_A \exp(-\kappa_A s)\\ &\leq K_B \exp(-\kappa_B t) + K_A \exp(-\kappa_A s).\end{aligned}$$

This completes the proof. ∎

Now we are in a position to state the result about the exponential decay of $\psi^{(i,j)}(\cdot,\cdot)$.

THEOREM 3.8. *Under the conditions of Theorem* 3.1, *for each i, j with* $0 \leq i + j \leq n+1$, *there exist polynomials $c_A^{i,j}(\tau,\mu)$ and $c_B^{i,j}(\tau,\mu)$, and positive numbers κ_A and κ_B such that*

$$\left|\psi^{(i,j)}(\tau,\mu)\right| \leq c_A^{(i,j)}(\tau,\mu)\exp(-\kappa_A\tau) + c_B^{(i,j)}(\tau,\mu)\exp(-\kappa_B\mu). \quad (3.28)$$

Proof. Note that $\sum_{i=1}^m p_i^0 = 1$, and $\sum_{i=1}^m \varphi_i^{(0,0)}(0) = 1$. It follows that $\sum_{i=1}^m \psi_i^{(0,0)}(0,0) = \sum_{i=1}^m p_i^0 - \sum_{i=1}^m \varphi_i^{(0,0)}(0) = 0$. That is, $\psi_i^{(0,0)}(0,0)$ is orthogonal to $\mathbb{1}_m$. Let $\overline{P}_A$ and $\overline{P}_B$ be defined in Lemma 3.6.

Recall the forms of $\overline{P}_A$ and $\overline{P}_B$. Then

$$\begin{aligned}\overline{P}_A\overline{P}_B &= \begin{pmatrix} \lambda_1(0)\mathbb{1}_{m_0}\pi(0) & \ldots & \lambda_l(0)\mathbb{1}_{m_0}\pi(0)\\ & \ldots & \\ \lambda_1(0)\mathbb{1}_{m_0}\pi(0) & \ldots & \lambda_l(0)\mathbb{1}_{m_0}\pi(0)\end{pmatrix}\\ &= \mathbb{1}_m(\lambda_1(0)\pi(0),\ldots,\lambda_l(0)\pi(0)) = \mathbb{1}_m\pi_\lambda(0).\end{aligned}$$

As a Consequence, $\psi^{(0,0)}(0,0)\overline{P}_A\overline{P}_B = \psi^{(0,0)}(0,0)\mathbb{1}_m\pi_\lambda(0) = 0$, i.e., $\psi^{(0,0)}(0)$ is orthogonal to $\pi_\lambda(0)$. By virtue of Lemma 3.7,

$$\begin{aligned}\left|\psi^{(0,0)}(\tau,\mu)\right| &= \left|\psi^{(0,0)}(0,0)\exp(A(0)\tau + B(0)\mu)\right|\\ &= \left|\psi^{(0,0)}(0,0)\overline{P}_A\overline{P}_B \quad - \psi^{(0,0)}(0,0)\overline{P}_A\overline{P}_B\right.\\ &\qquad \left. + \psi^{(0,0)}(0,0)\exp(A(0)\tau + B(0)\mu)\right|\\ &\leq \left|\psi^{(0,0)}(0,0)\overline{P}_A\overline{P}_B\right|\\ &\qquad + \left|\psi^{(0,0)}(0,0)(\exp(A(0)\tau + B(0)\mu) - \overline{P}_A\overline{P}_B)\right|\\ &= \left|\psi^{(0,0)}(0,0)(\exp(A(0)\tau + B(0)\mu) - \overline{P}_A\overline{P}_B)\right|\\ &\leq \left|\psi^{(0,0)}(0,0)\right|(K_A\exp(-\kappa_A\tau) + K_B\exp(-\kappa_B\mu))\\ &\leq \widetilde{K}_A\exp(-\kappa_A\tau) + \widetilde{K}_B\exp(-\kappa_B\mu). \end{aligned} \quad (3.29)$$

That is, $\psi^{0,0}(\tau,\mu)$ decays exponentially fast.

Next consider $\psi^{(0,1)}(\tau,\mu)$. Recall that $\psi^{(0,1)}(\tau,\mu) = H^{(0,1)}(\mu)\exp(A(0)\tau + B(0)\mu)$, where

$$H^{(0,1)}(\mu)=\frac{1}{2}\mu^2\big(p^0-\varphi^{(0,0)}(0)\big)\frac{dB(0)}{dt}+\mu\big(p^0-\varphi^{(0,0)}(0)\big)\widehat{Q}(0)-\varphi^{(0,1)}(0).$$

