Generalized parton distributions for large $x$

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Abstract

The $t$-dependence of generalized parton distributions for $x \rightarrow 1$ is discussed. Based on a Fock space expansion, we argue that models, where the $t$-dependence for $x \rightarrow 1$ is through the product $(1-x)t$, are inconsistent. Instead we suggest a leading dependence in terms of $(1-x)^n t$, where $n \geq 2$, for $x \rightarrow 1$.

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1. Introduction

Generalized parton distributions (GPDs) [1–3] are a very powerful theoretical tool, which allows linking parton distributions with form factors as well as many other hadronic matrix elements (for a recent review, see Ref. [4]). Unfortunately, they cannot be measured directly but instead they appear in convolution integrals of the form

$$\text{Amplitude}(\xi, t) \sim \int dx \frac{GPD(x, \xi, t)}{x - \xi \pm i\epsilon}. \quad (1)$$

Since these convolution integrals cannot be easily inverted, GPDs are often ‘extracted’ from the data by writing down a model ansatz with various free parameters which are then fitted to the data. In order to reduce the arbitrariness in this procedure, it is important to incorporate as many theoretical constraints as possible into the ansatz. For the intermediate and large $x$ region a commonly used ansatz for GPDs starts from a simple model for light-cone wave functions. For example, for the case of the pion one writes down a 2-particle wave function of the form

$$\psi(x, k_\perp) \sim f(x) \exp(-\text{const} \cdot M), \quad (2)$$

where one conveniently chooses $M$ such that wave function components with a high kinetic energy are suppressed

$$M = \frac{m^2 + k_\perp^2}{x} + \frac{m^2 + (1-x)q_\perp^2}{1-x}. \quad (3)$$

Upon inserting this type of ansatz into the convolution equations for GPDs [5] at $\xi = 0$

$$H(x, 0, t) = \int d^2k_\perp \psi^\ast(x, k_\perp) \psi(x, k_\perp + (1-x)q_\perp), \quad (4)$$

one finds [3] a $t$-dependence ($t \equiv -q_\perp^2$) that is suppressed by one power of $(1-x)$ for $x \rightarrow 1$.
\[ H(x,0,t) = q(x) \exp\left(\frac{at-x}{x}\right). \]  

Generalizations of Eqs. (2) and (3) to more than two constituents (higher Fock components, baryon) yield the same kind of \( t \)-dependence as Eq. (5).

Obviously, Eq. (5) gives rise to the wrong behavior for \( x \to 0 \) (transverse size grows like \( 1/\sqrt{x} \)), but this does not come as a surprise since one would not expect a good description at small \( x \) from a valence model. Moreover, when \( Q^2 \) is a few GeV\(^2 \), the small \( x \) behavior of GPDs is practically irrelevant for form factors and Compton scattering. Therefore, we will not concern ourselves here with the flaws of the above ansatz at small \( x \).

However, it is widely believed [3,6] that Eq. (5) provides a qualitatively reasonable description in the region of intermediate and large \( x \), where a valence model for hadrons has a chance to make sense.

In this Letter, we argue that at large \( x \) and in particular for \( x \to 1 \) the behavior of Eqs. (2)–(5) is inconsistent.

2. Transverse size

Upon Fourier transforming Eq. (5) to impact parameter space [7–11], one finds

\[
q(x, b_\perp) = \int \frac{d^2 q_\perp}{(2\pi)^2} e^{i b_\perp \cdot q_\perp} H(x, 0, -q_\perp^2) \\
= q(x) \frac{1}{4\pi a} \frac{x}{1-x} \exp\left(-\frac{b_\perp^2 x}{4a (1-x)}\right),
\]

which is exponential. The important feature that led to Eq. (8) was the fact that the dependence on \( t \) was through the combination \( (1-x)t \) for \( x \to 1 \).