Note that $\varphi^{(0,1)}(0)\mathbb{1} = \sum_{i=0}^{m}\varphi_i^{(0,1)}(0) = 0$. Since $\widehat{Q}(0)$ and B(0) are generators, $\widehat{Q}(0)\mathbb{1}_m = 0$, and $\frac{dB(0)}{dt}\mathbb{1}_m = \frac{d(B(0)\mathbb{1}_m)}{dt} = 0$. Thus we obtain

$$\begin{aligned}H^{(0,1)}(\mu)\mathbb{1}_m &= \left(\frac{1}{2}\mu^2(p^0-\varphi^{(0,0)}(0)\right)\frac{dB(0)}{dt}\\ &\quad+\mu\big(p^0-\varphi^{(0,0)}(0)\big)\widehat{Q}(0)-\varphi^{(0,1)}(0))\mathbb{1}_m\\ &=\frac{1}{2}\mu^2\big(p^0-\varphi^{(0,0)}(0)\big)\frac{dB(0)}{dt}\mathbb{1}_m\\ &\quad+\mu\big(p^0-\varphi^{(0,0)}(0)\big)\widehat{Q}(0)\mathbb{1}_m-\varphi^{(0,1)}(0)\mathbb{1}_m=0;\end{aligned}$$

i.e., $H^{(0,1)}(\mu)$ is orthogonal to $\mathbb{1}_m$. Hence,

$$\begin{aligned}\big|\psi^{(0,1)}(0,0)\big| &= \big|H^{(0,1)}(\mu)\exp(A(0)\tau+B(0)\mu)\big|\\ &=\big|H^{(0,1)}(\mu)\overline{P}_A\overline{P}_B+H^{(0,1)}(\mu)\big(\exp(A(0)\tau+B(0)\mu)-\overline{P}_A\overline{P}_B\big)\big|\\ &=\big|H^{(0,1)}(\mu)\big(\exp(A(0)\tau+B(0)\mu)-\overline{P}_A\overline{P}_B\big)\big|\\ &\le\big|H^{(0,1)}(\mu)\big|\big|\exp(A(0)\tau+B(0)\mu)-\overline{P}_A\overline{P}_B\big|\\ &=\big|H^{(0,1)}(\mu)\big|(K_A\exp(-\kappa_A\tau)+K_B\exp(-\kappa_B\mu))\\ &\le c_A^{(0,1)}(\mu)\exp(-\kappa_A\tau)+c_B^{(0,1)}(\mu)\exp(-\kappa_B\mu),\end{aligned}$$

where $c_A^{(0,1)}(\mu)$ and $c_B^{(0,1)}(\mu)$ are polynomials of degree 2. Thus, $\psi^{(0,1)}(\tau,\mu)$ also decays exponentially fast.

Next we show the exponential decay of $\psi^{(1,0)}(\tau,\mu)$. Recall that

$$\psi^{(1,0)}(\tau,\mu) = H^{(1,0)}(\tau)\exp(A(0)\tau + B(0)\mu),$$

where

$$H^{(1,0)}(\tau) = \big(p^0-\varphi^{(0,0)}(0)\big)\int_0^\tau \exp(A(0)s)\frac{dA(0)}{dt}\exp(-A(0)s)s\,ds - \varphi^{(1,0)}(0).$$

Since $\varphi^{(1,0)}(0)$ is orthogonal to $\mathbb{1}_m$, and

$$\begin{aligned}\frac{dA(0)}{dt}\exp(-A(0)s)\mathbb{1} &= \frac{dA(0)}{dt}\left(\sum_{n=0}^{\infty}\frac{(-1)^n s^n}{n!}(A(0))^n\right)\mathbb{1}_m\\ &=\frac{dA(0)}{dt}\sum_{n=0}^{\infty}\frac{(-1)^n s^n}{n!}((A(0))^n\mathbb{1}_m)\end{aligned}$$

$$= \frac{dA(0)}{dt}\left(I_m \mathbb{1}_m - \frac{s}{1!}(A(0))\mathbb{1}_m + \frac{s^2}{2!}(A(0))^2\mathbb{1}_m + \cdots\right)$$
$$= \frac{dA(0)}{dt}(I_m \mathbb{1}_m) = \frac{dA(0)}{dt}\mathbb{1}_m = 0,$$

$$H^{(1,0)}(\tau)\mathbb{1}_m = \left((p^0 - \varphi^{(0,0)}(0))\int_0^\tau \exp(A(0)s)\frac{dA(0)}{dt}\exp(-A(0)s)s\,ds \right.$$
$$\left. - \varphi^{(1,0)}(0)\right)\mathbb{1}_m$$
$$= (p^0 - \varphi^{(0,0)}(0))\int_0^\tau \exp(A(0)s)\frac{dA(0)}{dt}\exp(-A(0)s)\mathbb{1}\,s\,ds$$
$$- \varphi^{(1,0)}(0)\mathbb{1}_m = 0.$$

So we have shown that $H^{(1,0)}(\tau)$ is orthogonal to $\mathbb{1}_m$. Thus we have

$$\begin{aligned}
\left|\psi^{(1,0)}(0,0)\right| &= \left|H^{(1,0)}(\mu)\exp(A(0)\tau + B(0)\mu)\right| \\
&= \left|H^{(1,0)}(\mu)(\exp(A(0)\tau + B(0)\mu) - \overline{P}_A\overline{P}_B)\right| \\
&\le \left|H^{(1,0)}(\mu)\right|\left|\exp(A(0)\tau + B(0)\mu) - \overline{P}_A\overline{P}_B\right| \\
&= \left|H^{(1,0)}(\mu)\right|(K_A\exp(-\kappa_A\tau) + K_B\exp(-\kappa_B\mu)) \\
&\le c_A^{(1,0)}(\mu)\exp(-\kappa_A\tau) + c_B^{(1,0)}(\mu)\exp(-\kappa_B\mu),
\end{aligned}$$

where $c_A^{(1,0)}(\mu)$ and $c_B^{(1,0)}(\mu)$ are polynomials of degree 2.

The proof for the cases of $2 \le i + j \le n + 1$ are similar to the above. Hence we omit the details. The theorem is proved. ∎

Since the growth of $c_A^{(1,0)}(\mu)$ and $c_B^{(1,0)}(\mu)$ are much slower than exponential, we have the following two corollaries.

COROLLARY 3.9. *For $0 \le i + j \le n + 1$, we have $\left|\psi^{(i,j)}(\tau,\mu)\right| \le K_A\exp(-\kappa_A) + K_B\exp(-\kappa_B)$.*

COROLLARY 3.10. *Suppose there exist constants $h_1 > 0$ and $h_2 > 0$ such that $h_1\varepsilon \le \delta \le h_2\varepsilon$. Then for any integers $0 \le i + j \le n + 1$, $|\tau\psi^{(i,j)}(\tau,\mu)| \le K_1$, and $|\mu\psi^{(i,j)}(\tau,\mu)| \le K_2$, for some constants $K_1 > 0$ and $K_2 > 0$.*

4. ERROR ANALYSIS

In this section we will validate the asymptotic expansion. Recall that the operator $L^{\varepsilon,\delta}$ is defined by (2.5). We first prove a lemma.

LEMMA 4.1. *Suppose that for some* $0 \le k \le n+1$, $\sup_{t\in[0,T]} |L^{\varepsilon,\delta} v^{\varepsilon,\delta}(t)| = O(\varepsilon^k + \delta^k)$ *and* $v^{\varepsilon,\delta}(0) = 0$. *Then* $\sup_{t\in[0,T]} \left|v^{\varepsilon,\delta}(t)\right| \le O(\varepsilon^k + \delta^k)$.

Proof. Let $\eta^{\varepsilon,\delta}(t)$ be a vector-valued function satisfy $\sup_{t\in[0,T]} \left|\eta^{\varepsilon,\delta}(t)\right| O(\varepsilon^k + \delta^k)$. Consider the differential equation

$$L^{\varepsilon,\delta} v^{\varepsilon,\delta}(t) = \eta^{\varepsilon,\delta}(t), \;\; v^{\varepsilon,\delta}(0) = 0. \tag{4.1}$$

Now the solution of the above equation is given by $v^{\varepsilon,\delta}(t) = \int_0^t \eta^{\varepsilon,\delta}(s) X^{\varepsilon,\delta}(t,s)\, ds$, where $X^{\varepsilon,\delta}(t,s)$ is a principal matrix solution. Since $X^{\varepsilon,\delta}(t,s)$ is a transition probability matrix, $|X^{\varepsilon,\delta}(t,s)| \le K$, for all $t, s \in [0,T]$. Therefore, we have the inequality

$$\sup_{t\in[0,T]} \left|v^{\varepsilon,\delta}(t)\right| \le K \sup_{t\in[0,T]} \int_0^t \left|\eta^{\varepsilon,\delta}(s)\right| ds \le K(\varepsilon^k + \delta^k).$$

This completes the proof. ∎

In view of (3.1), $e^{\varepsilon,\delta,k}(t)$ is defined as $e^{\varepsilon,\delta,k}(t) = p^{\varepsilon,\delta}(t) - \varphi^{\varepsilon,\delta,k}(t) - \psi^{\varepsilon,\delta,k}(t/\varepsilon, t/\delta)$, where $p^{\varepsilon,\delta}(\cdot)$ is the solution to (2.2), and $\varphi^{\varepsilon,\delta,k}(\cdot)$ and $\psi^{\varepsilon,\delta,k}(\cdot,\cdot)$ are constructed in previous sections. We will estimate the error term $e^{\varepsilon,\delta,n}(t)$. We have the following result.

PROPOSITION 4.2. *Assume* (A1) *and* (A2). *Suppose there exist constants* $h_1 > 0$ *and* $h_2 > 0$ *such that* $h_1\varepsilon \le \delta \le h_2\varepsilon$. *Then, for* $0 \le i \le n$, $\sup_{t\in[0,T]} \left|e^{\varepsilon,\delta,i}(t)\right| = O(\varepsilon^{i+1} + \delta^{i+1})$.