While it has been realized that Sudakov effects need to be included for extremely large \( x \) (\( x \sim 0.9 \)) [12], the ansatz (5) is still being widely used for \( -t < 30 \text{ GeV}^2 \), where only \( x < 0.8 \) contribute significantly to a form factor based on Eq. (5). However, already for less extreme values of \( x \), the power law growth of the transverse size \( x \to 1 \) is not only bizarre, but in fact it makes the logic behind the valence ansatz (2), (3) for the light-cone wave function inconsistent: first of all, if a \( q\bar{q} \) pair is separated by a large \( \perp \) distance then its potential energy is very high and therefore one cannot neglect the potential energy in Eq. (3). In the next section we will provide an estimate of that potential energy. Secondly, a \( q\bar{q} \) pair that has a \( \perp \) separation must be connected by a \( \perp \) gauge string. Even if one does not put in such a gauge string “by hand”, any non-perturbative diagonalization of a light-cone Hamiltonian for QCD should yield such a string in the wave function for finite energy hadrons. The presence of such a \( \perp \) gauge string automatically implies that this component of the wave function also contains gluon degrees of freedom in contradiction with the valence ansatz that was used as a starting point (2) and (3). In fact, since the \( \perp \) separation between the quark and the antiquark diverges as \( x \to 1 \), such a state would have to contain an infinite number of gluons as \( x \to 1 \). Similar reasoning applies to a nucleon valence ansatz analogous to Eq. (3).

In summary, the whole logic that starts from a valence ansatz for the light-cone wave function, in which the \( \perp \) momentum dependence is only governed by the kinetic energy, becomes inconsistent for large \( x \). The above light-cone wave function model is the only justification for writing down a \( t \)-dependence of the form \( \exp(atb_\perp^2) \) for large \( x \). Hence we are led to abandon Eq. (5) for large \( x \). Of course, for intermediate \( x \), Eq. (5) may still provide a reasonable and consistent description. However, since form factors and Compton amplitudes, are rather sensitive to the behavior of
\( H(x, 0, Q^2) \) for \( x > 0.5 \) when \( Q^2 > 10 \text{ GeV}^2 \), it becomes necessary to improve on the \( x \to 1 \) behavior of Eq. (5).

3. An improved ansatz for \( H(x, 0, t) \)

If one really wants to know \( H(x, 0, t) \) for \( x \to 1 \), one has to solve QCD. Since we are not yet ready to perform such calculations for this observable with the required accuracy, we want to propose in this section an improved model ansatz for \( H(x, 0, t) \) for \( x \to 1 \).

Even without doing any calculation, it is clear that in order to cure the problem of increasing size as \( x \to 1 \), the \( t \)-dependence must be suppressed with a higher power of \( (1-x) \): a finite \( \perp \) size as \( x \to 1 \) is achieved if and only if the dependence on \( t \) for \( x \to 1 \) is of the form \( t(1-x)^n \) with \( n \geq 2 \), such as

\[
H(x, 0, t) = q(x) \exp \left( at \frac{(1-x)^2}{x} \right), \quad (9)
\]
or

\[
H(x, 0, t) = q(x) \exp \left( at(1-x) \ln \frac{1}{x} \right), \quad (10)
\]

In this section, we would like to present additional plausibility arguments that support this kind of behavior.

In the previous section we indicated already that the infinite transverse size is inconsistent with neglecting the potential energy in the original ansatz (5). In order to estimate the potential energy contribution to the light-cone Hamiltonian for a \( q\bar{q} \) pair that is separated by a \( \perp \) distance \( r_\perp \), we extend the valence picture to include some of the effects from the glue. As a result the model is no longer a strict valence model (which we argued is inconsistent).

Since the gluon string must result in a linearly rising static potential at large distances, we (under-)estimate the effective mass of the QCD string connecting the \( q\bar{q} \) pair to be at least

\[
m_q = \sigma |r_\perp|, \quad (11)
\]

The quantity that enters the light-cone Hamiltonian is the invariant mass of the glue divided by the light-cone momentum carried by the glue. Without actually solving (i.e. diagonalizing) the light-cone Hamiltonian one cannot know how the light-cone momentum is divided among the antiquark and the string of glue, but obviously the glue cannot carry more than momentum fraction \( 1-x \) if \( x \) is the momentum fraction carried by the active quark. This motivates us to consider the following (conservative) ansatz for the light-cone energy of the system, including the effects of the gluon string at large separations,

\[
\hat{M} = M + \frac{\sigma^2 r^2_\perp}{1-x}, \quad (12)
\]

where \( \sigma \approx (440 \text{ MeV})^2 \) is the string tension. This ansatz is conservative because, as we explained above, it only underestimates the light-cone energy of the glue.