Proof. We first prove the result for

$$e^{\varepsilon,\delta,1}(t) = p^{\varepsilon,\delta}(t) - \sum_{i+j=0}^{1} \varepsilon^i \delta^j \varphi^{(i,j)}(t) - \sum_{i+j=0}^{1} \varepsilon^i \delta^j \psi^{(i,j)}\left(\frac{t}{\varepsilon}, \frac{t}{\delta}\right).$$

It is easy to see that $e^{\varepsilon,\delta,1}(0) = 0$, and hence the condition of Lemma 4.1 on the initial data holds. By the definition of $L^{\varepsilon,\delta}$, $L^{\varepsilon,\delta} p^{\varepsilon,\delta}(t) = 0$. Consequently, we have

$$\begin{aligned}
L^{\varepsilon,\delta} e^{\varepsilon,\delta,1}(t) &= L^{\varepsilon,\delta} p^{\varepsilon,\delta}(t) - L^{\varepsilon,\delta}\left(\sum_{i+j=0}^{1} \varepsilon^i \delta^j \varphi^{(i,j)}(t)\right) \\
&\quad - L^{\varepsilon,\delta}\left(\sum_{i+j=0}^{1} \varepsilon^i \delta^j \psi^{(i,j)}\left(\frac{t}{\varepsilon}, \frac{t}{\delta}\right)\right) \\
&= -L^{\varepsilon,\delta}\left(\sum_{i+j=0}^{1} \varepsilon^i \delta^j \varphi^{(i,j)}(t)\right) - L^{\varepsilon,\delta}\left(\sum_{i+j=0}^{1} \varepsilon^i \delta^j \psi^{(i,j)}\left(\frac{t}{\varepsilon}, \frac{t}{\delta}\right)\right) \\
&= -\left[\frac{d}{dt}\left(\sum_{i+j=0}^{1} \varepsilon^i \delta^j \varphi^{(i,j)}(t)\right) - \sum_{i+j=0}^{1} \varepsilon^i \delta^j \varphi^{(i,j)}(t) Q^{\varepsilon,\delta}(t)\right]
\end{aligned}$$

$$-\left[\frac{d}{dt}\left(\sum_{i+j=0}^{1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\right)\right.$$
$$\left.-\sum_{i+j=0}^{1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)Q^{\varepsilon,\delta}(t)\right].$$

Based on the smoothness of $\varphi^{(i,j)}(\cdot)$ on $[0,T]$ and the defining equation (3.4),

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i+j=0}^{1}\varepsilon^i\delta^j\varphi^{(i,j)}(t)\right)-\sum_{i+j=0}^{1}\varepsilon^i\delta^j\varphi^{(i,j)}(t)Q^{\varepsilon,\delta}(t)\\
&=\varepsilon\left(\frac{d\varphi^{(1,0)}(t)}{dt}-\varphi^{(1,0)}(t)\widehat{Q}(t)\right)+\delta\left(\frac{d\varphi^{(0,1)}(t)}{dt}-\varphi^{(0,1)}(t)\widehat{Q}(t)\right)\\
&=O(\varepsilon+\delta).
\end{aligned}$$

Now let us estimate the terms containing $\psi^{(i,j)}(\cdot,\cdot)$. Recall that

$$\begin{aligned}
Q^{\varepsilon,\delta}(t)=\frac{1}{\varepsilon}A(0)+\frac{1}{\delta}B(0)+\left(\tau\frac{dA(0)}{dt}+\mu\frac{dB(0)}{dt}+\widehat{Q}(0)\right)\\
+O(\varepsilon\tau^2+\delta\mu^2+\delta\mu),
\end{aligned}$$

where $\tau=t/\varepsilon$ and $\mu=t/\delta$. According to the exponential decay property of $\psi^{(i,j)}(\cdot,\cdot)$ and the defining equations (3.15) and (3.16),

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i+j=0}^{1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\right)-\sum_{i+j=0}^{1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)Q^{\varepsilon,\delta}(t)\\
&=\frac{d}{dt}\left(\sum_{i+j=0}^{1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\right)-\sum_{i+j=0}^{1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\\
&\quad\times\left(\frac{1}{\varepsilon}A(0)+\frac{1}{\delta}B(0)+\left(\tau\frac{dA(0)}{dt}+\mu\frac{dB(0)}{dt}+\widehat{Q}(0)\right)+O(\varepsilon\tau+\delta\mu)\right)\\
&=-(\varepsilon\psi^{(1,0)}(\tau,\mu)+\delta\psi^{(0,1)}(\tau,\mu))\left(\tau\frac{dA(0)}{dt}+\mu\frac{dB(0)}{dt}+\widehat{Q}(0)\right)\\
&\quad-\sum_{i+j=0}^{1}\varepsilon^i\delta^j\psi^{(i,j)}(\tau,\mu)O(\varepsilon\tau^2+\delta\mu^2+\delta\mu).
\end{aligned}$$