Nevertheless, let us estimate the effect of adding such a term in the effective Hamiltonian for a model of the pion which includes a string of gluons.

For \( x \to 1 \), the variables \( k_\perp^2 \) and \( r_\perp^2 \) (which are Fourier conjugate to each other) appear symmetrically in \( \hat{M} \) (that is up to the factor \( 2\sigma^2 \)), which provides the scale: both are divided by one power of \((1-x)\). In order to understand the implications of this result, we consider an ansatz for the effective light-cone Hamiltonian of a \( q\bar{q} \) pair (in this ansatz we are only concerned about the singularity in the energy for \( x \to 1 \))

\[
H = \frac{m^2}{x(1-x)} + \frac{k_\perp^2 + \sigma^2 r_\perp^2}{x(1-x)} + h_L(x), \quad (13)
\]

where \( h_L(x) \) acts on the longitudinal degrees of freedom only. To solve this Hamiltonian we make the ansatz

\[
\psi(x, k_\perp) \to \frac{1}{\chi(x)} \phi(k_\perp), \quad (14)
\]

where \( \phi(k_\perp) \) is one of the eigenfunctions for the harmonic oscillator in 2 space dimensions \( \hat{h}_\text{ho} = \frac{1}{2}k^2_\perp + \frac{1}{2}\sigma^2 r^2_\perp \) with eigenvalue \( E_{\text{ho}} = (n+1)\sigma \) and \( n = 0, 1, 2, \ldots \). This yields an eigenstate of (13) provided \( \chi(x) \) is an eigenstate of

\[
H = \frac{m^2_{\text{eff}}}{x(1-x)} + h_L(x), \quad (15)
\]

where \( m^2_{\text{eff}} = m^2 + 2\sigma \). For a specific example see Ref. [13].

Inserting (14) into the convolution formula (4) yields GPDs where the \( t \)-dependence is suppressed by 2 powers of \((1-x)\) near \( x \to 1 \).
Our analysis has thus illustrated that, even for large $x$ it is dangerous to suppress in an ansatz of the light-cone wave function only components that have a high kinetic energy but not those that have a high potential energy (2)—especially if the resulting state yields a potential energy that diverges badly as $x \to 1$. Our improved ansatz (13) at least builds in some of the effects from the gluon field and thus gives rise to a finite size for $x \to 1$. Of course that this result also demonstrates the difficulties that one encounters in a Fock space expansion. Even at large $x$, where one might have hoped that a Fock space expansion is a good approximation, the finite size as $x \to 1$ indicates the presence of gluons, which is why we were forced to include the gluon string into the model.

Of course, we neglected many things in our analysis. For example, while we included an estimate for the energy from the glue, which we paid attention only to the string tension, we most likely underestimated the energy from the glue, which leaves open the possibility that the suppression of the potential energy that diverges badly as $x \to 1$. However, a more detailed analysis than we are currently capable of doing would be required to specify the correct value of $n$ uniquely.

It is quite possible that the actual behavior of GPDs at large $x$ is even more complicated than the semi-factorized form in Eqs. (9), (10). Because of this possibility, one should regard Eqs. (9), (10) only as one possibility to illustrate the differences to the previously used ansatz. Nevertheless, our main point, i.e., the fact that the $t$-dependence of GPDs for $x \to 1$ should be suppressed by at least 2 powers of $(1-x)$ should be model-independent.

This general result is also supported by perturbative QCD [14], where it was found that the $t$-dependent terms near $x \to 1$ are suppressed by an additional power of $(1-x)^2$ near $x \to 1$.