Based on the exponential decay of $\psi^{(i,j)}(\cdot,\cdot)$ and Corollary 3.10, we obtain

$$\frac{d}{dt}\left(\sum_{i+j=0}^{1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\right)-\sum_{i+j=0}^{1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)Q^{\varepsilon,\delta}(t)=O(\varepsilon+\delta).$$

Thus we have proved that $L^{\varepsilon,\delta}e^{\varepsilon,\delta,1}(t) = O(\varepsilon+\delta)$ uniformly in $[0,T]$. By Lemma 4.1, we have $e^{\varepsilon,\delta,1}(t) = O(\varepsilon+\delta)$ uniformly in $[0,T]$. Next we show the result holds for $i = j = 0$. Note that

$$e^{\varepsilon,\delta,1}(t) = e^{\varepsilon,\delta,0}(t) - \varepsilon\varphi^{(1,0)}(t) - \delta\varphi^{(0,1)}(t) - \varepsilon\psi^{(1,0)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right) - \delta\psi^{(0,1)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right).$$

By Corollary 3.10 we have

$$\varepsilon\varphi^{(1,0)}(t) + \delta\varphi^{(0,1)}(t) + \varepsilon\psi^{(1,0)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right) + \delta\psi^{(0,1)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right) = O(\varepsilon+\delta),$$

uniformly in $t \in [0,T]$. Thus $e^{\varepsilon,\delta,0}(t) = O(\varepsilon+\delta)$ uniformly in $t \in [0,T]$.

Similarly, we can estimate $e^{\varepsilon,\delta,n}(t)$. In fact, from

$$e^{\varepsilon,\delta,n+1}(t) = p^{\varepsilon,\delta}(t) - \sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\varphi^{(i,j)}(t) - \sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right),$$

we see that $e^{\varepsilon,\delta,n+1}(0) = 0$, and

$$\begin{aligned}
L^{\varepsilon,\delta}e^{\varepsilon,\delta,n+1}(t) &= L^{\varepsilon,\delta}\left(p^{\varepsilon,\delta}(t) - \sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\varphi^{(i,j)}(t) - \sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\right)\\
&= -L^{\varepsilon,\delta}\left(\sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\varphi^{(i,j)}(t)\right) - L^{\varepsilon,\delta}\left(\sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\right)\\
&= -\left[\frac{d}{dt}\left(\sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\varphi^{(i,j)}(t)\right) - \sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\varphi^{(i,j)}(t)Q^{\varepsilon,\delta}(t)\right]\\
&\quad -\left[\frac{d}{dt}\left(\sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\right)\right.\\
&\quad\left. - \sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)Q^{\varepsilon,\delta}(t)\right].
\end{aligned}$$

It follows from the defining equations of $\varphi^{(i,j)}(\cdot)$,

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\varphi^{(i,j)}(t)\right) - \sum_{i+j=0}^{n+1} \varepsilon^i\delta^j\varphi^{(i,j)}(t)Q^{\varepsilon,\delta}(t)\\
&\quad = \varepsilon^{n+1}\left(\frac{d\varphi^{(1,0)}(t)}{dt} - \varphi^{(1,0)}(t)\widehat{Q}(t)\right)
\end{aligned}$$

$$+\varepsilon^n\delta\left(\frac{d\varphi^{(n,1)}(t)}{dt} - \varphi^{(n,1)}(t)\widehat{Q}(t)\right)$$

$$+\cdots+\delta^{n+1}\left(\frac{d\varphi^{(0,n+1)}(t)}{dt} - \varphi^{(0,n+1)}(t)\widehat{Q}(t)\right).$$

Based on the smoothness of $\varphi^{(i,j)}(\cdot)$ on $[0,T]$ and the assumption $h_1\varepsilon \le \delta \le h_2\varepsilon$, we have

$$\frac{d}{dt}\left(\sum_{i+j=0}^{n+1}\varepsilon^i\delta^j\varphi^{(i,j)}(t)\right) - \sum_{i+j=0}^{n+1}\varepsilon^i\delta^j\varphi^{(i,j)}(t)Q^{\varepsilon,\delta}(t) = O\big(\varepsilon^{n+1}+\delta^{n+1}\big)$$

uniformly in t.

Now we estimate the terms containing $\psi^{(i,j)}(\cdot,\cdot)$. According to the exponential decay property and the expansion of $Q^{\varepsilon,\delta}(\cdot,\cdot)$,

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i+j=0}^{n+1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\right) - \sum_{i+j=0}^{n+1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)Q^{\varepsilon,\delta}(t)\\
&= \frac{d}{dt}\left(\sum_{i+j=0}^{n+1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\right) - \sum_{i+j=0}^{n+1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)\\
&\quad\times\sum_{k=0}^{n+1}\left(\varepsilon^{k-1}\frac{\tau^k}{k!}\frac{d^kA(0)}{dt^k}+\delta^{k-1}\frac{\mu^k}{k!}\frac{d^kB(0)}{dt^k}+\delta^k\frac{\mu^k}{k!}\frac{d^k\widehat{Q}(0)}{dt^k}\right)\\
&\quad-\sum_{i+j=0}^{n+1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)O(\varepsilon^{n+1}\tau^{n+2}+\delta^{n+1}\mu^{n+2})\\
&= O(\varepsilon^{n+1}+\delta^{n+1}) + \sum_{i+j=0}^{n+1}\varepsilon^i\delta^j\psi^{(i,j)}\left(\frac{t}{\varepsilon},\frac{t}{\delta}\right)O\big(\varepsilon^{n+1}\tau^{n+2}+\delta^{n+1}\mu^{n+2}\big)\\
&= O(\varepsilon^{n+1}+\delta^{n+1}),
\end{aligned}$$

so we have $L^{\varepsilon,\delta}e^{\varepsilon,\delta,n+1}(t) = O(\varepsilon^{n+1}+\delta^{n+1})$ uniformly in $[0,T]$. By Lemma 4.1, we have $e^{\varepsilon,\delta,n+1}(t) = O(\varepsilon^{n+1}+\delta^{n+1})$ uniformly in $[0,T]$. Finally, from the expression of $e^{\varepsilon,\delta,n+1}(t)$, we have $e^{\varepsilon,\delta,n+1}(t) = e^{\varepsilon,\delta,n}(t) + O(\varepsilon^{n+1}+\delta^{n+1})$. This implies that $e^{\varepsilon,\delta,n}(t) = O(\varepsilon^{n+1}+\delta^{n+1})$ uniformly in t. This completes the proof of the corollary and hence the proof of Theorem 3.1. ∎

5. FURTHER REMARKS

5.1. *Asymptotic Expansions for $\varepsilon/\delta = o(1)$*

In the previous sections we have constructed expansions for Markov chains with generators that have two small independent parameters such that ε/δ is bounded. In this section, we consider the case $\varepsilon/\delta = o(1)$; i.e., ε goes to zero much faster than δ. To be more specific, let $\varepsilon = \delta^2$.

Suppose that the generator is of the form (2.3) with $\varepsilon = \delta^2$, where $A(t)$, $B(t)$, and $\widehat{Q}(t)$ are as before. We seek asymptotic expansion of the form $p^{\delta}(t) = \varphi^{\delta,n}(t) + \psi^{\delta,n}(t/\delta^2) + e^{\delta,n}(t)$, where the regular part the initial layer corrections are

$$\varphi^{\delta,n}(t) = \sum_{i=0}^{n} \delta^i \varphi^{(i)}(t), \quad \psi^{\delta,n}\Big(\frac{t}{\delta^2}\Big) = \sum_{i=0}^{n} \delta^i \psi^{(i)}\Big(\frac{t}{\delta^2}\Big),$$

respectively, and $e^{\delta,n}(t)$ is the remainder.

5.1.1. *Outer Expansion*

Consider the differential equation $\frac{d\varphi^{\delta,n}(t)}{dt} = \varphi^{\delta,n}(t)Q^{\delta}(t)$; that is,

$$\sum_{i=0}^{n} \delta^i \frac{d\varphi^{(i)}(t)}{dt} = \sum_{i=0}^{n} \delta^i \varphi^{(i)}(t)\Big(\frac{1}{\delta^2}A(t) + \frac{1}{\delta}B(t) + \widehat{Q}(t)\Big).$$

Equating the coefficients of δ^i, we have the following system of equations

$$\begin{aligned}
\delta^{-2} &: \quad 0 = \varphi^{(0)}(t)A(t),\\
\delta^{-1} &: \quad 0 = \varphi^{(1)}(t)A(t) + \varphi^{(0)}(t)B(t),\\
\delta^{0} &: \quad \frac{d\varphi^{(0)}(t)}{dt} = \varphi^{(2)}(t)A(t) + \varphi^{(1)}(t)B(t) + \varphi^{(0)}(t)\widehat{Q}(t),\\
\delta^{1} &: \quad \frac{d\varphi^{(1)}(t)}{dt} = \varphi^{(3)}(t)A(t) + \varphi^{(2)}(t)B(t) + \varphi^{(1)}(t)\widehat{Q}(t),\\
&\quad \dots .
\end{aligned} \tag{5.1}$$

It is easy to see that $\varphi^{(0)}(t) = \pi_{\lambda}(t) = (\lambda_1(t)\pi(t), \dots, \lambda_l(t)\pi(t))$, where $\lambda(\cdot)$ and $\pi(\cdot)$ are as defined previously. In addition, we have, for any $t \in [0, T]$, $\sum_{i=1}^{m} \varphi_i^{(0)}(t) = 1$.