Recent lattice gauge theory calculations of the r.m.s. radii for the lowest moments of $H(x,0,t)$ indicate a strong suppression for the r.m.s. radii of subsequent moments. In Ref. [15] the r.m.s. radii of non-singlet moments were evaluated for $m_\pi = 897$ MeV. The result indicates a rather rapid decrease of the transverse size with $n$ ($n = 1, 2, 3$): $(b^2_1)^{(3)}/(b^2_1)^{(1)} \approx 0.15$. While the average value of $x$ in $q(x)$ in this calculation is about 0.2, the average value of $x$ in $x^2q(x)$ is about 0.4. The interpretation of this result in Ref. [15] is that $(b^2_1)$ drops by a factor 0.15 as $x$ increases from 0.2 to 0.4. A slightly less dramatic drop is observed for $m_\pi = 744$ MeV. We do not compare this result with the width derived from Eq. (5), since Eq. (5) is unphysical for small $x$. A more realistic model which has both linear behavior as $x \to 1$ and only a logarithmic growth is a model where the $\perp$ width [i.e., the $x$-dependent term multiplying $t$ in Eq. (5)] is $\propto \ln(1/x)$. For such a model the width would only decrease by a factor 0.69 as $x$ increases from 0.2 to 0.4. Similarly, if one attempts to fit simple parameterizations of $H(x,0,t)$ to the lattice data, then a very steep decrease of the $\perp$ width as a function of $x$ is observed. However, r.m.s. radii for higher moments (high enough so they are dominated by $x > 0.5$) are needed to confirm that this rapid decrease as a function of $x$ continues for larger values of $x$.

Recent transverse lattice calculations [16] indicate a shrinking $\perp$ size as $x \to 1$, i.e. $n > 2$, in the case of the pion.

It is interesting to note that GPDs, which have a $t$-dependence that comes with a factor of $(1-x)^3$ near $x \to 1$ naturally give rise to Drell–Yan–West duality between parton distributions at large $x$ and the form factor. Indeed, the ansatz

$$H(x,0,t) = (1-x)^{2N_\pi - 1} \exp\left[\alpha t(1-x)^2\right]$$

yields

$$F(t) = \int dx\ H(x,0,t)$$

$$\Gamma(N_\pi) \frac{1}{2N_\pi} \frac{1}{(t)^{N_\pi}}.$$ (19)

Here only the behavior near $x \to 1$ matters and DYW duality would also arise (with a different coefficient) if the exponential function were being replaced by any function that falls rapidly for $t \to -\infty$, $x$ fixed.

We should also point out that there is an interesting connection between the value of $n$ and the occurrence of color transparency [17], where color transparency does not occur for $n < 2$. 

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4. Summary

We have provided plausibility arguments that a commonly employed valence ansatz for light-cone wave functions, where the $k_\perp$-dependence is only through the light-cone kinetic energy of the quarks, is inconsistent for large $x$ because it leads to a divergent transverse size for those wave function components where one quark carries $x \to 1$.

The origin for this divergent size problem is the fact that the distance between the active quark and the center of momentum of the spectators $r = r_\perp q - r_\perp \bar{q}$ is related to the impact parameter $b_\perp$ (the distance from the active quark to the center of momentum) via $r_\perp = b_\perp / (1 - x)$. In order for the hadron to have a finite size as $x \to 1$, one must have $\langle b_\perp^2 \rangle \sim (1 - x)^n$ with $n \geq 2$, which in turn requires that the leading dependence on $t$ is through the product $(1 - x)^n t$ with $n \geq 2$.

While we are not able to predict what the actual behavior is for $x \to 1$, we made an attempt to include the gluonic energy into an ansatz for the light-cone wave function. With such an ansatz, large size configurations are naturally suppressed and we find GPDs where the $x \to 1$ dependence on $t$ is through the combination $(1 - x)^n t$ with $n = 2$. Of course the estimate that led to this result was very crude and therefore one should not take the value $n = 2$ as a rigorous prediction. Nevertheless, we believe that $n \geq 2$ is a much more reasonable choice in parameterizations than $n = 1$ since $n = 2$ is consistent with hadrons that have a finite size (actually for $n > 2$ it would even be consistent with a vanishing size) for $x \to 1$.

Another interesting observation concerns the Drell–Yan–West duality between the form factor and structure functions since the case $n = 2$ naturally leads to the same duality relation as quark counting rules.

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