To find the solution $\varphi^{(1)}(t)$, we first notice that $\varphi^{(0)}(t)B(t) = \pi_{\lambda}(t)B(t) = 0$. Now from the second equation of (5.1), we have $\varphi^{(1)}(t)A(t) = 0$. Hence the vector-valued function $\varphi^{(1)}(t) = (x_1(t)\pi(t), \dots, x_l(t)\pi(t))$ is a solution to the second equation in (5.1), where $x(t) \in \mathbb{R}^{1\times l}$ is any vector-valued function that satisfies the condition $\sum_{i=1}^{l} x_i(t) = 0$.

To obtain $\varphi^{(2)}(t)$, we examine the third equation in (5.1). By substituting $\varphi^{(0)}(t)$ and $\varphi^{(1)}(t)$ into the equation, we have

$$\begin{aligned}
&\Big(\frac{d\lambda_1(t)}{dt}\pi(t), \dots, \frac{d\lambda_l(t)}{dt}\pi(t)\Big) + \Big(\lambda_1(t)\frac{d\pi(t)}{dt}, \dots, \lambda_1(t)\frac{d\pi(t)}{dt}\Big)\\
&\quad = \varphi^{(2)}(t)A(t) + (x_1(t)\pi(t), \dots, x_l(t)\pi(t))B(t)\\
&\qquad + (\lambda_1(t)\pi(t), \dots, \lambda_l(t)\pi(t))\widehat{Q}(t).
\end{aligned} \tag{5.2}$$

To solve (5.2), it suffices to solve the following two equations derived from it,

$$\left(\lambda_1(t)\frac{d\pi(t)}{dt},\ldots,\lambda_1(t)\frac{d\pi(t)}{dt}\right)=\varphi^{(2)}(t)A(t)\quad\text{and}\tag{5.3}$$

$$\begin{aligned}\left(\frac{d\lambda_1(t)}{dt}\pi(t),\ldots,\frac{d\lambda_l(t)}{dt}\pi(t)\right)&=(x_1(t)\pi(t),\ldots,x_l(t)\pi(t))B(t)\\&\quad+(\lambda_1(t)\pi(t),\ldots,\lambda_l(t)\pi(t))\widehat{Q}(t).\end{aligned}\tag{5.4}$$

We first look at (5.4). It can be reduced to

$$\begin{aligned}\left(\frac{d\lambda_1(t)}{dt},\ldots,\frac{d\lambda_l(t)}{dt}\right)&=(x_1(t),\ldots,x_l(t))(\bar{q}_{ij}(t))\\&\quad+(\lambda_1(t),\ldots,\lambda_l(t)(t))(\hat{q}_{ij}(t)).\end{aligned}$$

Note that $(\frac{d\lambda_1(t)}{dt},\ldots,\frac{d\lambda_l(t)}{dt})\mathbb{1}_l=0$, and $(\hat{q}_{ij}(t))\mathbb{1}_l=0$. By the weak irreducibility of $(\bar{q}_{ij}(t))$, we can uniquely solve the equation for $x(t)$ with the condition $\sum_{i=1}^{l}x_i(t)=0$. The last equality implies $\sum_{i=1}^{m}\varphi_i^{(1)}(t)=0$. Similarly, we can solve (5.3) for $\varphi^{(2)}(t)$ so that $\sum_{i=1}^{m}\varphi_i^{(2)}(t)=0$. For $i\geq 3$, $\varphi^{(i)}(t)$ can be obtained similarly.

5.1.2. *Initial Layer Correction*

To construct the boundary layer terms, we consider another time scale $\tau=t/\delta^2$. Consider the differential equation

$$\frac{d\psi^{\varepsilon,n}(t/\delta^2)}{dt}=\psi^{\varepsilon,n}(t/\delta^2)\left(\frac{1}{\delta^2}A(t)+\frac{1}{\delta}B(t)+\widehat{Q}(t)\right).\tag{5.5}$$

Taking Taylor expansion of $Q^\delta(t)$ at $t=0$, we have

$$\begin{aligned}Q^\delta(\delta^2\tau)&=\sum_{k=0}^{n+1}\left(\delta^{2k-2}\frac{\tau^k}{k!}\frac{d^kA(0)}{dt^k}+\delta^{2k-1}\frac{\tau^k}{k!}\frac{d^kB(0)}{dt^k}+\delta^{2k}\frac{\tau^k}{k!}\frac{d^k\widehat{Q}(0)}{dt^k}\right)\\&\quad+R^{(n+1)}(\delta^2\tau),\end{aligned}$$

where $R^{(n+1)}(t)=O(t^{n+2})$. Drop the term $R^{(n+1)}(\delta^2\tau)$ and substitute the rest of the terms into (5.5),

$$\begin{aligned}\frac{d}{d\tau}\left(\sum_{i=0}^{n+1}\delta^i\psi^{(i)}(\tau)\right)&=\sum_{i=0}^{n+1}\delta^i\psi^{(i)}(\tau)\sum_{k=0}^{n+1}\left(\delta^{2k-2}\frac{\tau^k}{k!}\frac{d^kA(0)}{dt^k}\right.\\&\quad\left.+\delta^{2k-1}\frac{\mu^k}{k!}\frac{d^kB(0)}{dt^k}+\delta^k\frac{\mu^k}{k!}\frac{d^k\widehat{Q}(0)}{dt^k}\right).\end{aligned}$$

Equating the coefficients of δ^i, we obtain

$$\begin{aligned}
&\frac{d}{d\tau}\psi^{(0)}(\tau) = \psi^{(0)}(\tau)A(0),\\
&\frac{d}{d\tau}\psi^{(1)}(\tau) = \psi^{(1)}(\tau)A(0) + \psi^{(0)}(\tau)B(0),\\
&\frac{d}{d\tau}\psi^{(2)}(\tau) = \psi^{(2)}(\tau)A(0) + \psi^{(1)}(\tau)B(0) + \psi^{(0)}(\tau)\\
&\qquad\qquad \times\left(\tau\frac{dA(0)}{dt} + \widehat{Q}(0)\right),\\
&\cdots.
\end{aligned} \tag{5.6}$$

The solutions to (5.6) are

$$\begin{aligned}
\psi^{(0)}(\tau) &= \psi^{(0)}(0)\exp(A(0)\tau),\\
\psi^{(1)}(t) &= \psi^{(1)}(0)\exp(A(0)\tau)\\
&\quad + \int_0^\tau \psi^{(0)}(s)B(s)\exp(A(0)(\tau - s))\,ds,\\
\psi^{(2)}(t) &= \psi^{(2)}(0)\exp(A(0)\tau)\\
&\quad + \int_0^\tau (\psi^{(1)}(s)B(s) + \psi^{(0)}(s)D(s))\\
&\quad \times \exp(A(0)(\tau - s))\,ds,\\
&\cdots,
\end{aligned} \tag{5.7}$$

where $D(s) = (s\frac{dA(0)}{dt} + \widehat{Q}(0))$. The initial values of $\psi^{(i)}(0)$ satisfy $\psi^{(0)}(0) = p^0 - \varphi^{(0)}(0)$, $\psi^{(i)}(0) = -\varphi^{(i)}(0)$, for $i \geq 1$.

Denote $\overline{P}_A = \operatorname{diag}(\mathbb{1}_{m_0}\nu(0), \ldots, \mathbb{1}_{m_0}\nu(0))$. Similar to the previous sections, $|\exp(A(0)s) - \overline{P}_A| \leq K_A\exp(-\kappa_A s)$ for some $\kappa_A > 0$. In addition, we have the following exponential decay properties of $\psi^{(i)}(\cdot)$. The proof of the following proposition is similar to that of Lemma 3.7 and is thus omitted.

PROPOSITION 5.1. *For each $i = 0, 1, \ldots, n+1$, there exist a polynomial $c^{(i)}(\tau)$ and positive number κ such that $\left|\psi^{(i)}(\tau)\right| \leq c^{(i)}(\tau)\exp(-\kappa\tau)$.*

By virtue of Proposition 5.1, we have for any $i = 1, \ldots, n+1$, $|\tau^k\psi^{(i)}(\tau)| \leq K$, for some $K > 0$ and $i = 1, \ldots, n+1$.

5.1.3. *Asymptotic Validation*

Now we give the estimate for the error term $e^{\delta,n}(t) = p^\delta(t) - \sum_{i=0}^n \varphi^{(i)}(t) - \sum_{i=0}^n \psi^{(i)}(t/\delta^2)$. The next proposition is the asymptotic property of the expansion.

PROPOSITION 5.2. *Assume* (A1) *and* (A2). *Then*, *for* $0 \leq i \leq n$, $\sup_{t\in[0,T]} |e^{\delta, i}(t)| = O(\delta^{i+1})$.

The proof of the proposition follows from Proposition 3.7 and the proof of Proposition 5.1. Thus we omit the details here. As a corollary of the above proposition, we have $\lim_{\delta\to 0} p^{\delta}(t) = \varphi^{(0)}(t) = \pi_{\lambda}(t)$.

5.2. *Concluding Remarks*

In this paper, we have developed asymptotic expansions of the probability vector. The results obtained will be useful for many optimal control problems that involve singularly perturbed Markovian models with multiple time scales. The choice of the generators $A(t)$ and $B(t)$ is largely motivated by the applications in control and optimization of manufacturing systems. If $A(t)$ and $B(t)$ are both weakly irreducible (i.e., consisting of one block), the asymptotic expansion similar to the development discussed in this paper can be obtained. Such a treatment can be used for certain reducible matrices having different forms than those of this paper. However, if $A(t)$ and $B(t)$ have different partitions in the most general form, it appears that the construction of the asymptotic expansion cannot be done as here since the algebraic and differential equations involved may not be consistent. Future study can be directed to the investigation of further probabilistic properties of the model.

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